Developing a practical robust long term yield curve model

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Abstract

This paper describes the initial development and testing of the Black-corrected version of a workhorse 3-factor Gaussian yield curve (term structure) model, the economic factor model (Dempster et al., 2010) which we have used for many years with Monte Carlo scenario simulation for structured derivative valuation, investment modelling and asset liability management with various time steps and currencies. In common with most alternative approaches in the literature to generating non-negative yields using Black’s idea, we propose a simple approximation to the Black mathematical model using the nonlinear unscented Kalman filter. However, its calibration, unlike that of the current computationally intensive alternatives, requires not significantly more computing time than is needed for the linear Kalman filter with the underlying affine shadow rate model. Initial empirical testing of the new Black EFM model both in- and out-of-sample shows acceptable accuracy, sometimes improved over the affine EFM model, which can be improved by UKF tuning in future research. Migration of the system to the cloud can reduce calibration times for both models from a few hours to a few minutes by exploiting massive parallelization of the computationally intensive step.

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Appendix
Developing a practical robust long term yield curve model

1. Introduction

Since the 2007-2008 financial crisis low interest rates have prevailed in all the world's major developed economies, presaged by more than a decade in Japan. This has posed a problem for the widespread use of diffusion based yield curve models for derivative and other structured financial product pricing and for forward rate simulation for systematic investment and asset liability management. Indeed, while Gaussian models remained sufficiently accurate for pricing and discounting in relatively high rate environments, their tendency to produce an unacceptable proportion of negative forward rates at short maturities with Monte Carlo scenario simulation from initial conditions in low rate environments has called their current use into question. The implications for this question of negative nominal rates in deflationary regimes remain to be seen, as does the necessity for currently fashionable multi-curve models. Be that as it may, beginning with work in the Bank of Japan in the early 2000s, there has recently been a flurry of research in universities, central banks and financial services firms to develop yield curve models whose simulation produces non-negative rate scenarios.

This paper concerns the preliminary development of a robust long term yield curve model which is a (mildly) nonlinear version of our workhorse Gaussian 3-factor affine yield curve model, the Economic Factor Model (EFM), which we have used for Monte Carlo scenario generation over many years in practical structured derivative pricing, investment and asset liability management. We have employed the EFM model in the past using time steps from daily to semi-annual in the five major currency areas: EUR, CHF, GBP, USD and JPY. Its 14 parameters are calibrated to market data using the expectation-maximization (EM) algorithm which alternates the linear Kalman filter (KF) with maximum likelihood parameter estimation (MLE) to convergence. Here we implement a Black(1995) corrected version of the EFM model (Black EFM) using the nonlinear unscented Kalman filter (UKF) in place of the ordinary KF. This represents an approximation to the mathematical Black model in the presence of Black's piecewise linear 0-strike call option nonlinearity which suppresses negative rates. Indeed, while we use the EFM affine closed form expressions for yields at all maturities, it should be noted that the zero lower bound will be inactive for all but those of short, but not necessarily the shortest, maturities.

Our approach appears promising both in calibration and forecasting accuracy relative to both market data and the EFM model, even at short maturities, but further work is necessary in both empirical testing and the development of the UKF.

The remainder of the paper is structured as follows. After enunciating our guiding design requirements, Section 2 summarizes existing yield curve (term structure) models developed for both pricing and forecasting. It illustrates the nature of the difficulties encountered with each model in terms of our requirements. In Section 3, the details of the basic EFM model are set out and the Black correction is defined. Section 4 contains a survey of recent contributions to approximating Black-corrected affine yield curve models whose calibrations are all (too) heavily computationally intensive. These models are often (in our view mis-) designated "shadow rate models", a term introduced by Black for the underlying Gaussian model to be corrected for non-negative rates. Our approximate, but accurate, Black EFM model is described in Section 5, which presents the unscented Kalman filter in detail, including our HPC implementation still under development. Section 6 presents the empirical evaluation of the model in terms of both in-sample calibration and out-of-sample scenario generation, using data for the above five major currency areas. Although there remains room for improvement, the Black EFM model is found to be sufficiently accurate for both purposes.
Conclusions are presented in Section 7, in which it is noted that on average the runtime of its computationally intensive calibration is only double that of the original EFM model -- a very significant improvement on the current alternatives in the literature. An appendix gives a step of the EM algorithm for the EFM model in pseudo code form.

2. Existing models and their drawbacks

The range of yield curve models discussed in the literature is vast. The task of choosing a suitable model for a variety of purposes, including trading, systematic investment, asset liability management and structured product valuation, is non-trivial. There are relatively few papers in the literature that measure (as opposed to just discussing) the comparative advantages of more than a handful of different models, as the implementation of the more complex ones is a time consuming process. Therefore in developing a suitable new model it is important to start out from a clear formulation of model requirements, so that the set of possible suitable choices is limited.

Requirements for model development

The principal applications of the model we envisage are the following:

- Scenario simulation for a diverse set of (predominantly long-term) asset liability management (ALM) problems for multiple currencies
- Valuation of complex structured derivatives and loans (which often have embedded derivatives)
- Risk assessment of portfolios and structured products.

The problem of model selection has been discussed in Dempster et al. (2010) and 2014. We shall borrow some of the requirements for the new yield curve model from these works and extend them here. These include:

1. A continuous-time framework to allow the flexibility of using different time steps, including uneven time steps
2. Exhibit mean-reverting behaviour
3. Allow both pricing and dynamic evolution under the market (real world) measure, i.e. the model should reflect the market risk premium\(^1\)
4. Reproduce a wide range of yield curve shapes and dynamics (to allow for realistic risk assessment, for example), including steepening, flattening\(^2\), inverted and humped yield curves
5. Incorporate realistic modelling of the zero-lower bound (ZLB) empirically observed for zero-coupon bond yields
6. Calculate bond prices computationally feasibly and economically
7. Calibrate the model to market data by estimating parameters similarly
8. Allow estimation and use of the model for multiple correlated yield curves and currency exchange rates

\(^1\) As argued in Nawalkha and Rebonato (2011), this is especially relevant for the buy-side practitioner. For sell-side banks it usually suffices to do pricing and initial hedging calculations under the risk-neutral measure from the forward market data on the day. Having an exact fit to the observed yields is thus more important for the sell-side.

\(^2\) Products based on these properties of the yield curve are traded on the NYSE, e.g. US Treasury Flattener ETN (ticker: FLAT) and iPath US treasury Steepener ETN (ticker: STPP), although admittedly these are not very popular.
9. Parsimony
10. Time-homogeneity.

Clearly, the requirements related to ease and speed of calculations (6-8) contradict the requirements for model realism (2-5), so a compromise is necessary.

The choice of the model made when the prevailing short-term yields are near-zero could be different from that in a high-rate environment, but ideally the chosen model should cover all rate environments. However, the present global low-rate environment is the principal motivation for our work to improve on our workhorse affine yield curve model (Dempster et al., 2010). The most important enhancement of our existing EFM model (see Section 3) is a better way of dealing with the zero-lower bound for initial low short rates.

If we assume that both normal and low rate environments are probable in the medium to long term future, and that once the situation reverts to normal levels away from zero it will be reasonably similar to the nominal rate environment before the crisis, a prudent decision would be to try to construct a model that is suitable for both environments.

Overview of available modelling options

To analyse the potential choices we can first divide most yield curve models into three broad overlapping classes:

- Short rate models
- Models in Heath-Jarrow-Morton (HJM) framework
- Market models.

Short rate models are based on modelling a process for the instantaneous interest rate which is then used to derive zero coupon bond prices, i.e. discount factors, or their yields at different maturities. This class of models is the oldest and probably the most extensively researched.

The HJM models start from modelling instantaneous forward rates directly. A feature of this framework is using the no-arbitrage property to derive constraints on the structure of forward rates. This framework is very general and convenient for studying arbitrage-free properties in theory. However, some of the models in this framework may be non-Markovian and most practical models coming from the HJM framework are either well-known short rate models or market models.

The class of market models is focussed on describing the dynamics of the observable quantities (e.g. LIBOR and SWAP market models). They are especially useful for derivative pricing. However, under the current actually occurring low interest rate conditions, the popular LIBOR Market Model (see de Jong et al., 2001) may require parameter estimates that are unrealistic (e.g. simulated cash returns more volatile than actual equity returns, with significant weight assigned to interest rates of more than 10,000%).

Some authors also categorize stochastic volatility models (such as SABR) separately.

Model and computational complexity considerations, as well as the applications envisaged, suggest that short rate models are the most suitable class for our needs. There are other factors influencing our considerations. First, we have long successful experience with utilization of the economic factor model (EFM model) (see Section 3).
in situations in which the rate zero-lower bound is not binding. We have confidence in
the performance of the EFM yield curve model in these situations, so we would
prefer our new model not to deviate too far from it.

Secondly, most of the current research on zero lower bounds is done in the
framework of short rate models. Having a way of estimating the level of "shadow"
rates may be useful, not least because some policy makers appear to monitor them.
There have been attempts in the research literature to use the "shadow" short rate
and its distance to zero as a forecast of the estimated time until the low-rate regime
is lifted, see Ueno et al. (2006) and Wu and Xia (2014). The Federal Reserve Bank
of Atlanta publishes the Wu-Xia Shadow Federal Funds Rate based on the Wu and
Xia paper. It should be noted, however, that the level of the shadow rate is not a very
reliable indicator, as it is strongly model-dependent (see Bauer and Rudebusch
(2013) and Christensen and Rudebusch (2013)).

A review of the literature on short rate models shows that most popular sub-class of
short rate models in empirical research and applications are the Affine Term
Structure Models (ATSMs), due to their analytical tractability, flexibility and empirical
efficiency.

This class of models includes Vasicek (1977), Dothan (1978), Cox-Ingersoll-Ross
(1985), Ho-Lee (1986), Hull-White (1990) and many other one- and multi-factor
models.

**Analysis of one factor short rate models**

To illuminate the analysis that we undertake below for the more complex multi-factor
models, we first discuss the characteristics of the simpler one-factor models.

The stochastic differential equations (SDEs) governing the evolution of the short
rate under the risk-neutral or *pricing* measure $Q$ for the respective models are: $^3$

1. **Vasicek** (1977)
   \[ dX_t = \lambda(\theta - X_t)dt + \sigma dW_t \]  
   (1)

2. **Dothan** (1978)
   \[ dX_t = -\lambda X_t dt + \sigma X_t dW_t \]  
   (2)

3. **Cox-Ingersoll-Ross** (1985)
   \[ dX_t = \lambda(\theta - X_t)dt + \sigma \sqrt{X_t} dW_t \]  
   (3)

4. **Ho-Lee** (1986)
   \[ dX_t = \theta dt + \sigma dW_t \]  
   (4)

5. **Hull-White** (1990)
   \[ dX_t = \lambda(\theta_t - X_t)dt + \sigma_t dW_t \]  
   (5)

It is easy to see that Hull-White (also called extended Vasicek) is the most general
of these models. It can fit any term structure exactly, because of the time-dependent

$^3$We use boldface type in the sequel to denote stochastic entities, here conditionally.
equilibrium drift coefficients $\theta_t$. However, having such a time-dependent parameter, as in the Ho-Lee and Hull-White models, contradicts our requirements 9-10, i.e. parsimony and time homogeneity.

The Ho-Lee model lacks the desired mean-reversion property and the Vasicek, Ho-Lee and Hull-White diffusion models can all produce negative yields. The Dothan and Cox-Ingersoll-Ross models produce positive yields, but the short rate in these models never hits the zero lower bound. In other words, the zero lower bound in these models is repelling instead of absorbing. This is not consistent with the historical data observed in developed countries.

Note that the multi-factor square root (CIR) model and quadratic Gaussian models (QGMs) are also unable to reproduce the absorbing ZLB.

The Vasicek, Dothan and Cox-Ingersoll-Ross models do not satisfy our requirement 4, i.e. the yield curve shapes attainable with these models are constrained.

Choosing a time-homogenous structure for our model, by not using non-stationary parameters in the corresponding SDE, means that exact matching of the yield curve is not possible with a small number of factors.

**Analysis of multi-factor short rate models**

The number of factors necessary for adequate modelling of the whole term structure has been analyzed in Litterman & Sheinkman (1991). Their principal component analysis of data showed that 99% of the variance can be captured by 3 factors.

It is well-known (see e.g. Nawalkha and Rebonato, 2011) that single-factor and two-factor time-homogenous models deviate significantly from the initially observed bond prices. However, three to five factors produce a close fit. The Nelson-Siegel (1987) model widely-used in central banks uses three factors to estimate the entire yield curve but has time inhomogeneous parameters, except in the Diebold-Rudebusch arbitrage-free version of the model (see Rebonato, 2015).

Rebonato and Cooper (1995) argued that a two-factor affine or quadratic model cannot reproduce a realistic correlation structure of interest rate changes, but that three to five factors are sufficient for this purpose.

So it seems that a reasonable choice (taking into account additional computational complexities connected with introducing the ZLB property) would be an affine short rate model with 3 factors.

**Classification of 3 factor affine short rate models**

Duffie and Kan (1996) derive necessary and sufficient conditions on the SDEs to have an affine representation and Dai and Singleton (2000) analyze the different subfamilies of affine term structure models. Dai and Singleton's analysis for the 3-factor case shows that some affine subfamilies explain historical interest rate behaviour better than others.

The SDEs for the factors are of the form

$$dX_t = \Lambda (\Theta - X_t) dt + \Sigma \sqrt{S_t} dW_t , \quad (6)$$

with $X$ the K-dimensional state vector; $W$ K-dimensional Brownian motion; $\Theta$ a fixed point in $K$-dimensional space; $\Lambda$, $S_t$ and $\Sigma$ $K \times K$ matrices and $S_t$ a diagonal matrix with diagonal elements satisfying
\[
[S_i]_{ii} = \alpha_i + \beta_i^t X_i, \quad i = 1, \ldots, K, \tag{7}
\]

with prime denoting transpose.

For an **admissible** parametrization, the bond prices can be calculated as

\[
P_i(\tau) = e^{A(\tau) - B(\tau)^t X_i}, \tag{8}
\]

where \( A \) and \( B \) are solutions of certain ODEs (see e.g. James and Webber, 2000). The **instantaneous short rate** is also an **affine** function of the state

\[
r_i = \phi_i^0 + \phi_i^E X_i. \tag{9}
\]

Zero coupon bond prices in terms of expectations under the risk-neutral \( Q \) measure are given by

\[
P_i(\tau) = E^Q \left[ \exp \left( - \int_\tau^\tau r_i ds \right) \right]. \tag{10}
\]

Zero coupon bond yields to maturity, termed **rates**, are linked with the bond prices by

\[
y_i(\tau) = -\log P_i(\tau) / \tau. \tag{11}
\]

There are models that lack affine structure (and thus forfeit simple formulae for bond prices) but a vector of \( K \) rates \( R_t \) of specified maturities may sometimes still be recovered as the numerical solution of the **Ricatti** equation

\[
\frac{\partial R_i(\tau)}{\partial \tau} = \Lambda R_i(\tau) - \frac{1}{2} R_i(\tau) \Sigma \Sigma^t R_i(\tau) + r, \tag{12}
\]

where \( 1 \) is the \( K \) vector of ones.

Dai and Singleton (2000) denote different affine subfamilies by \( A_m(n) \) with \( n \) the number of factors and \( m \leq n \) the number of bounded factors. They perform empirical tests on the different subclasses for \( n \) equal to 3.

Dempster et al. (2014) also analyze different 3-factor models with requirements similar to ours to uncover a variety of shortcomings with the models evaluated. They were led to introduce a Black-corrected affine model which always produces non-negative rates.\(^4\) In their paper, they summarized in a table the stylized features satisfied by the alternative models they evaluated, which is reproduced here as Table 1.

Most recent papers considering Black-corrected models have been based on the \( A_3(3) \) class. The EFM 3 factor model described in the next section, which has proven itself in a variety of different applications, also belongs to this class.

\(^4\)We shall describe the Black correction in the next section.
In summary, we have stated a number of desirable requirements for a yield curve model and we have briefly analyzed the range of available models. We determined that the short rate class is the most suitable for our needs and within this class it seems that the most reasonable decision \textit{a priori} is to evaluate the family of models with 3 factors, in particular, within the $A_3(3)$ affine class.

### Table 1. Properties of evaluated yield curve models with regard to stylized facts

<table>
<thead>
<tr>
<th>Stylized Fact Properties</th>
<th>Yield Curve Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CIR</td>
</tr>
<tr>
<td>Mean Reverting Rates</td>
<td>Yes</td>
</tr>
<tr>
<td>Nonnegative Rates</td>
<td>Yes</td>
</tr>
<tr>
<td>Stochastic Rate Volatility</td>
<td>Yes</td>
</tr>
<tr>
<td>Closed Form Bond Prices</td>
<td>Yes</td>
</tr>
<tr>
<td>Replicates All Observed Curves</td>
<td>No</td>
</tr>
<tr>
<td>Good for Long Term Simulations</td>
<td>No</td>
</tr>
<tr>
<td>State Dependent Risk Premia</td>
<td>No</td>
</tr>
<tr>
<td>+ve Rate/Volatility Correlation</td>
<td>No</td>
</tr>
<tr>
<td>Effective in Low Rate Regimes</td>
<td>No</td>
</tr>
</tbody>
</table>

*Rate volatilities are piecewise constant punctuated by random jumps to 0 at rate 0 boundary hitting points.

3. **Towards a long term low rate nonlinear Black model**

\textit{Around the turn of the last century, a famous Austrian economist, Eugen von Bohm-Bawerk (1851-1914), declared that the cultural level of a nation is mirrored by its rate of interest: the higher a people’s intelligence and moral strength, the lower the rate of interest.} (Homer and Sylla, 2005).

As a low rate environment has prevailed in most major developed countries since 2008, in Japan since the early 1990s, it is crucial to realistically model rates behaviour in these circumstances. We will present a Black-corrected version of the economic factor model discussed in Medova \textit{et al.} (2006), Yong (2007) and Dempster \textit{et al.} (2010).
**Basic EFM model**

We first present the original economic factor model (EFM) of the yield curve which we have used previously in a variety of applications in the five principal currency areas with various time steps from daily to quarterly.\(^5\)

The evolution under the risk-neutral measure \(Q\) of the 3 unobservable (i.e. latent) factors of the model is governed by the SDEs

\[
\begin{align*}
\frac{dX_t}{X_t} &= \lambda_X \left( \theta_X - X_t \right) dt + \sigma_X dW^X_t \\
\frac{dY_t}{Y_t} &= \lambda_Y \left( \theta_Y - Y_t \right) dt + \sigma_Y dW^Y_t \\
\frac{dR_t}{R_t} &= k \left( X_t + Y_t - R_t \right) dt + \sigma_R dW^R_t ,
\end{align*}
\]

with fixed pair-wise correlations of the standard Brownian motion innovations given by

\[
\rho_{Xt} dt, \rho_{Xt} dt, \rho_{Xt} dt .
\]

The stochastic evolution of the factors under the market (i.e. real-world) measure \(P\) involving the market prices of risk of the 3 factors is governed by

\[
\begin{align*}
\frac{dX_t}{X_t} &= \lambda_X \left( \theta_X + \frac{\gamma_X \sigma_X}{\lambda_X} - X_t \right) dt + \sigma_X dW^X_t \\
\frac{dY_t}{Y_t} &= \lambda_Y \left( \theta_Y + \frac{\gamma_Y \sigma_Y}{\lambda_Y} - Y_t \right) dt + \sigma_Y dW^Y_t \\
\frac{dR_t}{R_t} &= k \left( X_t + Y_t + \frac{\gamma_R \sigma_R}{k} - R_t \right) dt + \sigma_R dW^R_t .
\end{align*}
\]

The first factor \(X_t\) represents the long rate and the third factor \(R_t\) the short rate. The second factor \(Y_t\) represents minus the slope of the yield curve between the long rate and the unobservable instantaneous short rate. Thus the sum of the first two factors \(X_t + Y_t\) represents the unobservable stochastic instantaneous short rate about which the observable short rate \(R_t\) mean reverts.

Note that as the \(X\) and \(Y\) equations have the same form the factor dynamics under \(Q\) given by (13) are not econometrically identified, i.e. the parameters are not uniquely determined in that different sets will generate the same factor dynamics. However, the factor dynamics under \(P\) given by (15) are identified by virtue of expressing the factor market prices of risk in volatility units. Also note that the rates of mean reversion of the three factors are identical under \(P\) and \(Q\). As a result, the parameters of the dynamics must be estimated from market data under the \(P\) measure and the resulting market price of risk estimates set to 0 to generate the dynamics of the \(Q\) measure for pricing.

The SDEs (13) and (15) have explicit solutions. Substituting the explicit solutions of the SDEs (13) into the sum of the first two factors and using (10) and (11) produces closed-form formulae for bond prices and yields in affine functions of the SDE parameters (see e.g. Medova et al. (2006), and Yong (2007) for the factor

\(^5\)It is interesting to note that this model originated at Long Term Capital Management and was first brought to our attention by Lehman Brothers under the auspices of Pioneer Investments of UniCredit Bank. It has been attributed to Chi Fu Huang but we have been unable to verify this.
covariance matrix). In particular, denoting the 3 latent factors at time $t$ in vector form by $x_t := (X_t, Y_t, R_t)'$, the yields of the $K$ different maturity zero coupon bonds are given in affine form by

$$y_t = Bx_t + d,$$

(16)

where $B$ and $d$ are closed-form deterministic affine functions (matrix or vector-valued) of the SDE parameters.

**Basic EFM calibration**

Calibration of the EFM model is a non-trivial task even without the Black correction. The parameters of the model are estimated using a version of the expectation-maximization (EM) algorithm (Dempster et al., 1977) which iterates to parameter convergence the Kalman filter (KF) to generate sample paths and maximum likelihood estimation (MLE) of parameters for each path. Given a fixed set of parameters, the Kalman filter produces estimates for the unobserved states of the factors and prediction for the yields from (16). These are then used as the observed sample for the next numerical parameter optimization step of MLE.

Note that for the EFM model in state space form, MLE is trying to fit all of the observed rates approximately, in contrast to other approaches which often fit a small number of rates (equal to the number of factors) exactly.

**KF transition equation**

Taking the discretization time step $\Delta t := 1$, the Euler approximation of the SDEs for the 3 factor state variables becomes the state variable transition equation

$$x_{t+1} = Ax_t + c + \eta_t,$$

(17)

where $\eta_t \sim N(0, G)$ is the Gaussian innovation, with $A$, $c$ and $G$ deterministic matrix or vector-valued functions of the SDE coefficients.

**KF measurement equation**

The corresponding measurement equation is

$$y_{t}^{\text{obs}} = Bx_t + d + \varepsilon_t,$$

(18)

where $y_{t}^{\text{obs}}$ corresponds to the yields observed in the market and $B$ and $d$ are defined above. The centred measurement error process $\varepsilon_t$ is a $K$ vector serially independent Gaussian noise with covariance matrix $H$.\(^6\)

Given a data series for the observed yields $y_{t}^{\text{obs}}$, the Kalman filter generates an estimated expected path of the Gaussian state variables, and their conditional

\(^6\)But see Dempster and Tang (2011) regarding handling measurement error serial correlation, which we intend to implement in future research.
covariance matrix $\Sigma_{t-1}$. The filter is initialized using unconditional moments. Following Harvey (1993) gives

$$\hat{x}_0 := (I - A)^{-1} c$$

$$\text{vec}(\Sigma_0) := (I - A \otimes A)^{-1} \text{vec}(G) ,$$

where $\otimes$ is the Kronecker product, $\text{vec}(.)$ is the operation of writing out a matrix as a vector and $G$ is the covariance matrix of the factor dynamics innovations $\eta$. The matrix $A$ and the vector $c$ are the entities in the transition equation (17) and the elements of $\Sigma_0$ can be computed analytically.

**KF prediction**

$$\hat{x}_{t|t-1} = A\hat{x}_{t-1} + c$$

$$\hat{y}_{t|t-1} = B\hat{x}_{t|t-1} + d$$

$$\Sigma_{t|t-1} = A\Sigma_{t-1}A^T + G$$

**KF update**

$$v_t := y_{t}^{obs} - \hat{y}_{t|t-1} = y_{t}^{obs} - B\hat{x}_{t|t-1} - d$$

$$F_t = B\Sigma_{t|t-1}B^T + H$$

$$\hat{x}_t = \hat{x}_{t|t-1} + \Sigma_{t|t-1}B^TF_t^{-1}v_t$$

$$\Sigma_t = \Sigma_{t|t-1} - \Sigma_{t|t-1}B'F_t^{-1}B\Sigma_{t|t-1}$$

**Quasi MLE parameter estimation**

Letting $\Theta$ denote the 14 SDE model parameters of the transition equation and defining $\psi := \{\Theta, H\}$, the log-likelihood is given by

$$\log \ L(\Theta, H) = -\frac{TK}{2}\log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log \det F_t - \frac{1}{2} \sum_{t=1}^{T} v_t'F_t^{-1}v_t .$$

where $K$ is the total number of maturities used, $T$ is the number of time steps and $v$ and $F$ are computed using (20).

The maximization of the log likelihood is performed in two steps, alternatively optimizing $\Theta$ and $H$ to convergence. There are two phases of the numerical optimization: a global phase using the DIRECT global optimization algorithm (Jones et al., 1993) to locate the region of the maximum, followed by a local phase using an approximate conjugate gradient algorithm (Powell, 1963) to locate the maximum itself.

**Black correction**

The distribution of the instantaneous short rate $r_t$ is Gaussian in most yield curve models, therefore it is easy to see that it can become negative when initialized at a low level. Black (1995) suggested a way of solving this problem. He argued that
nominal rates cannot become negative, because there is always the option of investing in the (0-yielding) currency instead. Black started from a process \( S_t \), which can take negative values, which he called the \textit{shadow short rate}, and the \textit{nominal short rate} is then defined as

\[ r_t = \max(0, S_t). \]  

(23)

This modification makes all the yields calculated through the bond price formula non-negative. A similar model was independently discussed by Rogers (1995).

Unfortunately, the shadow short rates implied by affine models lose their linearity when modified using this idea. This makes the resulting models difficult to calibrate. We shall discuss different approaches to calibration in the next section.

4. Alternative approaches to calibrating Black models

Although Black's idea was proposed in the 90s, the first implementation followed seven years later in the work of Gorovoi and Linetsky (2002). Active work on extensions to multi-factor models started only after the crisis of 2008. There are two main reasons for such a timeline. First, the zero-lower bound was not observed in the U.S. from the Great Depression until 2008; only in Japan from the mid-1990s did rates in a major economy start hitting zero. Perhaps more importantly, the implementation of the Black correction is considerably more difficult (both theoretically and computationally) than implementation of the usual affine term-structure models. The main problem is the lack of a closed form formula for the bond price given by

\[ P_t(r) = E^Q_t \left[ \exp \left( -\int_t^\tau \max(s_u, 0)du \right) \right]. \]  

(24)

1-factor Black models

Gorovoi and Linetsky (2002) showed that for a shadow short rate following a 1-dimensional diffusion process, the zero-coupon bond price can be calculated as the Laplace transform (at the unit value of the transform parameter) of the area functional of the shadow rate process. They applied the method of eigen function expansions (see Linetsky, 2002; Davydov and Linetsky, 2003; Linetsky, 2004) to derive the quasi-analytic formulae (relying on Weber-Hermite parabolic cylinder functions) for the bond price in the Vasicek and shifted CIR process cases. Unfortunately their method works only in the scalar case.

Gorovoi and Linetsky applied their method to estimating yield curve models for a single time point. However, their method was used by Ueno \textit{et al.} (2006) for the Japanese market who applied the method to a \textit{dynamic} model with a market price of risk.

The shadow rate \( S_t \) in these models follows a diffusion, therefore in state space form the single discretized transition equation takes the form

\[ x_{t+1} = ax_t + c + \eta_t. \]

(25)
The mapping that links observed yields and the shadow rate is no longer linear, so it takes the piecewise linear form
\[ y_{\text{obs}} = h(x_t) . \]  

Ueno et al. (2006) applied the Kalman filter with conditional linearization of (25) to calibrate the model. However, it was clear from their results that further work in developing the shadow rate models would be needed. For example, the shadow rates in their analysis reach the implausibly low levels of -15%, which suggests model misspecification.

### 2-factor Black models

Both Bomfim (2003) and Kim and Singleton (2011) relied on a numerical (finite-difference) method for solving a 2-dimensional parabolic quasilinear bond price PDE given by
\[
\frac{\partial P_t}{\partial \tau} - \frac{1}{2} \text{tr} \left( \frac{\partial^2 P_t}{\partial x \partial x'} \Sigma' \Sigma \right) - \frac{\partial P_t}{\partial x} K(\theta - x) + \max \left[ 0, s(x) \right] \frac{\partial P_t}{\partial \tau} = 0
\]

with boundary condition \( P(\tau = 0, x) = 1 \). Here the short rate \( s(x) \) is an affine function of the state 2-dimensional state \( x \).

Bomfim (2003) estimated the parameters of his model on the subset of data where rates were safely above zero, using an analytical approximation, i.e. the usual affine model. Kim and Singleton (2011) used the extended Kalman filter with quasi-maximum likelihood to estimate the parameters. Ueno et al. (2007) performed a sensitivity analysis of the 2-factor Black-corrected model without estimating the parameters.

Kim and Singleton and Ueno et al. report superior performance of the shadow rate models compared to their standard affine term structure equivalents (for the shadow rates). Two-factor models also produce more plausible levels of the shadow rate. However, the analysis of Section 2 suggests that 3 factors would be preferred for realistic modelling. Unfortunately, the alternating direction implicit finite difference scheme used by Kim and Singleton cannot easily be extended to the corresponding PDE in 3 dimensions.

Krippner (2013) applies a different method, which can be seen as an approximation to the Black model. The advantage of his method is that the forward rates have closed-form formulae. In the Black model, the price of a bond can be expressed as
\[
P_t^*(\tau) = P_t^S(\tau) - C_t^A(\tau, \tau; 1) ,
\]

where \( P_t^S(\tau) \) is the shadow bond price (i.e. the price of a bond in a market where currency is not available) and \( C_t^A(\tau, \tau; 1) \) is the value of an American call option at time \( t \) with maturity in \( \tau \) years and strike 1, written on the shadow bond maturing in \( \tau \) years. There is no analytic formula for \( C_t^A(\tau, \tau; 1) \), but Krippner argues that the American option can be approximated by an analytically tractable European one and introduces an auxiliary bond price equation
\[
P_t^{\text{aux}}(\tau, \tau + \delta) = P_t^S(\tau + \delta) - C_t^E(\tau, \tau + \delta; 1) ,
\]
where $C_t^E (\tau, \tau + \delta; 1)$ is the value of a European call option at time $t$ with maturity at time $t + \tau$ and strike 1 written on a shadow bond maturing at $t + \tau + \delta$. Krippner then takes the limit with $\delta \rightarrow 0$ to obtain the non-negative (due to future currency availability immediately before maturity) instantaneous forward rate as

$$f_t^0 (\tau) = \lim_{\delta \rightarrow 0} \left[ -\frac{\partial}{\partial \delta} \ln P_t^{\max} (\tau, \tau + \delta) \right].$$

(30)

The non-negative yield with maturity $\tau$ in Krippner's framework is calculated as

$$y_t^s (\tau) = \frac{1}{\tau} \int_0^\tau f_t^0 (s) ds = y_t^s (\tau) + \frac{1}{\tau} \int_0^\tau \lim_{\delta \rightarrow 0} \left[ \frac{\partial}{\partial \delta} C_t^E (\tau, \tau + \delta; 1) \right] P_t^{\max} (s + \delta) ds.$$  

(31)

Here $y_t^s (\tau)$ are the shadow bond yields. Unfortunately, closed-form analytic expressions for the bond prices and yields are still not available, but they can be evaluated through calculating integrals that are numerically tractable. More importantly, Krippner's approach is not fully arbitrage-free. The short rates are identical under the market measure $P$ in the Black and Krippner frameworks but different under the risk-neutral measure $Q$. Krippner's approach is extendible to 3-factors.

### 3-factor Black models

There are several approaches to calibrating 3-factor shadow rate models. Most of the differences between them can be classified in terms of the following:

1. Method of calculating bond prices
   a. Monte-Carlo simulation
   b. PDE solution
   c. approximate formulas
2. Method of inferring states from the observed yields
   a. inverse mapping (or least-squares)
   b. extended Kalman filter (EKF)
   c. iterated extended Kalman filter (IEKF)
   d. unscented Kalman filter (UKF)
3. Method of optimizing the QMLE objective

### Monte-Carlo simulation or PDE solution for bond pricing

Dempster et al. (2012) report the development of a 3-factor Black-corrected model using a combination of analytical closed form yield calculations and Monte Carlo simulation for bond pricing. They suggest using the unconditional likelihood function and multiple starting points for parameter optimization, as there will be numerous local maxima for the problem.

Bauer and Rudebusch (2014) use Monte-Carlo simulations (circa 500 paths of the shadow short rate) to calculate the bond prices. They employ the EKF to infer the states from the observed yields. However, they report using the same workaround as Bomfim (2003), i.e. estimating the parameters of the model on the subset of data where the ZLB is not important, to compare the shadow rate and affine models in practice.
Richard (2013) estimates the full Black shadow rate model. He notes that calibration "requires a long time, literally a month, on large and fast computers to estimate"\(^7\). He solves the 3 dimensional PDE for bond prices using an implicit numerical scheme.

Lemke and Vladu (2014) have applied the Monte-Carlo bond price calculation method and the EKF to construct yield curves in the Eurozone.

Krippner (2013) suggests using the Krippner framework results as control variates in Monte Carlo simulations for calculating the true Black model bond prices.

**Krippner approximation**

Christensen and Rudebusch (2013 a,b) apply the Krippner framework to estimate a 3-factor shadow-rate model. They argue that the divergence of Krippner approach from the fully arbitrage-free Black approach is not very significant and well compensated by much greater tractability. Wu and Xia (2014) apply an approach equivalent to the Krippner framework in discrete time.

**Cumulant approximation**

Priebsch (2013) proposes to view the quantity

\[
\log P_r(\tau) = \log E_Q^\tau \left[ \exp \left( -\int_t^{t+\tau} \max(0, s_u) du \right) \right] .
\]

as the value at -1 of conditional cumulant-generating function of the random variable \( S_r(\tau) = \int_t^{t+\tau} \max(0, s_u) du \) under \( Q \). It can be expanded as

\[
\log E_Q^\tau \left[ \exp( -S_r(\tau) ) \right] = \sum_{j=1}^{\infty} (-1)^j \frac{\kappa^Q_j}{j!}
\]

where \( \kappa^Q_j \) is the \( j \)-th cumulant of \( S_r(\tau) \) under \( Q \) and an approximation can be computed by taking a finite number of terms in this series.

The method of Ichiue and Ueno (2013) is equivalent to using the first term approximation in (28). Priebsch (2013) evaluates both 1- and 2-term approximations by analytically deriving the expression for the first two moments of \( S_r(\tau) \). He shows that this technique is sufficiently fast and accurate to fit the term-structure within a half basis point for a single time step. Priebsch notes that the Krippner approximation tends to underestimate the arbitrage-free yields of the Black model, while the first order cumulant approximation tends to overestimate these yields, suggesting a systematic error. The errors of second order cumulant approximation do not appear to have a discernible bias in any direction.

Andreasen and Meldrum use cumulant approximation to compare shadow rate models with quadratic term structure models to find that shadow rate models are better at out of sample forecasts.

**Filters**

Christensen and Rudebusch (2013), Bauer and Rudebusch (2014) and Lemke and Vladu (2014) use the EKF for parameter estimates.

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\(^7\)In the latest version of the paper the search time has been reduced to 3 days but the methods used to achieve this are not specified.
However, Krippner (2013) uses IEKF to fit his shadow rate approximation for the case of 2 and 3 factors, because he found that using the EKF was not robust.

Priebsch (2013) uses the unscented Kalman filter (UKF). Christoffersen et al. (2014) perform a series of comparisons of EKF with UKF and the particle filter. They conclude that the UKF significantly outperforms the EKF and performs well compared to the significantly more computationally expensive particle filter.

**Likelihood optimization**

Most of the papers on shadow rate models omit discussion of the optimization methods used. Richard (2013) mentions that he maximizes the likelihood function by using Powell (1964) local search combined with Nelder-Mead global search.

**Summary**

Implementing the Black correction leads to non-linearity of the measurement equation, i.e. of the mapping of factors/states to yields, so that the classical Kalman filter is no longer applicable. Taking account of the information in the literature reviewed above, we will use the unscented Kalman filter (Julier et al., 1995; Julier & Uhlmann, 1997) for our shadow rate model. To calculate the bond prices, we will use the measurement equation approximation

\[ y_{t,\text{obs}} = 0 \vee (Bx_t + d) + \epsilon, \]

where \( \vee \) denotes coordinate-wise maximum at each step of the UKF dynamics. It will be demonstrated in the sequel that the computational times for our approach are very acceptable relative to those of the basic linear Kalman filtering algorithm and the nonlinear KF alternatives.

5. **Unscented Kalman filter EM algorithm for the Black EFM model**

As stated above, we will use the unscented Kalman filter for our development.

**Unscented Kalman filter**

We initialize the filter at the unconditional mean and variance of the state variables under the \( P \) measure in the EFM model. This can be justified by the fact that most of our datasets start before the onset of low-rate regimes.

Since only the measurement equation is non-linear, the state prediction step is the same as that of the linear Kalman filter in (20).

For the factor path update step of the UKF, the state is first augmented with the expected measurement error (here 0) of the linear KF to give

\[ x_{\text{aug}}_{t-1} = \left[ \hat{x}_{t-1}', E[\epsilon] \right] ', \]

and the state innovation conditional covariance matrix is augmented with the measurement error covariance matrix to give
\[
\Sigma_{\tilde{\eta}^{-1}} = \begin{bmatrix} \Sigma_{\tilde{\eta}^{-1}} & 0 \\ 0 & H \end{bmatrix}.
\]

(36)

Next, a set of perturbed \textit{sigma-points} is constructed as

\[
\begin{align*}
\chi_{\eta^{-1}}^0 &= \hat{x}_{\eta^{-1}} \\
\chi_{\eta^{-1}}^j &= \hat{x}_{\eta^{-1}} + \left( \sqrt{(L+\lambda)\Sigma_{\tilde{\eta}^{-1}}} \right)_j, & j = 1, \ldots, L \\
\chi_{\eta^{-1}}^j &= \hat{x}_{\eta^{-1}} - \left( \sqrt{(L+\lambda)\Sigma_{\tilde{\eta}^{-1}}} \right)_{-L}, & j = L+1, \ldots, 2L ,
\end{align*}
\]

(37)

where \(\sqrt{\cdot}\) denotes the matrix square root of the symmetric positive definite augmented matrix (36), whose \(j\) th column augments the conditional state vector to give the \textit{augmented conditional state vector}. Here \(L\) is the \textit{dimension} of the augmented state and the scalar parameter \(\lambda\) is defined as

\[
\lambda := \alpha^2 (L + \kappa) - L ,
\]

(38)

where \(\alpha\) and \(\kappa\) control the \textit{spread} of the sigma points in an elliptical configuration around the conditional augmented state vector. The choice of these parameters is very important for the results of the calibration. We used a (NAG) code which sets \(\alpha\) equal to 1 and \(\kappa\) equal to 0, but we shall see that this is probably not the best choice.

Next, the (here piecewise linear) measurement equation is evaluated at the \(2L\) (\(= 34\)) sigma points to obtain \(2L\) estimates of the augmented observations as

\[
\gamma_{\eta^{-1}}^j = h(\chi_{\eta^{-1}}^j) = 0 \vee (B \chi_{\tilde{\eta}^{-1}}^j + d) \quad j = 1, \ldots, 2L .
\]

(39)

These \(2L\) sigma point results are then combined to obtain the predicted (here yield) measurements, measurements covariance matrix and predicted state-measurement cross-covariance matrix

\[
\begin{align*}
\hat{y}_{\eta^{-1}} &= \sum_{j=0}^{2L} W^j_i \gamma_{\eta^{-1}}^j \\
\Sigma_{\gamma'y} &= \sum_{j=0}^{2L} W^j_i [\gamma_{\eta^{-1}}^j - \hat{y}_{\eta^{-1}}] [\gamma_{\eta^{-1}}^j - \hat{y}_{\eta^{-1}}]' \\
\Sigma_{\gamma'y} &= \sum_{j=0}^{2L} W^j_i [\chi_{\eta^{-1}}^j - \hat{y}_{\eta^{-1}}] [\gamma_{\eta^{-1}}^j - \hat{y}_{\eta^{-1}}]' ,
\end{align*}
\]

(40)

where the weights \(W^j_i\) for combining sigma point estimates (predictions), are potentially different for the state vector and the covariance matrices. They are given by
Here $\beta$ is related to the higher moments of the state vector distribution and is usually set to 2, which is optimal for Gaussian innovations.

These results are used to compute UKF Kalman gain

$$K_t := \Sigma_{t_{\hat{y}_t}} \Sigma^{-1}_{t_{\hat{y}_t y_t}} ,$$

which defining $v_t := y_{t_{obs}} - \hat{y}_{t_{t-1}} = y_{t_{obs}} - BB_x \hat{y}_{t_{t-1}} - d$ gives the updated state estimate in observation prediction error feedback form as

$$\hat{x}_t := \hat{x}_{t_{t-1}} + K_t v_t$$

with updated state covariance matrix

$$\Sigma_t = \Sigma_{t_{t-1}} - K_t \Sigma_{t_{\hat{y}_t y_t}} K_t^\top .$$

**Choice of parameters for the UKF**

As noted above the choice of parameters $(\alpha, \kappa, \beta)$ for the UKF is very important and is not usually detailed in the literature (but see Tuner et al., 2012). Some nonlinear models are known to exhibit UKF algorithm divergence with certain parameter values. The other problem is inefficient estimation because of excessive spread of the sigma points. The first issue is not a problem here, but we aim to address the last issue in future research.

**Quasi maximum likelihood estimation**

Parameters estimates in the approximate Black corrected EM algorithm are calibrated from the current UHF data path prediction as before by maximizing the log likelihood function (21)

$$\log L(\Theta, H) = -\frac{TK}{2}\log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log \det F_t - \frac{1}{2} \sum_{t=1}^{T} v_t F_t^{-1} v_t ,$$

by alternating between the parameters $\Theta$ and $H$, except that now the measurement prediction errors in the last term of the log likelihood are those of the UKF and the calculation of the $F_t$ terms from (20) and (21) uses the UKF state covariance matrices $\Sigma_t$ of (44).

**Technical implementation**

A Numerical Algorithms Group Ltd (NAG) routine was used for the UKF implementation, but an updated NAG routine with more key parameter setting flexibility will be used in future research.
The EM calibration code was implemented in C++ with quasi MLE optimization using global search with DIRECT (Jones et al., 1993) followed by a local (approximate) conjugate gradient algorithm (Powell, 1964) which does not require derivatives.

The Black EFM model calibration run-times for currencies with 12 years of daily data are around 4.5 hours (scaling linearly with data length). This is approximately twice the time the basic Kalman filter takes for EFM model calibration to the same data.

**HPC implementation**

The development was coded in C and C++ under Linux with the use of MPI functionality. Calculations are performed on 5 compute nodes with 32 cores in total and the following hardware:

- **Node 1:**
  - Memory: 16GB
  - 4 x CPU Xeon (X5550) 2.66GHz quad core
  - OS is Centos 5.7
- **Nodes 2 to 5:**
  - 4 x CPU Xeon (TM) 3GHz
  - Memory: 16GB
  - OS Centos 5.7

The DIRECT global optimization algorithm to cope with the non-unimodal likelihood function was implemented in parallel in a master-slave configuration with synchronicity. The Powell local optimization algorithm is not parallelizable.

The functionality actually parallelized was the full Kalman filter algorithm path estimation iteration step of the EM algorithm. The master thread controls the optimization calculation synchronizing 31 slave threads (number of cores = number of threads = 32), i.e. providing them at each optimization step with best step values of the log likelihood objective function and calculated filter predictions.

![Parallelization Schema](image)

**Figure 1. Parallelization Schema**
When the Black EFM model UKF implementation is fully developed, we will migrate it to the cloud. We have already experimented with using the Amazon cloud for this model and have consulted on a compute intensive earlier Black model yield curve commercial development which uses this cloud (Dempster et al., 2014).


This section contains a preliminary empirical evaluation of our approach to developing a robust long term nonnegative yield curve model from the EFM Gaussian model using the Black correction. We will evaluate the Black-corrected EFM yield curve model against the original EFM model and the market data used to calibrate the models, both in-sample, for goodness-of-fit, and out-of-sample, for prediction accuracy.

**Data**

We use a combination of LIBOR data and fixed interest rate swap rates (the ISDA fix) for each of 4 currency areas (EUR, GBP, USD, JPY) to bootstrap the yield curve daily for 14 maturities:

- 3 month, 6 month, 1 year, 2 years, 3 years, 4 years, 5 years, 6 years,
- 7 years, 8 years, 9 years, 10 years, 20 years, 30 years.

In the case of the Swiss franc (CHF), only 12 maturities are available:

- 3 month, 6 month, 1 year, 2 years, 3 years, 4 years, 5 years, 6 years,
- 7 years, 8 years, 9 years, 10 years.

The calibration periods used for these 5 currencies are the following:

- EUR: 02.01.2001 to 02.01.2012
- CHF: 02.01.2001 to 31.05.2013
- GBP: 07.10.2008 to 31.05.2013
- USD: 02.01.2001 to 31.05.2013
- JPY: 30.03.2009 to 31.05.2013.

After the 2012 Libor scandals, ICAP (formerly InterCapital Brokers) lost to ICE Benchmark Administration Limited its role as administrator for the ISDA fix rates, data collection and calculation. Major reforms in the calculation methodology are being implemented (changing sources from polls of contributing banks to actual transaction quotes). This transfer process is not without difficulties for data providers.

The data was obtained from Bloomberg (indices US000**, BP000**, EE000**, JY000**, SF000** for LIBOR rates and USISDA**, BPISDB**, JYISDA**, SFISDA** for ISDAfix rates).

**Yield curve bootstrapping**

For the short rate maturities (\( T \) in years equal to 0.25, 0.5, 1) we use only the LIBOR data and the formula (Ron, 2000).
\[ y(T) = \ln(1 + \text{LiborRate}(T) \cdot 0.01 \cdot T) / T . \quad (45) \]

For the longer maturities we use the following process. If the coupons on the swaps are paid semi-annually (as is the case for USD, GBP and JPY), then we calculate the discount factor for one year as

\[ df(1) = 1 / (1 + \text{LiborRate}(1) / 100) . \quad (46) \]

If the coupon payments are annual, we use the formula

\[ df(1) = 1 / (1 + \text{LiborRate}(0.5) / 100 \cdot 0.5) + 1 / (1 + \text{LiborRate}(1) / 100) . \quad (47) \]

We then proceed to calculate the discount factors

\[ df(n) = \frac{1 - \text{swap}_\text{rate} \cdot \frac{1}{\text{freq}} \sum_{i=1}^{n-1} df(i)}{1 + \text{swap}_\text{rate} \cdot \frac{1}{\text{freq}}} , \quad (48) \]

where \( \text{freq} = 1 \) for annual coupons and \( \text{freq} = 2 \) for semi-annual coupons, \( \text{swap}_\text{rate} \) is the swap coupon paid.

Finally, the rates are given by

\[ y(T) = -\log(df(T)) / T . \quad (49) \]

**In-sample goodness-of-fit**

First, let us consider statistics for overall goodness-of-fit across the entire sample period for the five currency areas EUR, CHF, GBP, USD and JPY, ordered by average rates in the data period from highest to lowest average rate. Table 2 shows the comparative goodness-of-fit, in terms of optimal log likelihood and standard deviation (vol) of the sample measurement errors across all yields at the data maturities and all observations, of three models: the affine EFM estimated with both the Kalman and unscented Kalman filters and the Black EFM estimated with the UKF.
Table 2. Comparative model goodness of fit

<table>
<thead>
<tr>
<th>Currency</th>
<th>Observations</th>
<th>calibration</th>
<th>log likelihood</th>
<th>measurement error vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUR</td>
<td>2817</td>
<td>EFM</td>
<td>232652.00</td>
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</tr>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
<td>CHF</td>
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<td>EFM</td>
<td>232100.00</td>
<td>8 bp</td>
</tr>
<tr>
<td></td>
<td></td>
<td>EFM UKF</td>
<td>250391.00</td>
<td>10 bp</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Black EFM</td>
<td>221358.00</td>
<td>8 bp</td>
</tr>
<tr>
<td>GBP</td>
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<td>EFM</td>
<td>98021.40</td>
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</tr>
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<td></td>
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<td></td>
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<td>Black EFM</td>
<td>60642.70</td>
<td>60 bp</td>
</tr>
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<td>USD</td>
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<td>EFM</td>
<td>279114.00</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>EFM UKF</td>
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<td></td>
<td></td>
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<td>87035.10</td>
<td>8 bp</td>
</tr>
</tbody>
</table>

This allows an overall comparison of the fitting errors of the original and Black corrected model and also of the size of the fitting error relative to the average level of rates in the currency area. From Figures 1 to 5 below we can see that the total measurement error vol of the best fit is very respectably small for all currency areas, EUR, GBP and USD being the highest and CHF and JPY the lowest.

Although the three models in Table 2 have the same parameter set, their likelihoods are not generally comparable as the models are not nested in the econometric sense. However, the likelihoods of the affine EFM model estimated with the KF and the UKF are comparable and in all cases, except for Japan, the UKF likelihood exceeds the KF likelihood, a reflection of the general power of the UKF widely attested to in the literature.

We may nevertheless compare the likelihoods achieved with the UKF for the affine EFM and nonlinear Black EFM models. Here the Black EFM likelihood exceeds that of the EFM for JPY and the two likelihoods are close for CHF. In terms of total measurement error standard deviation the two models are also close, with the Black EFM giving the lowest value. However, as measured by both statistics, for EUR, GBP and USD the fits are significantly worse than the EFM for the Black-corrected model.

We feel that this is likely because the NAG unscented Kalman filter code we used had the α parameter set to 1 (appropriate for high curvature nonlinearities) instead of the generally recommended $10^{-3}$, which would give a much smaller displacement of sigma points and be much more appropriate for the simple piecewise linear option "hockey stick" nonlinearity we are handling here with the UHF. We should probably also have a β parameter which reflects the positive skew in the Black EFM yields.
Turning to yield curve fits on specific days, Figures 1 to 5 show the yield curve fits of the Black EFM model on representative days throughout the data period for all five currency areas relative to both the data and the original linear EFM alternative. (We have in fact developed software that can show these yield curve fits stepping through every (daily) observation in the data.) Each figure shows both a "good" fit and a "bad fit" of the Black EFM model for a single currency. Root mean square error based on quarterly evaluation of the yield curve rates over 30 years (10 for CHF) is calculated using the model expression for the yield at each quarterly maturity in terms of the estimated parameters.

Overall, the Black EFM model is broadly comparable to the original EFM model in all five jurisdictions. However, GBP and USD are in general fit worse than EUR by both models and much worse than CHF and JPY. However, for GBP and USD the Black model fits the short end kinks (see Figures 3 and 4) significantly better than the original model (although both models are based on 3 factors), and similarly for the low rate JPY (see Figure 5). It should be noted that such non-text book yield curve shapes in the data period may reflect an behavioural market excess demand for short term bonds or have resulted from market manipulation, or both.

On the other hand, the "bad" figures show that these short maturity rate spikes sometimes throw the Black-corrected model off completely. Again this appears to be due to the wide spread of the sigma points wired into the UKF code we have used which is most likely to affect the distributions sampled by the UKF at low short maturities. For our call option nonlinearity this produces too many sigma point zero estimates at short maturities which pull the UKF weighted average down and at longer maturities cause the weighted average to be too high due to the absence of enough low rate nonnegative paths. With a very much smaller α parameter this effect will likely be eliminated, which we hope to demonstrate with the updated NAG UKF code we will use in future research.

**Out-of-sample Monte Carlo projection**

Figures 7 to 10 show the results of monthly out-of-sample Monte Carlo scenario projection over a 30 year horizon using the EFM and Black EFM models calibrated to the last day of the data period, 31 May 2013. These are for 5 year rates for EUR and 10 year rates in 3 currency areas: EUR, GPB and JPY. The figures show the evolution of the paths of the quartiles and 1% and 5% tails of the 10,000 scenario
**Figure 2**

**EUR** "Good". Date: 02 Jan 2007.

<table>
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<th>RMSE</th>
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<tbody>
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<tr>
<td>Black EFM</td>
<td>6 bp</td>
</tr>
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</table>

**EUR** "Bad". Date: 18 Nov 2008.

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<th>RMSE</th>
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</thead>
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<tr>
<td>Black EFM</td>
<td>82 bp</td>
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</table>
**Figure 3**


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CHF “Bad”. Date: 25 Jan 2002.

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GBP "Good". Date: 02 Jan 2009.

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GBP "Bad". Date: 16 Jul 2012.

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<td>Black EFM</td>
<td>63 bp</td>
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**Figure 5**

USD “Good”. Date: 14 Oct 2008.

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<td>24 bp</td>
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**Figure 6**

**JPY "Good". Date: 03 Apr 2012.**

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<td>Black EFM</td>
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**JPY "Bad". Date: 08 Feb 2013.**

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</table>
distribution. The actual market data evolution is also plotted on these figures up to a more recent date for each maturity and currency area: EUR, 11 December 2014; GPB, 15 January 2015 and JPY, 24 January 2014⁹.

These figures demonstrate the basic negative scenario generation problem with the Gaussian EFM model (cf. Dempster et al., 2014) and the primary effectiveness of the non-negative Black correction for the longstanding low rate Japanese economy (Figure 10).

However, for EUR 5 and 10 year rates (Figures 7 and 8) and GPB 10 year rates (Figure 9), the dynamic effects of using too-wide sigma points as described above are visible. The scenarios therefore do not replicate the (short span of) market data in a period of declining rates (EUR) and quantitative easing (GBR). Again we hope to remedy this issue in future research. It is also interesting note that the problem effects are more severe for the shorter 5 year EUR maturity (taking account of the different vertical scales of Figures 7 and 8).

It should also be noted, by way of comparison of the spread of the 10 year rate scenario distributions as time evolves, that in spite of the difficulties the Black EFM produces a much tighter, more realistic, spread over a 30 year horizon for all three economies than the diffusion based affine EFM model.

7. Conclusion

This paper reports on the initial development and evaluation of a new approximation of the Black (1995) correction to ensure non-negative nominal rates of all maturities for a practically effective Gaussian 3-factor affine yield curve model -- the EFM model. Perhaps the most important feature of this novel approach is the demonstrated fact that the HPC calibration of the Black EFM model can be effected in only about double the runtime of that of the underlying shadow rate EFM model. Although some difficulties with the unscented Kalman filter code used for this report have been identified, the results presented here are promising, both in- and out-of-sample. We are confident that the issues identified here can be resolved in ongoing research.

Acknowledgements

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-13) under grant agreement nº 289032 (HPC Finance). The authors wish to acknowledge support and helpful comments from Martyn Byng of Numerical Algorithms Group, Grigorios Papamanousakis of Aberdeen Asset Management and, Giles Thompson, Senior Associate of Cambridge Systems Associates, particularly.

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⁹ After which date Bloomberg dropped JPY CMS swap data.
Figure 7

Euro 5 year rate EFM

Euro 5 year rate Black EFM
Figure 8

Euro 10 year rate EFM

Euro 10 year Black EFM
Figure 9

GPB 10 year rate EFM:

GBP 10 year Black EFM
Figure 10

JPY 10 year EFM

JPY 10 year Black EFM
References


Appendix

Given an initial set of parameters \((\Theta_0, H_0)\), the EM algorithm for estimation of the parameters of the EFM model from market data using the Kalman filter alternates between generating paths with the filter for the log likelihood function and optimizing this function in the model parameters.

Defining \(O(\Theta, H) := \log L(\Theta, H)\), a single step of the EM algorithm for quasi MLE can be presented in pseudo code as follows.

**Calculation of the log likelihood function**

1. **Input** \((\Theta_0, H_0)\)
2. **for** \(t=1\) to \(T\) **do**
3. KF predictions (19)
4. KF update (20)
5. Calculate a term of the log likelihood function (21)
6. **end for**
7. Compute \(O(\Theta, H)\) (21)
8. **Output** \(O(\Theta, H)\)

**Optimization of the log likelihood function**

The 2-phase optimization algorithm is the following.

1. Initialize parameters \((\Theta, H)\) from previous EM algorithm step
2. **while** \(\|\Delta O(\Theta, H)\| \geq \text{tolerance}_1\) **do**
3. DIRECT optimization of \(O(\Theta, H)\) with \(H\) fixed
4. DIRECT optimization step of \(O(\Theta, H)\) with \(\Theta\) fixed
5. **end while**
6. **while** \(\|\Delta O(\Theta, H)\| \geq \text{tolerance}_2\) **do**
7. Powell optimization of \(O(\Theta, H)\) with \(H\) fixed
8. Powell optimization of \(O(\Theta, H)\) with \(\Theta\) fixed
9. **end while**