The Impact of Jump Distributions on the Implied Volatility of Variance

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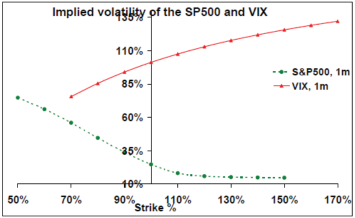
Motivation-1

- The empirical literature offers ample evidence indicating the presence of jumps both in stock prices and in their volatilities.
- In a seminal paper, [Duffie et al., 2000] introduce the class of affine jump-diffusion models, which allows for jumps both in asset prices and in their stochastic variances.
- Since then, affine models have been applied empirically in a number of studies, including [Broadie et al., 2007], [Chernov et al., 2003], [Eraker et al., 2003], and [Eraker, 2004] among others.

Looking at implied volatility curves



- For equity-options, jumps in the price level might explain the steep and negative skew of implied vols at short expiries. See [Gatheral, 2006].
- For volatility-options, jumps in the variance level might be helpful in reproducing the positive skews implied e.g. from VIX options.
- [Sepp, 2008a], [Sepp, 2008b], and [Lian and Zhu, 2011] propose augmenting the square-root dynamics of [Heston, 1993] to include exponential jumps in the instantaneous variance.

Equity vs Volatility Derivatives And Jump Distributions

- Despite the common notion that jumps are a necessary modelling ingredient, an examination of the **distribution** of jumps and its financial implications appears to be a matter of lesser relevance.
- In truth, a number of different jump specifications has been examined within the literature concerning equity-derivatives. See, for example, [Nicolato and Venardos, 2003] and [Carr et al., 2003].
- Affine models combining stochastic volatility with jumps seem to capture correctly the qualitative behavior of the equity-options skew almost independently of the particular choice of the jumps distribution.
- A similar analysis is absent from the literature on volatility derivatives and the exponential distribution appears to be the only considered candidate.

A little digression on returns as mean-variance mixtures

Basically in all option-pricing models for equities, the log-price X takes the following form

$$X \sim \text{drift} + \theta V + \sqrt{V} \cdot N(0,1)$$

where $N(0,1)$ is a standard normal and V is a distribution on the positive real line representing the random variance.

- In SV models, $V = \langle X \rangle = \int_0^{\cdot} v ds$;
- In most exponential Lévy models (VG, NIG, CGMY), V is a subordinator;

For any distribution of the random variance V:

- The "randomness" of V injects curvature
- The correlation between V-$N(0,1)$ and/or the parameter θ inject skewness in the associated smile

A toy example

We set $V \perp N(0,1)$ and consider two alternative specifications for the random variance V:

- The Gamma specification

$$V \sim \Gamma(\alpha, \beta) \quad \text{with density} \quad f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp\left(-\beta x\right),$$

  associated with VG model but also linked with the Heston model as well as with the VGSV

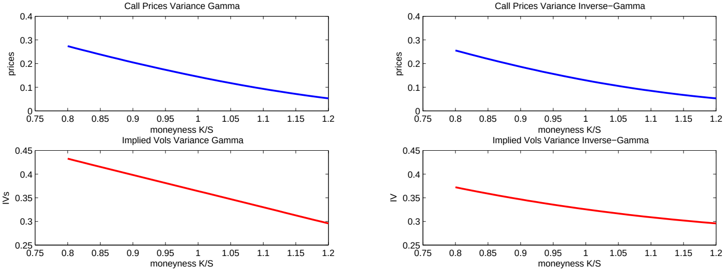
- versus the Inverse-Gamma specification

$$V \sim I\Gamma(\nu, \mu) \quad \text{with density} \quad f(x;\nu,\mu) = \frac{\mu^{\nu}}{\Gamma(\nu)} x^{-\nu-1} \exp\left(-\frac{\mu}{x}\right),$$

  associated with the 3/2 SV model

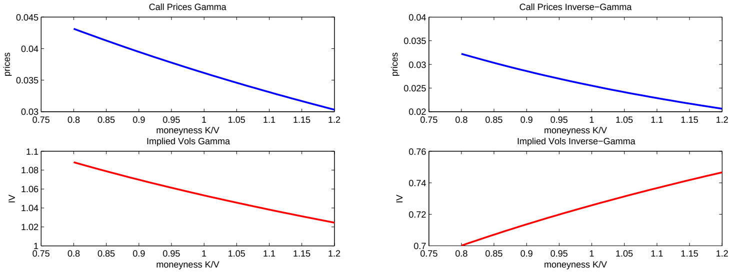
- We look at call prices written on the underlying mixed distribution X

VG versus VIΓcall prices and implied smiles



- Top: Call prices $E[(X-K)^{+}]$. Bottom: implied vols
- Left: VG case. Right: VIΓ case.
- Parameters chosen to match the first two moments of V. Same skewness parameter θ.

What if we look at calls written on the variance $E[(V-K)^+]$



- Top: Call prices $E[(V-K)^+]$. Bottom: implied vols
- Left: VG case. Right: VIΓ case.

The aim of this work

The aim of this work is to formalize the toy example in the context of affine models.

- We focus exclusively on volatility derivatives.
- We show that the particular distribution of jumps does, in fact, have a profound impact on the behavior of the associated implied volatility smile.
- The popular choice of exponentially distributed jump-sizes might not be the most suitable to capture upward sloping skews.

Realized Variance Options

- We consider contract written on the annualized realized variance V_T of an asset over a time interval $[0, T]$, i.e.

$$V_T = \frac{1}{T} \sum_{n=0}^{N} (X_{t_n} - X_{t_{n-1}})^2 \approx \frac{1}{T}[X]_T$$

  where $X_t = \log(S_t)$ denotes the log-price and $[X]_T$ is its quadratic variation. For a discussion on discrete vs continuous sampling see Broadie and Jain (2008).

- The price $C(K)$ of a call options maturing at T and striking at K is given by

$$C(K) = \mathbb{E}\left(V_T - K\right)^+$$

- At the core of our analysis is the implied volatility $I(K)$, which also in this case is obtained by matching the call price with the Black-Scholes formula:

$$C(K) = V_0 \Phi\left(\frac{x}{I(K)\sqrt{T}} + \frac{I(K)\sqrt{T}}{2}\right) + K\Phi\left(\frac{x}{I(K)\sqrt{T}} - \frac{I(K)\sqrt{T}}{2}\right).$$

SVJ-v Modelling Framework

The log-price X and its instantaneous variance v have risk-neutral dynamics

$$\begin{aligned} dX_t &= -\tfrac{1}{2}v_t\,dt + \sqrt{v_t}dW_t \\ dv_t &= \lambda(\eta - v_t)dt + \varepsilon\sqrt{v_t}dB + dJ_t \ . \end{aligned} \tag{1}$$

where (W, B) are Brownian motions with correlation ρ while J is a driftless subordinator independent of (W, B) with finite first moment.

- The SV model (1) is a particular case of the SVJJ model of [Duffie et al., 2000] where both v and X are affected by jumps. To emphasize that in (1) jumps are allowed only at the variance level, we refer to (1) as the **SVJ-v model**.

- It generalizes the [Heston, 1993] model by augmenting the CIR variance with jumps.

- For $\varepsilon = 0$ one obtains the OU-type variance process as in [Barndorff-Nielsen and Shephard, 2001] model.

The main advantage of the SVJ-v model is that realized variance V is given by

$$V_T = \frac{1}{T}\int_0^T v_t dt.$$

- We can explore alternative distributions for the jump process J and to analyze their impact on the price of RV derivatives.

- We deal with the tractable integrated variance: the Laplace transform of the RV is given by

$$\mathcal{L}_V(u) = \mathbb{E}[e^{-uV_T}] = e^{\alpha(u,T)+\beta(u,T)\,v_0} \tag{2}$$

  with functions $\alpha(u, T)$ and $\beta(u, T)$ available in closed-form

- Extensions to the full SVJJ class are possible, but technically more intricate.

In the SVJ-v framework, we analyze the asymptotic behavior of the implied volatility $I(K)$ at small or large strikes.

- We provide simple and easy-to-check sufficient conditions relating the distribution of the jump process J with the asymptotic behavior of the wings.
- Rather than deriving precise asymptotic estimates, we are interested in the qualitative behavior of the wings, e.g. upward or downward sloping, as this gives an indication of the overall smile shape.
- We provide numerical illustrations for a menagerie of positive distributions for J, showing how the commonly used exponential law might not be the optimal choice.

Ornstein-Uhlenbeck Specifications

- First, we examine the subclass of purely jumping OU-type processes, for which

$$dv_t = -\lambda v_t dt + dJ_t \quad \Rightarrow \quad v_t = e^{-\lambda t} v_0 + \int_0^t e^{-\lambda(t-s)} dJ_s$$

and

$$V_T = v_0 \epsilon(0, T) + \int_0^T \epsilon(t, T) dJ_t \quad \text{where} \quad \epsilon(t, T) = \frac{1 - e^{-\lambda(T-t)}}{\lambda T}.$$

- The main result in a nutshell is that the distributional properties of the jump process J are transferred basically unchanged to the realized variance V_T.
- In other words, the implied volatilities associated with

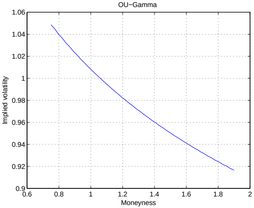
$$\mathbb{E}\left(V_T - K\right)^+ \quad \text{or} \quad \mathbb{E}\left(J_1 - K\right)^+$$

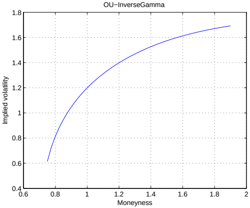
display the same qualitative behavior

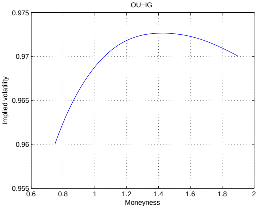
A Menagerie of Positive Distributions

Jump distribution	Small strikes	Large strikes
Gamma	Upward	Downward
Exponential	Upward	Downward
Inverse Gamma	Downward	Upward
Inverse Gaussian	Downward	Downward
Generalized Inverse Gaussian	Downward	Downward
Tempered Stable	Downward	Downward
Weibull	Upward	Downward

Numerical Illustrations - Alternative Distributions

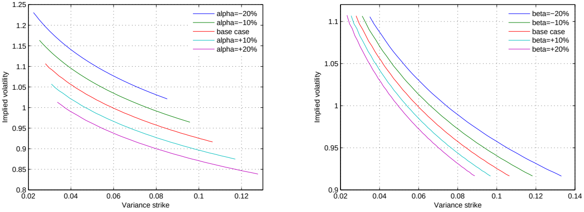




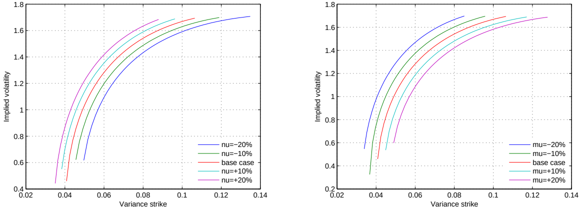


- 3-Months Implied volatilities of realized variance for the OU-Gamma with parameters $\alpha = 18$, $\beta = 22.8$ (Left), OU-InverseGamma with $\nu = 20$, $\mu = 15$ (Middle), and OU-IG with $a = 3.7697$, $b = 4.7749$ (Right).

- In all cases we take $\lambda = 8$, and $v_0 = 0$ and we obtain the parameters of the different jump specifications by matching the mean and the variance of J_1.



OU-Gamma parameter sensitivities. Base case parameters: $\alpha = 18$, $\beta = 22.8$, $\lambda = 8$, and $v_0 = 0$.



OU-Inverse Gamma parameter sensitivities. Base case parameters: $\nu = 20$, $\mu = 15$, $\lambda = 8$, and $v_0 = 0$

Full SVJv Specifications

- We extend the analysis to full specifications of the SVJ-v model

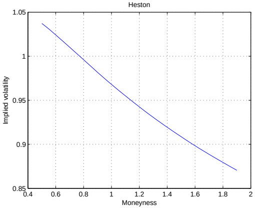
$$dv_t = \lambda(\eta - v_t)dt + \epsilon\sqrt{v_t}dB_t + dJ_t \quad \text{with} \quad J_t = \sum_{i=1}^{N(t)} Z_i, \quad Z_i \sim i.i.d.\ Z$$

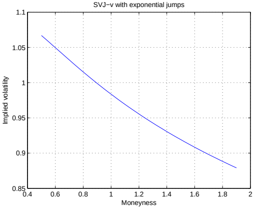
- Sepp (2008 a,b), Lin and Chang (2009), Lian and Zhu (2011): $Z \sim$ exp.

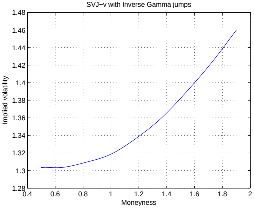
- Although to a lesser extent than in the OU case, the upper tail of V_T is qualitatively similar to that of Z.

- As a result, the right wing behavior of V_T is analogous to that of Z.

- However, jumps have no dramatic effect on the asymptotic behavior of the left wing, which is always downward-sloping to zero - also in the purely diffusive Heston case.







- Implied volatilities of variance for

  Left: Heston model with $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\epsilon = 0.39$

  Middle: SVJ-v model with exponential jumps with $l = 1.5$ and $1/b = 0.1^2$

  Right: SVJ-v model with Inverse Gamma jumps with $l = 1.5$, $\alpha = 3$ and $\beta = 2 \times 0.1^2$

- The inverse gamma parameters are obtained by matching the first and the second moment with those of the exponential jump size.

Few words about the theoretical results

Lee's moment formulas play a crucial role in the analysis. They relate the wings of the smile to the number of moments of the underlying distribution.

Theorem ([Lee, 2004])

Define

$$\tilde{p}_V = \sup \left\{ p : \mathbb{E}[V_T^{p+1}] < \infty \right\}, \quad \tilde{q}_V = \sup \left\{ q : \mathbb{E}[V_T^{-q}] < \infty \right\}$$

and let

$$\psi(x) = 2 - 4(\sqrt{x^2 + x} - x)$$

then

$$I^2(K) \sim \psi(\tilde{p}_V) \log(K/V_0)/T \quad \textit{as} \quad K \to \infty$$

and

$$I^2(K) \sim \psi(\tilde{q}_V) \log(V_0/K)/T \quad \textit{as} \quad K \to 0$$

- Lee's formulas are not informative when $\tilde{p}_V = \infty$ or $\tilde{q}_V = \infty$.
- In these cases, the results derived by Gulisashvili and coauthors turn out to be very useful.

Theorem ([Gulisashvili, 2012])

- *In the case of $\tilde{p}_V = \infty$, then $I(K) \sim \frac{1}{\sqrt{2T}} \log(K) \left(\log \frac{1}{C(K)} \right)^{-1/2}$ as $K \to \infty$, with $C(K)$ denoting the pricing function of the call.*
- *In the case of $\tilde{q}_V = \infty$, then $I(K) \sim \frac{1}{\sqrt{2T}} \log \frac{1}{K} \left(\log \frac{K}{P(K)} \right)^{-1/2}$ as $K \to 0$, with $P(K)$ denoting the pricing function of the put.*

We also employ some classical results such as

- Karamatas Theory of regular variation.
- Tauberians Theorems - which relate the behavior of a of a positive random variable with that of its Laplace transform.

A final remark

- The analysis of the smile-wings has attracted considerable attention during the last decade and a variety of results have been produced by means of analogous techniques.
- Among others, we refer to the studies of [Benaim and Friz, 2008], [Benaim and Friz, 2009], [Gulisashvili, 2009], [Gulisashvili, 2012] and the references therein.
- However, the focus has been on implied volatility smiles for equity options. So, most of the results are formulated in terms of the distribution of $\log V_T$ as the natural building block in that context is the log-price.
- In contrast, for variance options the relevant and tractable element is the variance V_T itself, which implies that a lot of the previous results available for equity options cannot be directly transferred to the case of variance smiles.

Conclusions

- We have considered options on realized variance in the SVJ-v model generalizing the seminal Heston model by augmenting it with jumps in the instantaneous variance.
- We provide sufficient conditions for the asymptotic behavior of the volatility of variance for small and large strikes - at least in the paper.
- We showed that by selecting alternative jump distributions, one obtains - ceteris paribus- fundamentally different shapes of the implied volatility smile.

Thank you for your attention!

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