Stock Price Dynamics, Dividends and Option Prices with Volatility Feedback

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Intuition: What is Volatility Feedback?

1. An unexpected increase in squared volatility $\rightarrow$
2. Investors require more return to compensate higher risk $\rightarrow$
3. An increase in an instantaneous (stochastic) discount rate $\rightarrow$
4. Dividends are discounted at a higher rate $\rightarrow$
5. The expectation of future discounted dividends decreases $\rightarrow$
6. The stock price decreases,
   price-dividend ratio decreases and
   instantaneous dividend yield increases $\rightarrow$
7. An increase in squared volatility can *decrease* the value of an in-the-money *call option* with a *convex* payoff function through stock price and dividend yield

   It can be important to model how volatility drives dividend yield making it time-varying

Assumption 1: Returns, Volatility, and Risk-Return Tradeoff

- The cumulative return from dividends and changes in prices satisfies

$$\mathrm{d}R_t = \frac{\mathrm{d}P_t + D_t \mathrm{d}t}{P_t}.$$

- The joint dynamics with its instantaneous volatility, x_t, evolves as

$$\mathrm{d}R_t = \left(r + \gamma x_t^2\right) \mathrm{d}t + x_t \mathrm{d}B_t^r$$
$$\mathrm{d}x_t = -\beta x_t \mathrm{d}t + \sigma_x \mathrm{d}B_t^x,$$

  where r, γ, β, and σ_x are constant positive real numbers. B^r and B^x are Brownian motions, $\mathrm{d}B_t^r \mathrm{d}B_t^x = \rho_{rx,t} \mathrm{d}t$, and $x_0 := x$, $x \in \mathbb{R}$.

- Consequently, squared volatility, $h_t = x_t^2$, follows the squared root process,

$$\mathrm{d}h_t = \kappa(\theta - h_t)\mathrm{d}t + \sigma_h \sqrt{h_t} \mathrm{d}B_t^x$$

  with $\kappa = 2\beta$, $\theta = \sigma_x^2/(2\beta)$, and $\sigma_h = 2\sigma_x$.

Assumption 2: Dividends

(*i*) The stochastic differential of dividends is given by

$$\mathrm{d}D_t = \alpha D_t \mathrm{d}t + y_t D_t \mathrm{d}B_t^d$$

with $\mathrm{d}B_t^d \mathrm{d}B_t^x = \rho_{dx}\mathrm{d}t$ and $D_0 := D$, $D > 0$.
The correlation coefficient of dividend growth and return volatility (leverage effect), $\rho_{dx} \in [-1, 1]$, and the expected rate of dividend growth, $\alpha \in \mathbb{R}$, are constant.
Dividend growth volatility, y_t, is stochastic.

(*ii*)

$$\operatorname{sign}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Cov}_t\left(D_t, (x_t^2)\right)|_{\tau=t}\right) = \operatorname{sign}(\rho_{dx}).$$

Assumption 3: Transversality Condition

The stock price, $p(D_t, x_t)$, can be expressed as the *expected value of discounted dividends*, conditional upon the present information:

$$
\begin{aligned}
p(D_t, x_t) &= \mathbb{E}_{D,x} \int_t^{\infty} \exp\left[-\int_t^s \left(r + \gamma x_u^2\right) \mathrm{d}u\right] D_s \mathrm{d}s \\
&= D_t \times \mathbb{E}_x \int_t^{\infty} \exp\left[\int_t^s \left(\alpha - r - \gamma x_u^2 - \frac{1}{2} y_u^2\right) \mathrm{d}u \right. \\
&\qquad\qquad\qquad\qquad \left. + \int_t^s y_u \mathrm{d}B_u^d\right] \mathrm{d}s < \infty.
\end{aligned}
$$

Here $r + \gamma x_u^2$ represents the instantaneous stochastic cost of capital at time u.

Price-Dividend Ratio

The price-dividend ratio, $f : \mathbb{R} \to \mathbb{R}_+$, satisfies $p(D_t, x_t) = D_t f(x_t)$. Hence,

$$f(x_t) = \mathbb{E}_x \int_t^\infty \exp\left[\int_t^s \left(\alpha - r - \gamma x_u^2 - \frac{1}{2} y_u^2 \right) \mathrm{d}u + \int_t^s y_u \mathrm{d}B_u^d \right] \mathrm{d}s.$$

- For all $x > 0$, $f(x) = f(-x)$ and so $p(D, x) = p(D, -x)$ (the sign of volatility does not matter)
- f is an even function, i.e., $f(x) = f(-x), f_x(x) = -f_x(-x)$, and $f_{xx}(x) = f_{xx}(-x)$ for all $x > 0$, where f_x and f_{xx} denote first and second order derivatives.
- For the stock price to be a continuously differentiable function, we can impose that $f_x(0) = 0$.

Stock Price Dynamics with Time-Varying Dividend Yield

Consequently, the stock price process, $P_t = p(D_t, x_t)$, $\{P_t; t \geq 0\}$, satisfies

$$\begin{aligned} \mathrm{d}P_t &= \left(r + \gamma x_t^2\right) P_t \mathrm{d}t - D_t \mathrm{d}t + x_t P_t \mathrm{d}B_t^r \\ &= \left(r + \gamma x_t^2 - \frac{1}{f(x_t)}\right) P_t \mathrm{d}t + x_t P_t \mathrm{d}B_t^r \\ &= \left(r + \gamma x_t^2 - \frac{1}{f(x_t)}\right) P_t \mathrm{d}t + y(x_t) P_t \mathrm{d}B_t^d + \sigma_x \frac{f_x(x_t)}{f(x_t)} P_t \mathrm{d}B_t^x, \end{aligned}$$

where $1/f(x_t)$ is the instantaneous dividend yield and $y(x_t)$ dividend growth volatility.

Solutions for Price-Dividend Ratio and Dividend Growth Volatility

- The price-dividend, $f(x)$ ratio satisfies the following relation:

$$(y(x)\sigma_x\rho_{dx} - \beta x)f_x(x) + \frac{1}{2}\sigma_x^2 f_{xx}(x) - (r + \gamma x^2 - \alpha)f(x) = -1,$$

  where dividend growth volatility, $y(x)$, is given by

$$y(x) = -\rho_{dx}\sigma_x \frac{f_x(x)}{f(x)} + \mathrm{sign}(x)\sqrt{x^2 - (1-\rho_{dx}^2)\left(\sigma_x \frac{f_x(x)}{f(x)}\right)^2}.$$

- For the interval $x \geq 0$, the boundary conditions are $f(x) = 0$ as $x \to \infty$ and $f_x(x) = 0$ at $x = 0$.
- Note that $y(x) = 0$ if and only if return volatility $x = 0$. Moreover, the paper shows that $\mathrm{sign}(y(x)) = \mathrm{sign}(x)$ and $y(-x) = -y(x)$ for any $x > 0$.
- We used the `bvp4c` solver in the MATLAB.

Numerical Illustration of Price-Dividend Ratio w.r.t. Volatility

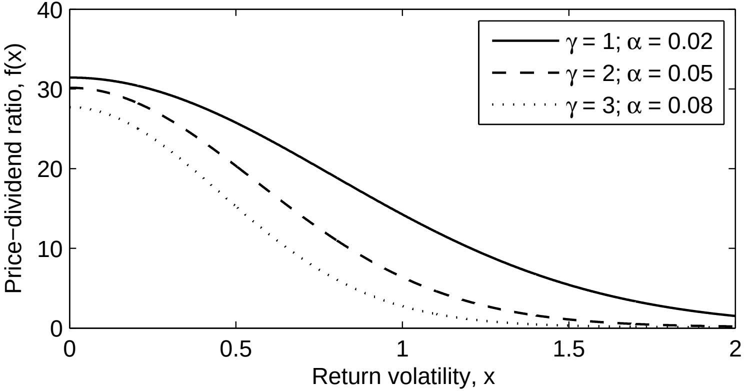


Figure: The price-dividend ratio, $f(x)$, with respect to return volatility, x. The parameters $r = 0.02$, $\beta = 0.5$, $\sigma_x = 0.2$, and $\rho_{dx} = -0.5$.

Return Volatility vs Dividend Growth Volatility

- Return volatility is higher than dividend growth volatility, $x^2 > y(x)^2$, if and only if
$$\rho_{dx} < -\frac{\sigma_x}{2x}\frac{f_x(x)}{f(x)}.$$
- Because the right hand side is always positive, returns fluctuate more than dividends if the correlation between dividends and return volatility is negative, $\rho_{dx} < 0$, which indicates the financial leverage effect.
- Therefore, the "excess volatility" can be explained by implementing volatility feedback and leverage effects in the same model.
- This is in contrast to Shiller's argument that volatility is "too" high.
- If γ is relatively high, volatility feedback can amplify a very small but nonzero dividend growth volatility to a relatively high return volatility (the next slide).

Return Volatility vs Dividend Growth Volatility (cont.)

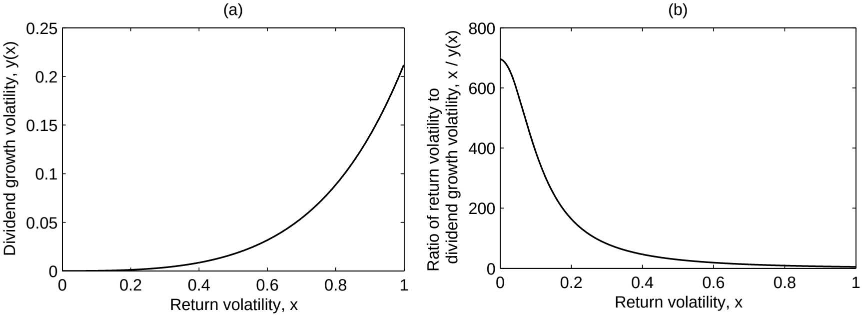


Figure: The parameters are $\gamma = 3.115$, $\alpha = 0.08$, $\sigma_x = 0.2$, $\beta = 0.5$, $r = 0.02$, and $\rho_{dx} = -0.5$. In (a), dividend growth volatility is plotted against return volatility, and in (b) the ratio of return volatility to dividend growth volatility is plotted.

Simulation

- In discrete time with $\epsilon^d, \epsilon^x \sim N(0,1)$, $\mathrm{Corr}(\epsilon^d, \epsilon^x) = \rho_{dx}$:

$$P_{t+\Delta t} = P_t \exp\left[\left(r + \gamma x_t^2 - \frac{1}{f(x_t)} - \frac{1}{2} x_t^2 \right) \Delta t \right. \\ \left. + y(x_t)\sqrt{\Delta t}\epsilon_t^d + \sigma_x \frac{f_x(x_t)}{f(x_t)} \sqrt{\Delta t} \epsilon_t^x \right],$$

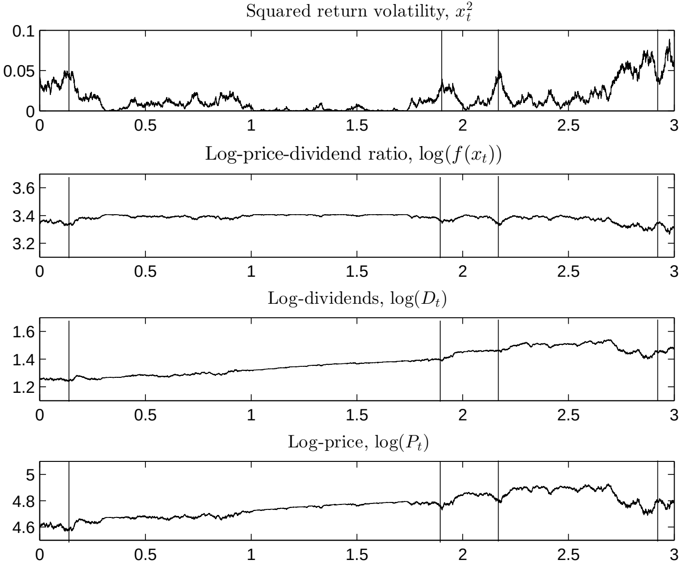
$$x_{t+\Delta t} = x_t \exp\left(-\beta \Delta t\right) + \sigma_x \sqrt{\frac{1 - \exp\left(-2\beta \Delta t\right)}{2\beta}} \epsilon_t^x.$$

- In each step, $f(x)$, $f_x(x)$, and $y(x)$ can be solved for a given x with the given PDE
- Note that the dividend process can be determined from $D_t = P_t / f(x_t)$ or, alternatively, simulated directly:

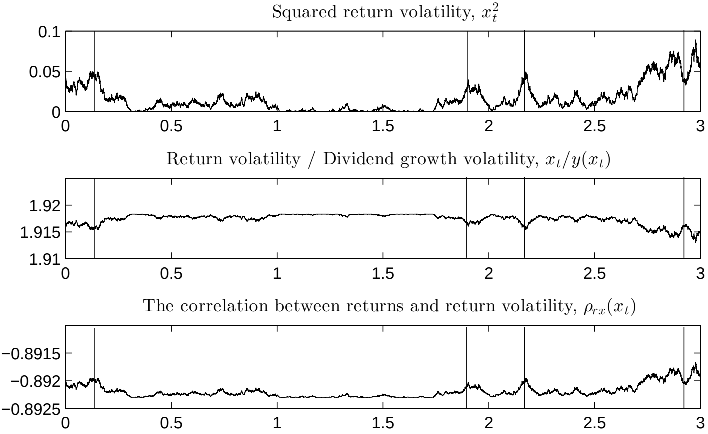
$$D_{t+\Delta t} = D_t \exp\left[\left(\alpha - \frac{1}{2} y(x_t)^2 \right) \Delta t + y(x_t)\sqrt{\Delta t}\epsilon_t^d \right].$$

- Alternatively, the dividend stream and return volatility can also be simulated together and the stock price is then given by $P_t = D_t f(x_t)$.

Sample Paths



Sample Paths



Risk-Neutral Dynamics

- Under the risk-neutral probability measure $\mathbb{Q}$,

$$\begin{aligned} \mathrm{d}R_t &= r\mathrm{d}t + x_t \mathrm{d}\tilde{B}_t^r, \\ \mathrm{d}x_t &= -\tilde{\beta}(x_t)x_t\mathrm{d}t + \sigma_x \mathrm{d}\tilde{B}_t^x, \end{aligned}$$

  where $\tilde{B}_t^r$ and $\tilde{B}_t^x$ are under the probability measure $\mathbb{Q}$, and $\tilde{\beta}$ is the speed of the mean reversion under $\mathbb{Q}$.
- The above is satisfied if

$$\begin{aligned} \mathrm{d}\tilde{B}_t^r &= \mathrm{d}B_t^r + \gamma x_t \mathrm{d}t, \\ \mathrm{d}\tilde{B}_t^x &= \mathrm{d}B_t^x + \frac{\lambda_x(x_t)}{\sigma_x} x_t \mathrm{d}t \end{aligned}$$

  with

$$x_t \mathrm{d}\tilde{B}_t^r = y(x_t)\mathrm{d}\tilde{B}_t^d + \sigma_x \frac{f_x(x_t)}{f(x_t)} \mathrm{d}\tilde{B}_t^x,$$

  where $\lambda_x(x_t) = \tilde{\beta}(x_t) - \beta$ represents the volatility risk premium.
- For simplicity, $\lambda_x = \tilde{\beta} - \beta$ is constant.

Option Valuation

- The price of a European call option can be computed as

$$c(t, P_t, x_t, T, K, r, \alpha; \theta) = \exp(-r(T-t))\mathbb{E}_t^Q\left[(P_T - K)^+\right],$$

  where T is the time of maturity, K the exercise price, and $\theta = \{\sigma_x, \beta, \tilde{\beta}, \gamma, \rho_{dx}\}$ the set of structural parameters.
- To price options, we *need* to know not only parameters from volatility process, $\{\sigma_x, \tilde{\beta}\}$, but also the price of diffusion return risk and volatility risk, $\{\gamma, \lambda_x\}$.
- If $\gamma = 0$, then $f = 1/(r - \alpha)$ an dividend yield equals $r - \alpha$. This corresponds an option pricing model with a constant dividend yield.

Option Valuation

- Specifically, even if $\{\gamma, \lambda_x\}$ do not directly appear in

$$
\begin{aligned}
P_{t+\Delta t} = P_t \exp \Bigg[& \left(r - \frac{1}{f(x_t)} - \frac{1}{2} x_t^2 \right) \Delta t \\
& + y(x_t) \sqrt{\Delta t} \tilde{\epsilon}_t^d + \sigma_x \frac{f_x(x_t)}{f(x_t)} \sqrt{\Delta t} \tilde{\epsilon}_t^x \Bigg],
\end{aligned}
$$

$$
x_{t+\Delta t} = x_t \exp\left(-\tilde{\beta}\Delta t\right) + \sigma_x \sqrt{\frac{1 - \exp\left(-2\tilde{\beta}\Delta t\right)}{2\tilde{\beta}}} \tilde{\epsilon}_t^x,
$$

  they are need to compute $f(x_t; \gamma, \beta, \sigma_x, \rho_{dx})$ every step (with $\beta = \tilde{\beta} + \lambda_x$).
- The price of diffusion return risk, γ, determines the sensitivity of dividend yield to return volatility and thereby *affects option prices* (vrt B-S).

Effect of Diffusion Return Risk on Call Prices

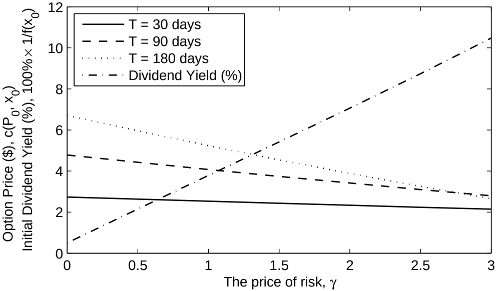


Figure: Parameters are $r = 0.02$, $\beta = \tilde{\beta} = 0.5$, $\alpha = 0.015$, and $\rho_{dx} = -0.5$. Moreover, $x_0 = 0.2$, $P_0 = \$100$, $K = \$100$, and $\Delta t = 1/252$ year.

Effect of Volatility Risk on Call Prices

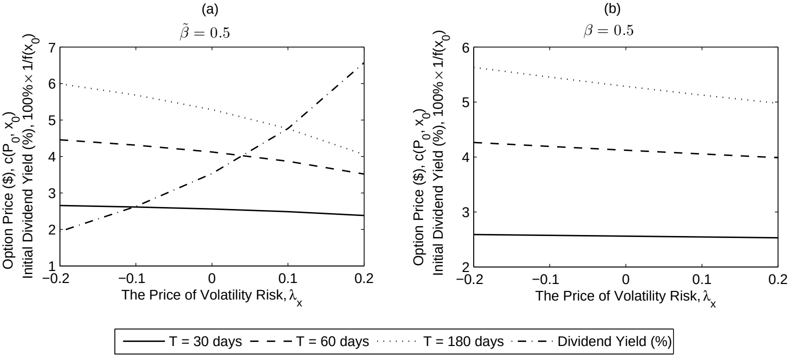


Figure: Parameters are $\gamma = 2, r = 0.02$, $\alpha = 0.05$, $\sigma_x = 0.2$, and $\rho_{dx} = -0.5$. In plot(a) $\tilde{\beta} = 0.5$ and in (b) $\beta = 0.5$. Moreover, $x_0 = 0.2$, $P_0 = \$100$, $K = \$100$, and $\Delta t = 1/252$ year.

Effect of return volatility on the price of a European call option

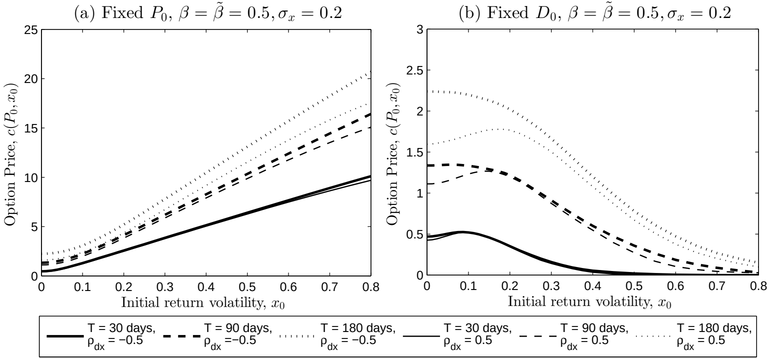


Figure: $\gamma = 2, r = 0.02$, $\alpha = 0.05$, $\beta = \tilde{\beta} = 0.5$, $\sigma_x = 0.2$. Moreover, $\rho_{dx} = -0.5$ (thicker lines) or $\rho_{dx} = 0.5$ (thinner lines).

Calibration on Option Prices

- Option mid-prices on S&P500 inedx call options used to calibrate the model.
- In-sample data: 11,267 observations (150 trading days)
- Out-of-sample data: 7,320 observations (100 trading days)
- The loss function we use is the square root dollar mean-squared error,

$$\sqrt{\frac{1}{n}\sum_{i}^{n}\left(\hat{c}(t_i, P_{t_i}, T_i, K_i) - c(t_i, P_{t_i}, x_{t_i}, T_i, K_i, r_i^{\text{T-bill}}, \bar{\alpha}; \theta)\right)^2},$$

- n denotes the number of contracts and $r_i^{\text{T-bill}}$ the observed T-Bill rate.
- $\hat{c}(\cdot)$ is the price of the i-th option, $c(\cdot;\theta)$ the corresponding model price,
- $x_{t_i} \equiv \text{VIX}_{t_i-1}$ denotes the volatility proxy used at time t_i.
- $\bar{\alpha} \approx 6.13\%$, from monthly data on S&P 500 dividends

Empirical Results

	(i) γ free	(ii) $\gamma = 0$	(iii) $\gamma = 0$ with $r_i = 2r_i^{\text{T-bill}}$, $\alpha = \bar{\alpha}/2$
$\tilde{\beta}$	1.2476	32.8570	1.3282
	(0.0420)	(2.5897)	(0.0474)
σ_x	0.2713	0.6659	0.2666
	(0.0032)	(0.0271)	(0.0035)
ρ_{dx}	-0.6410	0.4241	-0.8002
	(0.0069)	(0.0180)	(0.0033)
λ_x	-0.3376		
	(0.0128)		
γ	1.7929		
	(0.0099)		
$RMSEs			
Sample A	0.8111	1.1327	0.9114
Sample B	0.9355	1.6429	0.9859

Parameters of the variance process can be obtained by applying the relations: $\kappa = 2\beta = 2.4952$, $\sigma_h = 2\sigma_x = 0.5426$, and $\theta = \sigma_x^2/(2\beta) = 0.0288$ (spesification i).

Specification (i)

- ρ_{dx} is negative, indicating the existence of a leverage effect
- γ is about 1.8 and statistically significant
  - the expected returns are related to squared return volatility
  - the price-dividend ratio is sensitive to return volatility
- λ_x is negative, -0.338, and that it has been estimated accurately
  - Theoretical link between volatility risk premium and the price of diffusion return risk (see e.g. Bakshi and Kapadia RFS 2003):

$$\begin{aligned}\lambda_x(x_t) &= \gamma\sigma_x \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Corr}_t(x_t, R_t)|_{\tau=t} \\ &= \gamma\sigma_x\rho_{rx}(x_t),\end{aligned}$$

  - With our estimates, $\rho_{rx} \approx -0.7879$, $\sigma_x = 0.27$ and $\gamma = 1.79$ yields the right hand side of to be about -0.38.

Specification (ii)

- $\gamma = 0$ excludes volatility feedback and implying that the price-dividend ratio, and then also the dividend yield, do not depend on return volatility.
- Both the in-sample (sample A) and out-of-the-sample (sample B) RMSEs are substantially higher than in specification (i).
- The estimate of β is not reasonable, for it implies that κ is greater than 60
- Moreover, the ρ_{dx} is substantially positive and contradicts the leverage effect.
- On the other hand, $\gamma = 0$ is not consistent with the observed interest rates and the dividend growth rate. In particular, the average dividend growth rate, $\bar{\alpha} \approx 0.0613$, exceeds the T-bill rates that range from 0.054 to 0.0596 and imply that the dividend yield, $1/f = r - \alpha$, takes negative values.
- Therefore, we constructed a third specification, by which we ensured that the dividend yield is always a positive constant.

Specification (iii)

- To have strictly positive dividend yields with $\gamma = 0$, we use $r_i = 2r_i^{\text{T-bill}}$ with $\alpha = \bar{\alpha}/2$, where $r_i^{\text{T-bill}}$ is the T-bill rate at date i.
- Makes parameter estimates with $\gamma = 0$ more realistic and both the in-sample and out-of-the-sample RMSEs decrease yet remain greater than in specification (i).
- These adjusted interest rate and dividend growth rate values are ad-hoc and do not represent the true economic situation.
- Rather, this specification shows that the poor performance and unrealistic estimates of specification (ii) can be partly explained by the fact that the assumption of $\gamma = 0$ contradicts the empirical observations of dividend growth and risk-free interest rates.

Conclusions

- Under volatility feedback effect dividend yield becomes time-varying and endogenously determined by total return volatility
- This implies that
  - the stock price and dividend stream do not always move in the same direction
  - the ratio of total return volatility to dividend growth volatility can be relatively high
  - the market price of diffusion return risk (or equity risk-premium) affects option prices
  - the market price of diffusion return risk can be estimated from option data
  - the price of a call option can be decreasing in squared total return volatility
- Approximations or computational solutions needed to ease our progress
- Hedging under volatility feedback?

Thank you!