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New Thinking in Finance

Intuition: What is Volatility Feedback?

1. An unexpected increase in squared volatility →
2. Investors require more return to compensate higher risk →
3. An increase in an instantaneous (stochastic) discount rate →
4. Dividends are discounted at a higher rate →
5. The expectation of future discounted dividends decreases →
6. The stock price decreases, price-dividend ratio decreases and instantaneous dividend yield increases →
7. An increase in squared volatility can decrease the value of an in-the-money call option with a convex payoff function through stock price and dividend yield.

It can be important to model how volatility drives dividend yield making it time-varying.
Assumption 1: Returns, Volatility, and Risk-Return Tradeoff

- The cumulative return from dividends and changes in prices satisfies
  \[ dR_t = \frac{dP_t + D_t dt}{P_t}. \]

- The joint dynamics with its instantaneous volatility, \( x_t \), evolves as
  \[ dR_t = (r + \gamma x_t^2) dt + x_t dB_r^t \]
  \[ dx_t = -\beta x_t dt + \sigma_x dB_x^t, \]

where \( r, \gamma, \beta, \) and \( \sigma_x \) are constant positive real numbers. \( B^r \) and \( B^x \) are Brownian motions, \( dB_r^t dB_x^t = \rho_{rx,t} dt \), and \( x_0 := x, x \in \mathbb{R} \).

- Consequently, squared volatility, \( h_t = x_t^2 \), follows the squared root process,
  \[ dh_t = \kappa(\theta - h_t) dt + \sigma_h \sqrt{h_t} dB_x^t \]

with \( \kappa = 2\beta \), \( \theta = \sigma_x^2 / (2\beta) \), and \( \sigma_h = 2\sigma_x \).
Assumption 2: Dividends

(i) The stochastic differential of dividends is given by

\[ dD_t = \alpha D_t dt + y_t D_t dB_t^d \]

with \( dB_t^d dB_t^x = \rho dx dt \) and \( D_0 := D, D > 0 \).

The correlation coefficient of dividend growth and return volatility (leverage effect), \( \rho_{dx} \in [-1, 1] \), and the expected rate of dividend growth, \( \alpha \in \mathbb{R} \), are constant. Dividend growth volatility, \( y_t \), is stochastic.

(ii)

\[ \text{sign} \left( \frac{d}{d\tau} \text{Cov}_t (D_t, x_t^2) \big|_{\tau=t} \right) = \text{sign}(\rho_{dx}). \]
Assumption 3: Transversality Condition

The stock price, \( p(D_t, x_t) \), can be expressed as the \textit{expected value of discounted dividends}, conditional upon the present information:

\[
p(D_t, x_t) = \mathbb{E}_{D,x} \int_t^\infty \exp \left[ - \int_t^s \left( r + \gamma x_u^2 \right) du \right] ds
ds
\]

\[
= D_t \times \mathbb{E}_x \int_t^\infty \exp \left[ \int_t^s \left( \alpha - r - \gamma x_u^2 - \frac{1}{2} y_u^2 \right) du 
+ \int_t^s y_u dB_u^d \right] ds < \infty.
\]

Here \( r + \gamma x_u^2 \) represents the instantaneous stochastic cost of capital at time \( u \).
The price-dividend ratio, $f: \mathbb{R} \rightarrow \mathbb{R}_+$, satisfies $p(D_t, x_t) = D_t f(x_t)$. Hence,

$$f(x_t) = \mathbb{E}_x \int_t^\infty \exp \left[ \int_t^s \left( \alpha - r - \gamma x_u^2 - \frac{1}{2} y_u^2 \right) \, du + \int_t^s y_u \, dB_u^d \right] \, ds.$$

- For all $x > 0$, $f(x) = f(-x)$ and so $p(D, x) = p(D, -x)$ (the sign of volatility does not matter).
- $f$ is an even function, i.e., $f(x) = f(-x)$, $f_x(x) = -f_x(-x)$, and $f_{xx}(x) = f_{xx}(-x)$ for all $x > 0$, where $f_x$ and $f_{xx}$ denote first and second order derivatives.
- For the stock price to be a continuously differentiable function, we can impose that $f_x(0) = 0$. 
Consequently, the stock price process, $P_t = p(D_t, x_t)$, satisfies

$$dP_t = \left( r + \gamma x_t^2 \right) P_t dt - D_t dt + x_t P_t dB_t^r$$

$$= \left( r + \gamma x_t^2 - \frac{1}{f(x_t)} \right) P_t dt + x_t P_t dB_t^r$$

$$= \left( r + \gamma x_t^2 - \frac{1}{f(x_t)} \right) P_t dt + y(x_t) P_t dB_t^d + \sigma_x \frac{f_x(x_t)}{f(x_t)} P_t dB_t^x,$$

where $1/f(x_t)$ is the instantaneous dividend yield and $y(x_t)$ dividend growth volatility.
The price-dividend, $f(x)$ ratio satisfies the following relation:

$$(y(x)\sigma_x \rho_{dx} - \beta x) f_x(x) + \frac{1}{2} \sigma_x^2 f_{xx}(x) - (r + \gamma x^2 - \alpha) f(x) = -1,$$

where dividend growth volatility, $y(x)$, is given by

$$y(x) = -\rho_{dx} \sigma_x \frac{f_x(x)}{f(x)} + \text{sign}(x) \sqrt{x^2 - (1 - \rho_{dx}^2) \left(\sigma_x \frac{f_x(x)}{f(x)}\right)^2}.$$  

For the interval $x \geq 0$, the boundary conditions are $f(x) = 0$ as $x \to \infty$ and $f_x(x) = 0$ at $x = 0$.

Note that $y(x) = 0$ if and only if return volatility $x = 0$. Moreover, the paper shows that $\text{sign}(y(x)) = \text{sign}(x)$ and $y(-x) = -y(x)$ for any $x > 0$.

We used the bvp4c solver in the MATLAB.
Figure: The price-dividend ratio, $f(x)$, with respect to return volatility, $x$. The parameters $r = 0.02$, $\beta = 0.5$, $\sigma_x = 0.2$, and $\rho_{dx} = -0.5$. 
Return volatility is higher than dividend growth volatility, \( x^2 > y(x)^2 \), if and only if

\[
\rho_{dx} < -\frac{\sigma_x f_x(x)}{2x f(x)}.
\]

Because the right hand side is always positive, returns fluctuate more than dividends if the correlation between dividends and return volatility is negative, \( \rho_{dx} < 0 \), which indicates the financial leverage effect.

Therefore, the “excess volatility” can be explained by implementing volatility feedback and leverage effects in the same model.

This is in contrast to Shiller’s argument that volatility is ”too” high.

If \( \gamma \) is relatively high, volatility feedback can amplify a very small but nonzero dividend growth volatility to a relatively high return volatility (the next slide).
Figure: The parameters are $\gamma = 3.115$, $\alpha = 0.08$, $\sigma_x = 0.2$, $\beta = 0.5$, $r = 0.02$, and $\rho_{dx} = -0.5$. In (a), dividend growth volatility is plotted against return volatility, and in (b) the ratio of return volatility to dividend growth volatility is plotted.
Simulation

- In discrete time with $\epsilon^d, \epsilon^x \sim N(0, 1), \text{Corr}(\epsilon^d, \epsilon^x) = \rho_{dx}$:

  \[ P_{t+\Delta t} = P_t \exp \left[ \left( r + \gamma x_t^2 - \frac{1}{f(x_t)} - \frac{1}{2} x_t^2 \right) \Delta t \right. \]

  \[ \left. + y(x_t) \sqrt{\Delta t} \epsilon^d_t + \sigma_x \frac{f_x(x_t)}{f(x_t)} \sqrt{\Delta t} \epsilon^x_t \right], \]

  \[ x_{t+\Delta t} = x_t \exp (-\beta \Delta t) + \sigma_x \sqrt{\frac{1}{2\beta} \left( 1 - \exp (-2\beta \Delta t) \right)} \epsilon^x_t. \]

- In each step, $f(x), f_x(x), \text{and } y(x)$ can be solved for a given $x$ with the given PDE

- Note that the dividend process can be determined from $D_t = P_t / f(x_t)$ or, alternatively, simulated directly:

  \[ D_{t+\Delta t} = D_t \exp \left[ \left( \alpha - \frac{1}{2} y(x_t)^2 \right) \Delta t + y(x_t) \sqrt{\Delta t} \epsilon^d_t \right]. \]

- Alternatively, the dividend stream and return volatility can also be simulated together and the stock price is then given by $P_t = D_t f(x_t)$. 
Sample Paths

Squared return volatility, $x_t^2$

Log-price-dividend ratio, $\log(f(x_t))$

Log-dividends, $\log(D_t)$

Log-price, $\log(P_t)$
Sample Paths

Squared return volatility, $x_t^2$

Return volatility / Dividend growth volatility, $x_t/y(x_t)$

The correlation between returns and return volatility, $\rho_{rx}(x_t)$
Risk-Neutral Dynamics

- Under the risk-neutral probability measure \( Q \),

\[
\begin{align*}
    dR_t &= rd_t + x_t \tilde{B}_t^r, \\
    dx_t &= -\tilde{\beta}(x_t)x_t dt + \sigma_x d\tilde{B}_t^x,
\end{align*}
\]

where \( \tilde{B}_t^r \) and \( \tilde{B}_t^x \) are under the probability measure \( Q \), and \( \tilde{\beta} \) is the speed of the mean reversion under \( Q \).

- The above is satisfied if

\[
\begin{align*}
    d\tilde{B}_t^r &= dB_t^r + \gamma x_t dt, \\
    d\tilde{B}_t^x &= dB_t^x + \frac{\lambda_x(x_t)}{\sigma_x} x_t dt
\end{align*}
\]

with

\[
x_t d\tilde{B}_t^r = y(x_t)d\tilde{B}_t^d + \sigma_x \frac{f_x(x_t)}{f(x_t)} d\tilde{B}_t^x,
\]

where \( \lambda_x(x_t) = \tilde{\beta}(x_t) - \beta \) represents the volatility risk premium.

- For simplicity, \( \lambda_x = \tilde{\beta} - \beta \) is constant.
The price of a European call option can be computed as

\[ c(t, P_t, x_t, T, K, r, \alpha; \theta) = \exp(-r(T - t)) \mathbb{E}^Q_t [(P_T - K)^+] , \]

where \( T \) is the time of maturity, \( K \) the exercise price, and \( \theta = \{\sigma_x, \beta, \tilde{\beta}, \gamma, \rho_{dx}\} \) the set of structural parameters.

To price options, we need to know not only parameters from volatility process, \( \{\sigma_x, \tilde{\beta}\} \), but also the price of diffusion return risk and volatility risk, \( \{\gamma, \lambda_x\} \).

If \( \gamma = 0 \), then \( f = 1/(r - \alpha) \) an dividend yield equals \( r - \alpha \). This corresponds an option pricing model with a constant dividend yield.
Specifically, even if \{\gamma, \lambda_x\} do not directly appear in

\[
P_{t+\Delta t} = P_t \exp \left[ \left( r - \frac{1}{f(x_t)} - \frac{1}{2} x_t^2 \right) \Delta t \right.
\]
\[
+ y(x_t) \sqrt{\Delta t} \tilde{\epsilon}_t + \sigma_x \frac{f_x(x_t)}{f(x_t)} \sqrt{\Delta t} \tilde{\epsilon}_t \right],
\]

\[
x_{t+\Delta t} = x_t \exp \left( -\tilde{\beta} \Delta t \right) + \sigma_x \sqrt{\frac{1 - \exp \left( -2\tilde{\beta} \Delta t \right)}{2\tilde{\beta}}} \tilde{\epsilon}_t,
\]

they are need to compute \(f(x_t; \gamma, \beta, \sigma_x, \rho_{dx})\) every step (with \(\beta = \tilde{\beta} + \lambda_x\)).

The price of diffusion return risk, \(\gamma\), determines the sensitivity of dividend yield to return volatility and thereby affects option prices (vt B-S).
Effect of Diffusion Return Risk on Call Prices

Figure: Parameters are $r = 0.02$, $\beta = \tilde{\beta} = 0.5$, $\alpha = 0.015$, and $\rho_{dx} = -0.5$. Moreover, $x_0 = 0.2$, $P_0 = $100, $K = $100, and $\Delta t = 1/252$ year.
Effect of Volatility Risk on Call Prices

Figure: Parameters are $\gamma = 2, r = 0.02, \alpha = 0.05, \sigma_x = 0.2, \text{ and } \rho_{dx} = -0.5$. In plot (a) $\tilde{\beta} = 0.5$ and in (b) $\beta = 0.5$. Moreover, $x_0 = 0.2, P_0 = \$100, K = \$100, \text{ and } \Delta t = 1/252 \text{ year}$.
Effect of return volatility on the price of a European call option

Figure: \( \gamma = 2, r = 0.02, \alpha = 0.05, \beta = \hat{\beta} = 0.5, \sigma_x = 0.2 \). Moreover, \( \rho_{dx} = -0.5 \) (thicker lines) or \( \rho_{dx} = 0.5 \) (thinner lines).
Calibration on Option Prices

- Option mid-prices on S&P500 index call options used to calibrate the model.
- In-sample data: 11,267 observations (150 trading days)
- Out-of-sample data: 7,320 observations (100 trading days)
- The loss function we use is the square root dollar mean-squared error,

\[
\sqrt{\frac{1}{n} \sum_{i}^{n} (\hat{c}(t_i, P_{t_i}, T_i, K_i) - c(t_i, P_{t_i}, x_{t_i}, T_i, K_i, r_{i}^{\text{T-bill}}, \bar{\alpha}; \theta))^2},
\]

- \( n \) denotes the number of contracts and \( r_{i}^{\text{T-bill}} \) the observed T-Bill rate.
- \( \hat{c}(\cdot) \) is the price of the \( i \)-th option, \( c(\cdot; \theta) \) the corresponding model price,
- \( x_{t_i} \equiv \text{VIX}_{t_i-1} \) denotes the volatility proxy used at time \( t_i \).
- \( \bar{\alpha} \approx 6.13\% \), from monthly data on S&P 500 dividends.

\[
\sqrt{\frac{1}{n} \sum_{i}^{n} (\hat{c}(t_i, P_{t_i}, T_i, K_i) - c(t_i, P_{t_i}, x_{t_i}, T_i, K_i, r_{i}^{\text{T-bill}}, \bar{\alpha}; \theta))^2},
\]
Empirical Results

<table>
<thead>
<tr>
<th></th>
<th>(i) $\gamma$ free</th>
<th>(ii) $\gamma = 0$</th>
<th>(iii) $\gamma = 0$ with $r_i = 2r_i^{T-bill}$, $\alpha = \bar{\alpha}/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\beta}$</td>
<td>1.2476 (0.0420)</td>
<td>32.8570 (2.5897)</td>
<td>1.3282 (0.0474)</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>0.2713 (0.0032)</td>
<td>0.6659 (0.0271)</td>
<td>0.2666 (0.0035)</td>
</tr>
<tr>
<td>$\rho_{dx}$</td>
<td>-0.6410 (0.0069)</td>
<td>0.4241 (0.0180)</td>
<td>-0.8002 (0.0033)</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>-0.3376 (0.0128)</td>
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<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.7929 (0.0099)</td>
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$\text{RMSEs}$

<table>
<thead>
<tr>
<th></th>
<th>Sample A</th>
<th>Sample B</th>
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<tbody>
<tr>
<td></td>
<td>0.8111</td>
<td>0.9355</td>
</tr>
<tr>
<td></td>
<td>1.1327</td>
<td>1.6429</td>
</tr>
<tr>
<td></td>
<td>0.9114</td>
<td>0.9859</td>
</tr>
</tbody>
</table>

Parameters of the variance process can be obtained by applying the relations: $\kappa = 2\beta = 2.4952$, $\sigma_h = 2\sigma_x = 0.5426$, and $\theta = \sigma_x^2/(2\beta) = 0.0288$ (specification i).
Specification (i)

- $\rho_{dx}$ is negative, indicating the existence of a leverage effect.
- $\gamma$ is about 1.8 and statistically significant.
  - The expected returns are related to squared return volatility.
  - The price-dividend ratio is sensitive to return volatility.
- $\lambda_x$ is negative, -0.338, and that it has been estimated accurately.
  - Theoretical link between volatility risk premium and the price of diffusion return risk (see e.g. Bakshi and Kapadia RFS 2003):

$$\lambda_x(x_t) = \gamma \sigma_x \frac{d}{d\tau} \text{Corr}_t(x_t, R_t)|_{\tau=t}$$

$$= \gamma \sigma_x \rho_{rx}(x_t),$$

- With our estimates, $\rho_{rx} \approx -0.7879$, $\sigma_x = 0.27$ and $\gamma = 1.79$ yields the right hand side of to be about $-0.38$. 
Specification (ii)

• $\gamma = 0$ excludes volatility feedback and implying that the price-dividend ratio, and then also the dividend yield, do not depend on return volatility.

• Both the in-sample (sample A) and out-of-the-sample (sample B) RMSEs are substantially higher than in specification (i).

• The estimate of $\beta$ is not reasonable, for it implies that $\kappa$ is greater than 60.

• Moreover, the $\rho_{dx}$ is substantially positive and contradicts the leverage effect.

• On the other hand, $\gamma = 0$ is not consistent with the observed interest rates and the dividend growth rate. In particular, the average dividend growth rate, $\bar{\alpha} \approx 0.0613$, exceeds the T-bill rates that range from 0.054 to 0.0596 and imply that the dividend yield, $1/f = r - \alpha$, takes negative values.

• Therefore, we constructed a third specification, by which we ensured that the dividend yield is always a positive constant.
To have strictly positive dividend yields with $\gamma = 0$, we use $r_i = 2r_{i, \text{T-bill}}$ with $\alpha = \bar{\alpha} / 2$, where $r_{i, \text{T-bill}}$ is the T-bill rate at date $i$.

- Makes parameter estimates with $\gamma = 0$ more realistic and both the in-sample and out-of-the-sample RMSEs decrease yet remain greater than in specification (i).
- These adjusted interest rate and dividend growth rate values are ad-hoc and do not represent the true economic situation.
- Rather, this specification shows that the poor performance and unrealistic estimates of specification (ii) can be partly explained by the fact that the assumption of $\gamma = 0$ contradicts the empirical observations of dividend growth and risk-free interest rates.
Conclusions

- Under volatility feedback effect dividend yield becomes time-varying and endogenously determined by total return volatility
- This implies that
  - the stock price and dividend stream do not always move in the same direction
  - the ratio of total return volatility to dividend growth volatility can be relatively high
  - the market price of diffusion return risk (or equity risk-premium) affects option prices
  - the market price of diffusion return risk can be estimated from option data
  - the price of a call option can be decreasing in squared total return volatility
- Approximations or computational solutions needed to ease our progress
- Hedging under volatility feedback?
Thank you!