Optimal investment and contingent claim valuation under temporary price impacts and margin requirements

Teemu Pennanen
Department of Mathematics,
King's College London

- The cost of a market order depends nonlinearly on the traded amount.
- Much of trading consists of exchanging sequences of cash-flows (swaps, insurance contracts, coupon payments, dividends, . . .)
- We study hedging-based contingent claim valuation through asset-liability management (ALM).
- We extend basic results on optimal investment and contingent claim valuation to markets with nonlinear trading costs and portfolio constraints.
- We outline a numerical optimization scheme for approximating minimal reservation prices and indifference swap rates in incomplete markets.

Illiquidity

- Dalang, Morton, Willinger, *Equivalent martingale measures and no-arbitrage in stochastic securities market models*, Stoch. and Stoch. Rep., 1990.
- Hodges, Neuberger, *Optimal replication of contingent claims under transaction costs*, Rev. Fut. Markets, 1989.
- Pennanen, Perkkiö, *Stochastic programs without duality gaps*, Math. Progr., 2012.
- Pennanen, *Convex duality in optimal investment under illiquidity*, Math. Progr., to appear.
- Pennanen, *Optimal investment and contingent claim valuation in illiquid markets*, submitted.
- Hilli, Koivu, Pennanen, *Optimal construction of a fund of funds*, European Actuarial Journal, 2011.
- Hilli, Koivu, Pennanen, *Cash-flow based valuation of pension liabilities*, European Actuarial Journal, 2011.
- Aro, Pennanen, *Stochastic modeling of mortality and financial markets*, Scandinavian Actuarial Journal, 2012.

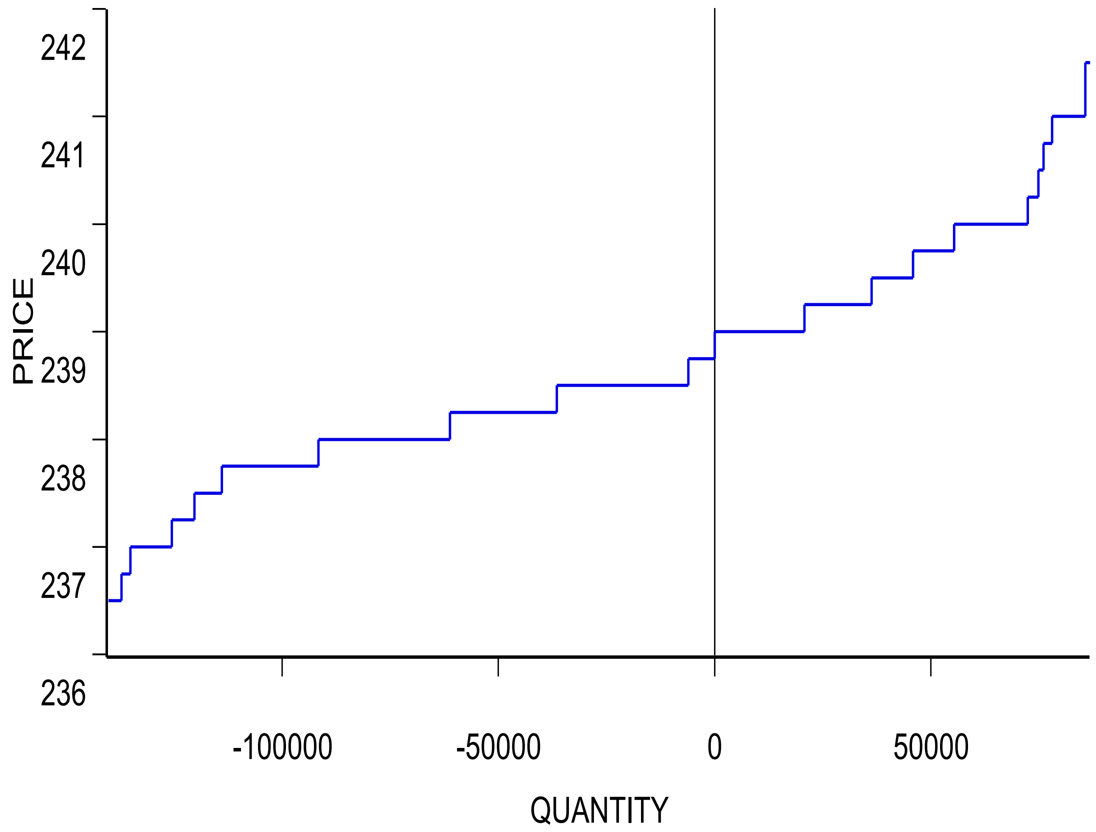
Limit order books

Limit order book of TDC A/S on 12 January 2005 at 13:58:19.43 (obtained from Copenhagen Stock Exchange order flow data using the rules of SAXESS trading protocol)

Bid		Ask	
Price	Quantity	Price	Quantity
238.75	140	239	3700
238.75	600	239	1000
238.75	3300	239	5000
238.75	2000	239	1000
238.5	10000	239	1000
238.5	3900	239	2500
238.5	15000	239	6600
238.5	1500	239.25	10000
238.25	10000	239.25	2500
238.25	1000	239.25	3000
238.25	3500	239.5	600
238.25	10000	239.5	5000
238.25	200	239.5	800
⋮	⋮	⋮	⋮

Limit order books

The corresponding marginal price curve. Negative quantity corresponds to a sale.



Limit order books

- Monotonicity of the marginal price curve $s : \mathbb{R} \to \mathbb{R}$ implies that the total cost

$$S(x) := \int_0^x s(z)dz$$

of a market order of x shares is convex.

- A negative x incurs a negative cost which just means that sales yield revenue.
- In perfectly liquid markets, s would be constant and S would be linear.
- Trading costs are often described in terms of the supply curve $x \mapsto S(x)/x$ but this misses the convexity of S (or equivalently, the monotonicity of s).

Outline

1. Market model with nonlinear trading costs and portfolio constraints.
2. Optimal investment problem parameterized by a sequence of cash-flows.
3. Reservation prices for financial liabilities.
4. Indifference pricing of general swap contracts.
5. Existence of solutions under an extended no-arbitrage condition.
6. Dual expressions for the optimal value, reserves and swap rates in terms of "consistent price systems".
7. A computational technique for ALM and pricing.
8. An example: Valuation of pension liabilities

Market model

Consider a financial market where a finite set J of assets can be traded at $t = 0, \dots, T$.

- Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space.
- The cost (in cash) of buying a portfolio $x \in \mathbb{R}^J$ at time t in state ω will be denoted by $S_t(x, \omega)$.
- We will assume that
  - $S_t(\cdot, \omega)$ is convex with $S_t(0, \omega) = 0$,
  - $S_t(x, \cdot)$ is $\mathcal{F}_t$-measurable.
- Such a sequence (S_t) will be called a convex cost process.

Market model

Example 1 (Liquid markets) *If* $s = (s_t)_{t=0}^T$ *is an* $(\mathcal{F}_t)_{t=0}^T$*-adapted* $\mathbb{R}^J$*-valued price process, then the functions*

$$S_t(x,\omega) = s_t(\omega) \cdot x$$

define a convex cost process.

Example 2 (Jouini and Kallal, 1995) *If* $(s_t^a)_{t=0}^T$ *and* $(s_t^b)_{t=0}^T$ *are* $(\mathcal{F}_t)_{t=0}^T$*-adapted with* $s^b \leq s^a$*, then the functions*

$$S_t(x,\omega) = \begin{cases} s_t^a(\omega)x & \text{if } x \geq 0, \\ s_t^b(\omega)x & \text{if } x \leq 0 \end{cases}$$

define a convex cost process.

Example 3 (Çetin and Rogers, 2007) *If $s = (s_t)_{t=0}^T$ is an $(\mathcal{F}_t)_{t=0}^T$-adapted process and ψ is a lower semicontinuous convex function on $\mathbb{R}$ with $\psi(0) = 0$, then the functions*

$$S_t(x, \omega) = x^0 + s_t(\omega)\psi(x^1)$$

define a convex cost process.

Example 4 (Dolinsky and Soner, 2013) *If $s = (s_t)_{t=0}^T$ is $(\mathcal{F}_t)_{t=0}^T$-adapted and $G_t(x, \cdot)$ are $\mathcal{F}_t$-measurable functions such that $G_t(\cdot, \omega)$ are finite and convex, then the functions*

$$S_t(x, \omega) = x^0 + s_t(\omega) \cdot x^1 + G_t(x^1, \omega)$$

define a convex cost process.

Market model

- We allow for portfolio constraints requiring that the portfolio held over $(t, t+1]$ in state ω has to belong to a set $D_t(\omega) \subseteq \mathbb{R}^J$.
- We assume that
  - $D_t(\omega)$ are closed and convex with $0 \in D_t(\omega)$.
  - $\{\omega \in \Omega \mid D_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}_t$ for every open $U \subset \mathbb{R}^J$.

Market model

- Models where $D_t(\omega)$ is independent of (t, ω) have been studied e.g. in [Cvitanić and Karatzas, 1992] and [Jouini and Kallal, 1995].
- In [Napp, 2003],

$$D_t(\omega) = \{x \in \mathbb{R}^d \mid M_t(\omega)x \in K\},$$

  where $K \subset \mathbb{R}^L$ is a closed convex cone and M_t is an $\mathcal{F}_t$-measurable matrix.
- General constraints have been studied in [Evstigneev, Schürger and Taksar, 2004], [Rokhlin, 2005] and [Czichowsky and Schweizer, 2012].

Optimal investment

Let $c \in \mathcal{M} := \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P)\}$ and consider the problem

$$\text{minimize} \quad \sum_{t=0}^T \mathcal{V}_t(S_t(\Delta x_t) + c_t) \quad \text{over} \quad x \in \mathcal{N}_D$$

- $\mathcal{N}_D = \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^J),\ x_t \in D_t,\ x_T = 0\}$,
- $\mathcal{V}_t : L^0 \to \overline{\mathbb{R}}$ are convex, nondecreasing and $\mathcal{V}_t(0) = 0$.

Example 5 *If $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$, the problem can be written*

$$\begin{array}{ll} \text{minimize} & \mathcal{V}_T(S_T(\Delta x_T) + c_T) \quad \text{over} \quad x \in \mathcal{N}_D \\ \text{subject to} & S_t(\Delta x_t) + c_t \le 0, \quad t = 0, \dots, T-1. \end{array}$$

Example 6 (Markets with a numeraire) *When*

$$S_t(x,\omega) = x^0 + \tilde{S}_t(\tilde{x},\omega) \quad \textit{and} \quad D_t(\omega) = \mathbb{R} \times \tilde{D}_t(\omega),$$

the problem can be written as

$$\text{minimize} \quad \mathcal{V}_T\left(\sum_{t=0}^T \tilde{S}_t(\Delta\tilde{x}_t) + \sum_{t=0}^T c_t\right) \quad \text{over} \quad x \in \mathcal{N}_D.$$

When $\tilde{S}_t(\tilde{x},\omega) = \tilde{s}_t(\omega)\cdot\tilde{x}$*,*

$$\sum_{t=0}^T \tilde{S}_t(\Delta\tilde{x}_t) = \sum_{t=0}^T \tilde{s}_t\cdot\Delta\tilde{x}_t = -\sum_{t=0}^{T-1} \tilde{x}_t\cdot\Delta\tilde{s}_{t+1}.$$

Optimal investment

We denote the optimal value function by

$$\varphi(c) = \inf_{x \in \mathcal{N}_D} \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t).$$

- We have

$$\varphi(c) = \inf_{d \in \mathcal{C}} \sum_{t=0}^{T} \mathcal{V}_t(c_t - d_t),$$

  where $\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \; S_t(\Delta x_t) + c_t \leq 0 \quad \forall t\}$.
- In the classical linear model,

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \; \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}.$$

Optimal investment

Lemma 7 *The value function φ is convex and*

$$\varphi(\bar{c}+c) \leq \varphi(\bar{c}) \quad \forall \bar{c} \in \mathcal{M},\ c \in \mathcal{C}^\infty.$$

where $\mathcal{C}^\infty = \{c \in \mathcal{M} \,|\, \bar{c} + \alpha c \in \mathcal{C} \quad \forall \bar{c} \in \mathcal{C},\ \forall \alpha > 0\}$*.*

- In particular, φ is constant with respect to the linear space $\mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$.
- If S_t are positively homogeneous and D_t are conical, then $\mathcal{C}$ is a cone and $\mathcal{C}^\infty = \mathcal{C}$.

Reservation price

- Reservation price: How much capital do we need to cover our liabilities at an acceptable level of risk?
- Indifference price: What is the least price we can sell a financial product for without worsening our position?
- The former is an important notion in accounting, financial reporting, supervision of financial institutions and in the Black–Scholes–Merton option pricing model.
- Unlike offered prices, the reservation price does not depend on a company's assets.
- It turns out that, in complete markets, reservation prices coincide with indifference prices.

Reservation price

- The reservation price for a liability $c \in \mathcal{M}$ is given by

$$\pi^0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \leq 0\}$$

where $p^0 = (1, 0, \dots, 0)$.

- If $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$, the reservation price $\pi^0(c)$ is given by the optimum value of

$$\begin{aligned} &\text{minimize} && S_0(x_0) \quad \text{over} \quad x \in \mathcal{N}_D,\\ &\text{subject to} && S_t(\Delta x_t) + c_t \leq 0, \quad t = 1, \dots, T-1,\\ &&& \mathcal{V}_T(S_T(\Delta x_T) + c_T) \leq 0. \end{aligned}$$

- If $\mathcal{V}_t = \delta_{L^0_-}$ for all t, the reservation price becomes the superhedging cost $\pi^0_{\sup}$. Let $\pi^0_{\inf}(c) = -\pi^0_{\sup}(-c)$.

Reservation price

Theorem 8 *The reservation price π^0 is convex and nondecreasing with respect to $\mathcal{C}^\infty$. We have $\pi^0 \le \pi^0_{\sup}$ and if $\pi^0(0) \ge 0$, then*

$$\pi^0_{\inf}(c) \le \pi^0(c) \le \pi^0_{\sup}(c)$$

with equalities throughout if $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$ for $\alpha \in \mathbb{R}$.

- π^0 may be interpreted much like a risk measure in [Artzner, Delbaen, Eber and Heath, 1999]. However, we do not assume the existence of a numeraire so π^0 operates on sequences of cash flows and it is not "cash invariant".
- When $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$, as e.g. in the BSM-model, the reservation price $\pi^0(c)$ is independent of P and $\mathcal{V}_t$.

Swap contracts

- In a swap contract, an agent receives a sequence $p \in \mathcal{M}$ of premiums and delivers a sequence $c \in \mathcal{M}$ of claims.
- Examples:
  - Swaps with a "fixed leg": $p = (1, \dots, 1)$, random c.
  - In credit derivatives (CDS, CDO, ...) and other insurance contracts both p and c are random.
  - Traditionally in mathematical finance:

$$p = (1, 0, \dots, 0) \quad \text{and} \quad c = (0, \dots, 0, c_T).$$

- Claims and premiums live in the same space

$$\mathcal{M} = \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R})\}.$$

Swap contracts

- If we already have liabilities $\bar{c} \in \mathcal{M}$, then

$$\pi(\bar{c}, p; c) := \inf\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

  gives the least swap rate that would allow us to enter a swap contract without worsening our financial position.

- Similarly,

$$\pi^b(\bar{c}, p; c) := \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} - c + \alpha p) \leq \varphi(\bar{c})\} = -\pi(\bar{c}, p; -c)$$

  gives the greatest swap rate we would need on the opposite side of the trade.

- When $p = (1, 0, \ldots, 0)$ and $c = (0, \ldots, 0, c_T)$, we get a nonlinear version of the indifference price of [Hodges and Neuberger, 1989].

Swap contracts

The super- and subhedging swap rates,

$$\pi_{\sup}(c) = \inf\{\alpha \,|\, c-\alpha p \in \mathcal{C}^\infty\},\ \pi_{\inf}(c) = \sup\{\alpha \,|\, \alpha p-c \in \mathcal{C}^\infty\}.$$

In the classical model with $p = (1, 0, \dots, 0)$, we recover the usual super- and subhedging costs.

Theorem 9 *If $\pi(\bar{c}, p; 0) \geq 0$, then*

$$\pi_{\inf}(c) \leq \pi_b(\bar{c}, p; c) \leq \pi(\bar{c}, p; c) \leq \pi_{\sup}(c)$$

with equalities if $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$.

- Agents with identical views P, preferences $\mathcal{V}$ and financial position $\bar{c}$ have no reason to trade with each other.
- Prices are independent of such subjective factors when $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$.

Example 10 (Linear models) *When $S_t(x) = s_t \cdot x$ and $D_t = \mathbb{R}^J$, we have $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ if there is an $x \in \mathcal{N}_D$ such that $s_t \cdot \Delta x_t + c_t = \alpha p_t$. The converse holds under the no-arbitrage condition $\mathcal{C} \cap \mathcal{M}_+ = \{0\}$.*

Example 11 (The classical model) *When $D_t = \mathbb{R}^J$, $S_t(x) = x_0 + \tilde{s}_t \cdot \tilde{x}$ and $p = (1, 0, \ldots, 0)$, we have $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ if $\sum_{t=0}^T c_t$ is attainable in the sense that*

$$\sum_{t=0}^{T} c_t = \alpha + \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}$$

for some $\alpha \in \mathbb{R}$ and $x \in \mathcal{N}_D$. The converse holds under the no-arbitrage condition.

Existence of solutions

Given a market model (S, D), let

$$S_t^\infty(x,\omega) = \sup_{\alpha>0} \frac{S_t(\alpha x,\omega)}{\alpha} \quad \text{and} \quad D_t^\infty(\omega) = \bigcap_{\alpha>0} \alpha D_t(\omega).$$

If S is sublinear and D is conical, then $S^\infty = S$ and $D^\infty = D$

Theorem 12 *Assume that $\mathcal{V}_t(c_t) = Ev_t(c_t)$, where v_t are bounded from below. If the cone*

$$\mathcal{L} := \{x \in \mathcal{N}_{D^\infty} \mid S_t^\infty(\Delta x_t) \leq 0\}$$

is a linear space, then φ is proper and lower semicontinuous in L^0 and the infimum is attained for every $c \in \mathcal{M}$.

Example 13 *In the classical perfectly liquid market model*

$$\mathcal{L} = \{x \in \mathcal{N} \mid s_t \cdot \Delta x_t \leq 0,\ x_T = 0\},$$

so the linearity condition coincides with the no-arbitrage condition. When $v_t = \delta_{\mathbb{R}_-}$, we have $\varphi = \delta_{\mathcal{C}}$ so we recover the key lemma from [Schachermayer, 1992].

Example 14 *When $D \equiv \mathbb{R}^J$, the linearity condition becomes the robust no-arbitrage condition: there exists a positively homogeneous arbitrage-free cost process $\tilde{S}$ with*

$$\tilde{S}_t(x,\omega) \leq S_t^\infty(x,\omega) \quad \forall x \in \mathbb{R}^J,$$
$$\tilde{S}_t(x,\omega) < S_t^\infty(x,\omega) \quad \forall x \notin \operatorname{lin} S_t(\cdot,\omega);$$

see [Schachermayer, 2004].

Existence of solutions

The linearity condition may hold even under arbitrage.

Example 15 *If* $S_t^\infty(x,\omega) > 0$ *for* $x \notin \mathbb{R}^J_-$, *then* $\mathcal{L} = \{0\}$.

Example 16 *In [Çetin and Rogers, 2007] with*

$$S_t(x,\omega) = x^0 + s_t(\omega)\psi(x^1)$$

one has $S_t^\infty(x,\omega) = x^0 + s_t(\omega)\psi^\infty(x^1)$. *When* $\inf\psi' = 0$ *and* $\sup\psi' = \infty$ *we have* $\psi^\infty = \delta_{\mathbb{R}_-}$, *so the condition in Example 15 holds.*

Example 17 *If* $S_t(\cdot,\omega) = s_t(\omega)\cdot x$ *for a componentwise strictly positive price process* s *and* $D_t^\infty(\omega) \subseteq \mathbb{R}^J_+$ *(infinite short selling is prohibited), then* $\mathcal{L} = \{0\}$.

Proposition 18 *Assume that φ is proper and lower semicontinuous. Then, for every $\bar{c} \in \operatorname{dom}\varphi$ and $p \in \mathcal{M}$, the conditions*

- $\sup_{\alpha>0}\varphi(\alpha p) > \varphi(0)$,
- $\pi(\bar{c}, p; 0) > -\infty$,
- $\pi(\bar{c}, p; c) > -\infty$ *for all* $c \in \mathcal{M}$,

are equivalent and imply that $\pi(\bar{c}, p; \cdot)$ is proper and lower semicontinuous on $\mathcal{M}$ and that the infimum

$$\pi(\bar{c}, p; c) = \inf\{\alpha \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

is attained for every $c \in \mathcal{M}$.

Duality

- Let $\mathcal{M}^p = \{c \in \mathcal{M} \,|\, c_t \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R})\}$.
- The bilinear form

$$\langle c, y \rangle := E \sum_{t=0}^{T} c_t y_t$$

puts $\mathcal{M}^1$ and $\mathcal{M}^\infty$ in separating duality.

- The conjugate of a function f on $\mathcal{M}^1$ is defined by

$$f^*(y) = \sup_{c \in \mathcal{M}^1} \{\langle c, y \rangle - f(c)\}.$$

- If f is proper, convex and lower semicontinuous, then

$$f(y) = \sup_{y \in \mathcal{M}^\infty} \{\langle c, y \rangle - f^*(y)\}.$$

Duality

Lemma 19 *The conjugate of φ can be expressed in terms of the support function $\sigma_{\mathcal{C}}(y) = \sup_{c\in\mathcal{C}}\langle c, y\rangle$ of $\mathcal{C}$ as*

$$\varphi^*(y) = E\sum_{t=0}^{T} v_t^*(y_t) + \sigma_{\mathcal{C}}(y).$$

Theorem 20 *If φ is lower semicontinuous, we have*

$$\varphi(c) = \sup_{y\in\mathcal{M}^\infty}\left\{\langle c, y\rangle - \sigma_{\mathcal{C}}(y) - E\sum_{t=0}^{T} v_t^*(y_t)\right\}.$$

In particular, when $\mathcal{C}$ is a cone,

$$\varphi(c) = \sup_{y\in\mathcal{C}^*}\left\{\langle c, y\rangle - E\sum_{t=0}^{T} v_t^*(y_t)\right\},$$

where $\mathcal{C}^* := \{y \in \mathcal{M}^\infty \mid \langle c, y\rangle \le 0\ \forall c \in \mathcal{C}\cap\mathcal{M}^1\}$ is the polar cone of $\mathcal{C}$.

Duality

Lemma 21 *If $S_t(x,\cdot)$ are integrable, then for $y \in \mathcal{M}^\infty_+$,*

$$\sigma_{\mathcal{C}}(y) = \inf_{v\in\mathcal{N}^1}\left\{\sum_{t=0}^{T} E(y_tS_t)^*(v_t) + \sum_{t=0}^{T-1} E\sigma_{D_t}(E[\Delta v_{t+1}|\mathcal{F}_t])\right\},$$

while $\sigma_{\mathcal{C}^1}(y) = +\infty$ for $y \notin \mathcal{M}^\infty_+$. The infimum is attained.

Example 22 *If $S_t(\omega,x) = s_t(\omega)\cdot x$ and $D_t(\omega)$ is a cone,*

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty \,|\, E[\Delta(y_{t+1}s_{t+1})\,|\mathcal{F}_t] \in D_t^*\}.$$

Example 23 *If $S_t(\omega,x) = \sup\{s\cdot x \,|\, s \in [s_t^b(\omega), s_t^a(\omega)]\}$ and $D_t(\omega) = \mathbb{R}^J$, then*

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty \,|\, ys \text{ is a martingale for some } s \in [s^b, s^a]\}.$$

Example 24 *In the classical model, $\mathcal{C}^*$ consists of positive multiples of martingale densities.*

Duality

Theorem 25 *Let* $\bar{c} \in \mathcal{M}^1$, $\mathcal{A}(\bar{c}) = \{c \,|\, \varphi(\bar{c}+c) \le \varphi(\bar{c})\}$ *and assume that* φ *is proper and lower semicontinuous. Then*

1. $\sup_{\alpha>0} \varphi(\alpha p) > \varphi(0)$,
2. $\pi(\bar{c}, p; 0) > -\infty$,
3. $\pi(\bar{c}, p; c) > -\infty$ *for all* $c \in \mathcal{M}$,
4. $\langle p, y\rangle = 1$ *for some* $y \in \operatorname{dom} \sigma_{\mathcal{A}(\bar{c})}$

are equivalent and imply that

$$\pi(\bar{c}, p; c) = \sup_{y \in \mathcal{M}^\infty} \left\{ \langle c, y\rangle - \sigma_{\mathcal{A}(\bar{c})}(y) \,\middle|\, \langle p, y\rangle = 1 \right\}.$$

Moreover, if $\inf \varphi < \varphi(\bar{c})$, *then*

$$\sigma_{\mathcal{A}(\bar{c})} = \sigma_{\mathcal{B}(\bar{c})} + \sigma_{\mathcal{C}},$$

where $\mathcal{B}(\bar{c}) = \{c \in \mathcal{M}^1 \,|\, \mathcal{V}(\bar{c}+c) \le \varphi(\bar{c})\}$.

Duality

Example 26 *In the classical model, with* $p = (1, 0, \dots, 0)$ *and* $v_t = \delta_{\mathbb{R}_-}$ *for* $t < T$, *we get*

$$\begin{aligned}
\pi(\bar{c}, p; c) &= \sup_{y \in \mathcal{M}^\infty} \left\{ \langle c, y \rangle - \sigma_{\mathcal{A}(\bar{c})}(y) \,\middle|\, \langle p, y \rangle = 1 \right\} \\
&= \sup_{Q \in \mathcal{Q}} \left\{ E^Q \sum_{t=0}^{T} (\bar{c}_t + c_t) - \sigma_{\mathcal{B}(\bar{c})} \left(E_t \frac{dQ}{dP} \right) \right\} \\
&= \sup_{Q \in \mathcal{Q}} \sup_{\alpha > 0} E^Q \left\{ \sum_{t=0}^{T} (\bar{c}_t + c_t) - \alpha \left[v_T^*(\frac{dQ}{dP}/\alpha) - \varphi(\bar{c}) \right] \right\}
\end{aligned}$$

where $\mathcal{Q}$ is the set of absolutely continuous martingale measures; see [Biagini, Frittelli, Grasselli, 2011] for a continuous-time version.

Duality

Theorem 27 (FTAP) *Assume that S^∞ is finite-valued and that $D \equiv \mathbb{R}^J$. Then the following are equivalent*

1. *S satisfies the robust no-arbitrage condition.*
2. *There is a strictly consistent price system: adapted processes y and s such that $y > 0$, $s_t \in \operatorname{ri}\operatorname{dom} S_t^*$ and ys is a martingale.*

- In the classical linear market model, $\operatorname{ri}\operatorname{dom} S_t^* = \{1, \tilde{s}_t\}$ so the above reduces to the Dalang–Morton–Willinger theorem.
- Robust no-arbitrage condition means that there exists a sublinear arbitrage-free cost process $\tilde{S}$ with $\operatorname{dom} \tilde{S}_t^* \subseteq \operatorname{ri}\operatorname{dom} S_t^*$.

Computations

- Both reserving and indifference pricing come down to one-dimensional line search with the optimal value of the (ALM) problem:
  - $\pi^0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \le 0\}$,
  - $\pi(\bar{c}, p; c) = \inf\{\alpha \,|\, \varphi(\bar{c} + c - \alpha p) \le \varphi(\bar{c})\}$.
- This is easy provided we can evaluate the optimal value function

$$\varphi(c) = \inf_{x \in \mathcal{N}_D} \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t).$$

  for given $c \in \mathcal{M}$.
- In general, we cannot, but we can use the Galerkin method to approximate $\varphi(c)$ and to optimize trading strategies.

Computations

- If $\{x^i\}_{i\in I} \subset \mathcal{N}_D$ is a finite collection of feasible trading strategies, the Galerkin method optimizes over all convex combinations of $\{x^i\}_{i\in I}$.
- Such a problem can be written

$$\text{minimize} \quad \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta \sum_{i\in I} \alpha^i x_t^i) + c_t) \quad \text{over} \quad \alpha \in \Delta^I,$$

  where $\Delta^I := \{\alpha \in \mathbb{R}^I_+ \mid \sum_{i\in I} \alpha^i = 1\}$.
- This is a finite-dimensional convex optimization problem.
- When $\mathcal{V}_t = Ev_t$, for given disutility functions on $\mathbb{R}$, we get a stochastic optimization problem so we can apply
  - quadrature approximations,
  - stochastic approximation algorithms.

An example

- Our aim is to calculate the minimum reserve for a pension insurance portfolio.
- The yearly claims c_t consist of aggregate old age, disability and unemployment pension benefits earned by the end of 2008.
- The claims depend on mortality and the price- and wage-inflation, etc.
- We will apply the Galerkin method with 529 strategies obtained from
  - buy and hold,
  - fixed proportion,
  - constant proportion portfolio insurance

  by varying their parameters.

An example

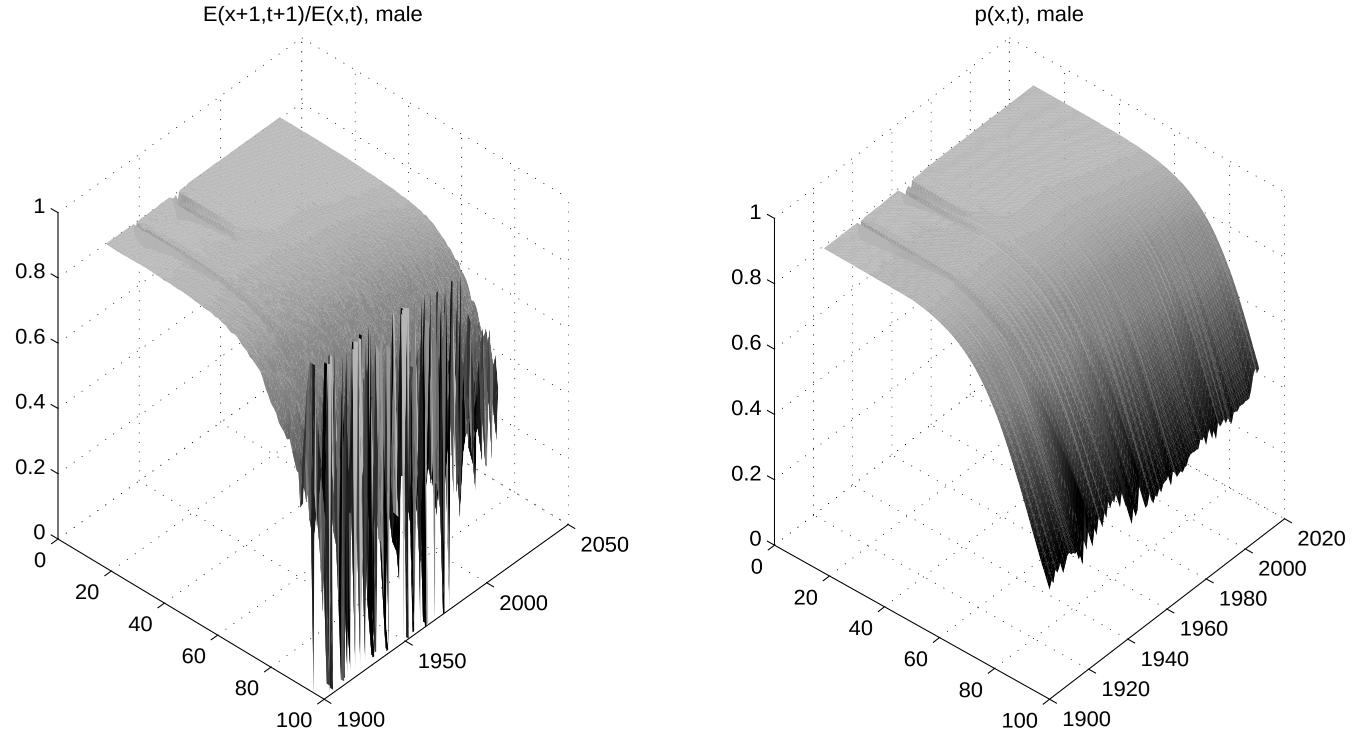
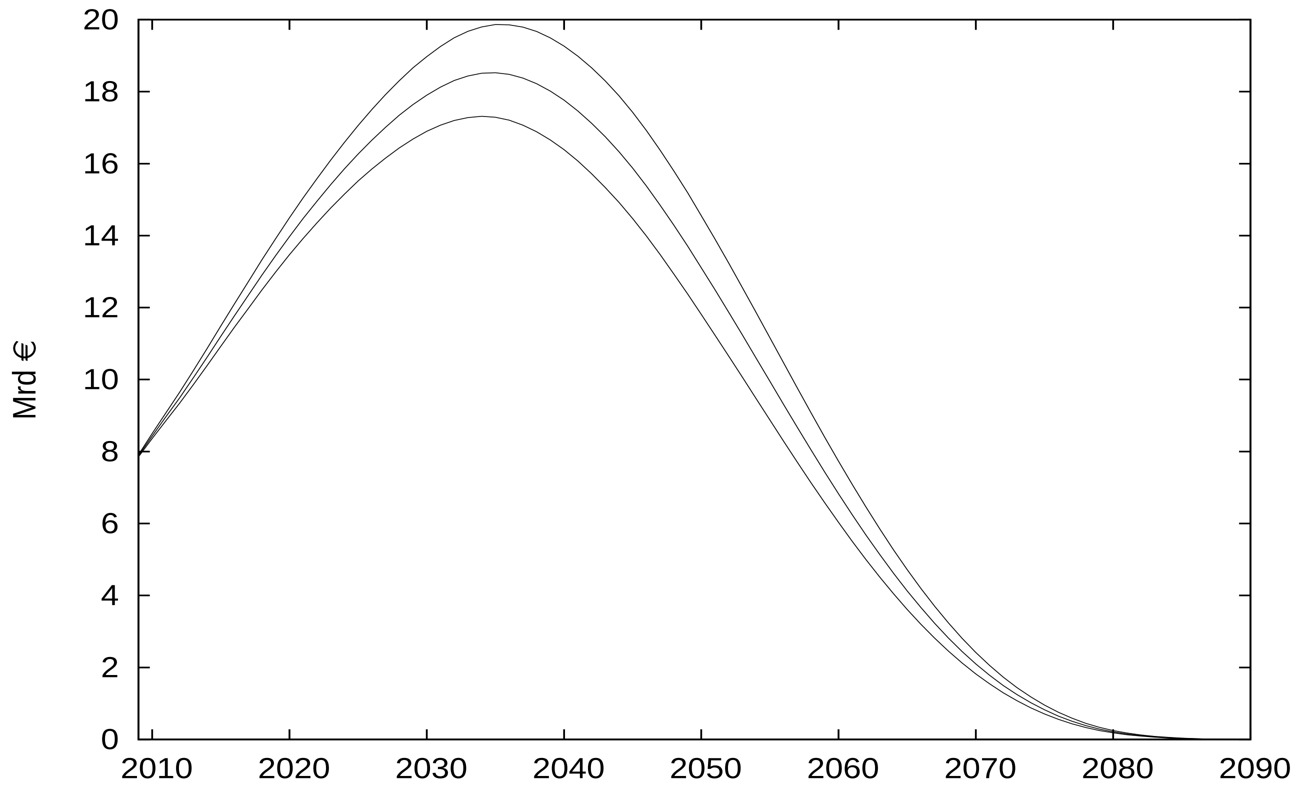


Figure 1: Survival rates of Finnish males

An example

Figure 2: Yearly claims



An example

- The traded assets consist of five equity indices and two bond indices.
- Yearly bond returns are modeled by

$$R_t = \exp(Y_t \Delta t - D \Delta Y_t),$$

  where Y is the yield to maturity and D the duration.
- Market risk factors are modeled together with the liability risk factors (mortality, price- and wage-inflation) by a stochastic difference equation of the form

$$\Delta \xi_t = A \xi_{t-1} + b + \varepsilon_t,$$

  where ξ is the vector of (transformed) risk factors.

An application

The first two columns give the nonzero Galerkin weights and the types of the active basis strategies. The last column gives the corresponding objective value.

Weight	Type	$CV@R_{97.5\%}$
0.665	BH	1569
0.029	BH	6567
0.104	BH	5041
0.022	CP	3324
0.039	PI	1420
0.099	PI	1907
0.042	PI	2417
	Best basis	1020
	Galerkin	251

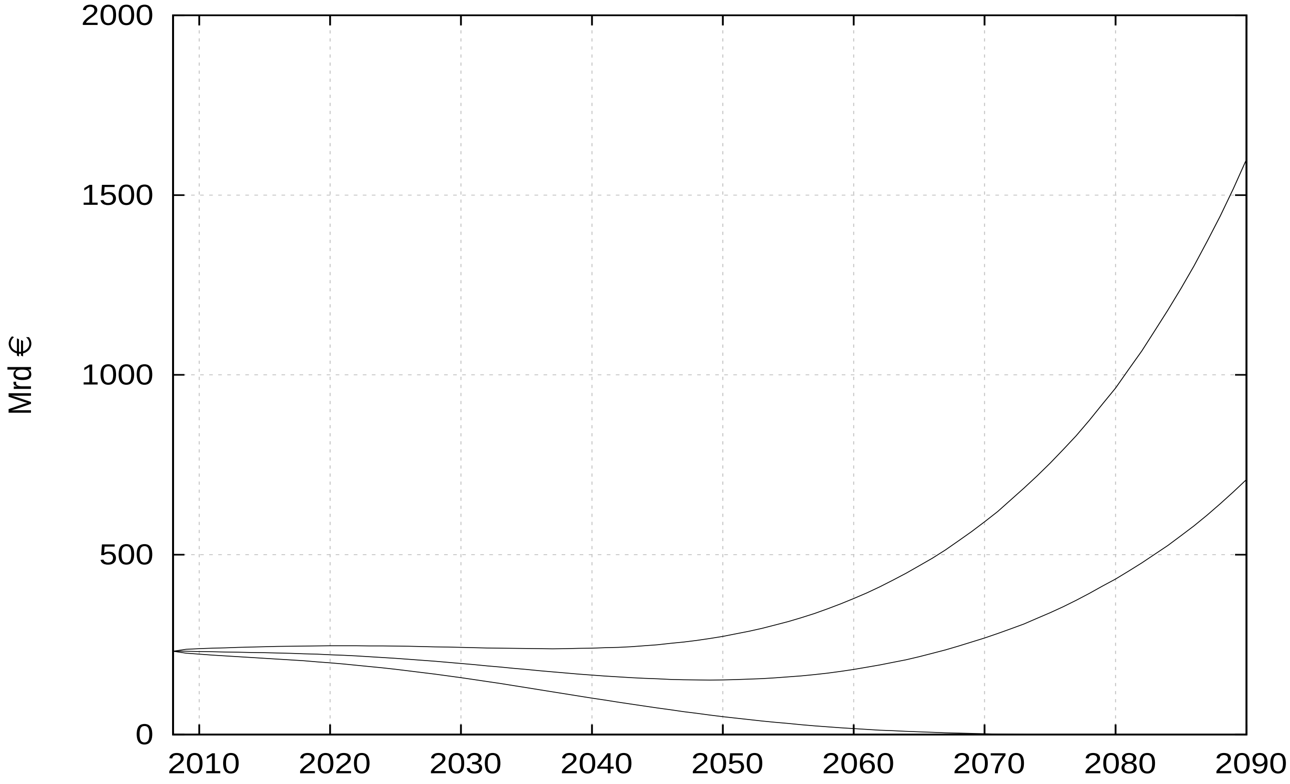
An example

	Confidence level				
	95%	90%	85%	80%	66%
Best basis	296	284	273	261	239
Optimized	288	271	254	236	202

Table 1: Reserves (10^9€) with varying risk tolerances

An example

Figure 3: The development of 34%, 50%- and 66%-quantiles of net wealth when $\pi^0(c)$ is defined with $\mathcal{V} = V@R_{66\%}$.



Summary

- Reservation prices and indifference swap rates/prices can be based on hedging arguments in asset-liability management.
- Financial contracts often involve sequences of cash-flows.
- In practice (in incomplete markets), adequacy of reserves and swap rates is subjective: they depend on views, risk preferences, trading expertise and the current financial position of the agent.
- Much of classical asset pricing theory can be extended to convex models of illiquid markets.
- The mathematics and computational techniques for hedging and pricing in illiquid markets combine techniques from stochastics and convex optimization.