Vol Target Options

Rob Coles

February 7, 2014
Outline

1. Introduction
2. Pricing Effects
   - Discretisation effect
   - Skew
   - Stochastic rates
3. Greeks
   - Vega
   - Theta
   - Delta & Gamma
   - Hedge P&L
   - Jump sensitivity
The Basic Idea

- Basket split between risky asset and cash
- Chose weight of risky asset $w$ to keep volatility of the basket constant.
  \[ w = \frac{\sigma_{\text{target}}}{\sigma_{\text{hist}}} \]
- Systematic differences between current and historical volatility prevent the basket behaving as true constant vol asset.
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Simple product

\[ I_{n+1} = I_n + \frac{w_n I_n}{S_n} (S_{n+1} - S_n) \]

\[ w_n = \frac{\sigma_{\text{target}}}{\sqrt{\frac{252}{m} \sum_{i=0}^{m-1} \log^2 \left( \frac{S_{n-i}}{S_{n-i-1}} \right)}} \]
The value of an option on a vol target is roughly the Black Scholes value with $\sigma^{eff} \approx \sigma^{target}$

$$V^{BS}(I, \sigma^{eff}, K, T)$$

Why derive closed form approximations to greeks and price?

- Improve Intuition.
- Hedge simulations
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Discretisation effect - Intuition

- Even in Black Scholes model $\sigma^{hist}$ is random.
  \[ \mathbb{E}[\sigma^{hist^2}] = \sigma^{BS^2} \]

- Convexity raises the effective volatility
  \[ \mathbb{E} \left[ \frac{1}{\sigma^{hist^2}} \right] = \left( \frac{m}{m - 2} \right) \frac{1}{\sigma^{BS^2}} \]
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Discretisation effect - Approximate result

- Business day volatility

\[ \sigma^{\text{target}} \rightarrow \sigma^{\text{target}} \sqrt{\frac{m}{m-2}} \]

- Calendar day vol. Proportion of variance that falls over weekend = \( \alpha \)

\[ \sigma^{\text{target}} \rightarrow \sigma^{\text{target}} \sqrt{\frac{m}{m-2 \left(5\alpha^2 + \frac{5}{4}(1-\alpha)^2\right)}} \]
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\]
Discretisation effect - Demonstration

Implied Vol discretisation premium 20% VT

- 261 day actual
- 365 day actual
- 261 day approx calc
- 365 day calc
Use a continuous time version of historical vol, over time period $\Delta$, to derive formulae without discretisation effect.

$$\sigma_{t}^{\text{hist}} = \sqrt{\frac{1}{\Delta} \int_{t-\Delta}^{t} \sigma_u^2 du}$$

Historical volatility averaging period

$$\Delta \approx \frac{m}{252}$$
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As underlying $S$ drifts down its volatility tends to increase.

- Historical volatility takes time to catch up.
- Risky weight $w$ too high and $\sigma^{\text{eff}}$ of the index is larger than $\sigma^{\text{target}}$ for paths where the index drops.
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Risky weight $w$ too high and $\sigma^{\text{eff}}$ of the index is larger than $\sigma^{\text{target}}$ for paths where the index drops.
Define skew to be the change in implied volatility for a 10% move in underlying. Assume constant skew in implied vol surface of $S$.

$$Skew \rightarrow Skew \frac{\Delta}{T}$$
Skew - Demonstration.

**Implied skew 30 day 20% Vol Target**

- **1y predicted**
- **6m predicted**
- **3m predicted**
- **1y measured**
- **6m measured**
- **3m measured**

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Skew - Demonstration.

3m Implied skew of 30 day Vol Target

- Underlying
- 20% VT
- 15% VT
- 10% VT

strike ratio

Implied Vol

70% 80% 90% 100% 110% 120% 130%

25% 20% 15% 10% 5% 0%
Skew simulations - Morrison, Tadrowski

Historical volatility from exponentially weighted mean. Equivalent to 0.8 year unweighted mean.

- Calibrate jump model to 1.5% skew on sx5e at 2 years.
- Effective Vol skew around 0.53%
- ATM vol bump 1.65%
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Skew simulations - Nelte, Roche

Calibrate jump model to skew on S&P500 at 3 years.

- $\sigma^{\text{eff}}$ skew under local vol $\approx \sigma^{\text{eff}}$ skew under local vol plus jumps.
- ATM vol bump $\approx 1.8\%$
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Risky and riskless assets grow at short rate following a Hull White process with normal volatility $\sigma_{\text{short}}$. Negligible speed of mean reversion.

$$
\sigma^{\text{eff}} \approx \sigma^{\text{target}} \sqrt{1 + \rho \frac{\sigma_{\text{short}}}{\sigma^{\text{target}}} T + \left(\frac{\sigma_{\text{short}}}{\sigma^{\text{target}}}\right)^2 \frac{T^2}{3}}
$$
Stochastic Rates - Approximate result

Stochastic Rate Effect
10% VT 50bp Hull White vol
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Volatility of underlying rises from $\sigma$ to $\sigma + \delta \sigma$.

$$V^{BS}(I, \sigma^{eff}(\delta \sigma), K, T - \epsilon)$$

- While historical volatility is too low the leverage is to high and the Index evolves with $\sigma^{eff}$ higher than $\sigma^{target}$
Vega - Approximate result

For an option maturing after $T$ years.

$$\sigma^{\text{eff}} \approx \sigma^{\text{target}} \sqrt{1 + \frac{\delta \sigma \Delta}{\sigma T}}$$

$$\text{Vega}^{BS} \rightarrow \text{Vega}^{BS} \sigma^{\text{target}} \frac{\Delta}{\sigma \left(2T\right)}$$
Vega - Demonstration.

Vega 30 day 20% vol target

- Red diamonds: measured 10% vol
- Green triangles: measured 20% vol
- Red line: predicted 10% vol
- Blue line: predicted 20% vol

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Theta - Intuition

Underlying remains constant for period $\epsilon$.

$$V^{BS}(I, \sigma^{\text{eff}}(\epsilon), K, T - \epsilon)$$

- Time decay of option will decrease its value.
- Period of zero volatility decreases $\sigma^{\text{hist}}$ so $\sigma^{\text{eff}}$ increases.
Theta - Intuition

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$$V^{BS}(I, \sigma^{\text{eff}}(\epsilon), K, T - \epsilon)$$

- Time decay of option will decrease its value.
- Period of zero volatility decreases $\sigma^{\text{hist}}$ so $\sigma^{\text{eff}}$ increases.
Theta - Approximate result

\[ \sigma_{\text{eff}} (\epsilon) = \sigma_{\text{target}} \sqrt{1 - \frac{\Delta}{T - \epsilon} \log \left(1 - \frac{\epsilon}{\Delta}\right)} \]

\[ \frac{\partial \sigma_{\text{eff}}}{\partial \epsilon} = \frac{\sigma_{\text{target}}}{2T} \]

\[ \theta^{BS} \rightarrow \theta^{BS} + \text{Vega}^{BS} \frac{\partial \sigma_{\text{eff}}}{\partial \epsilon} \]
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Delta - Intuition

Underlying moves from level at last fixing $S_0$ to $S$

$$V^{BS}(I(S), \sigma^{eff}(S), K, T)$$

- The value of $I$ changes in proportion to current weight.
- $\sigma^{eff}$ changes through addition of $\log^2 \left( \frac{S}{S_0} \right)$ term to $\sigma^{hist^2}\Delta$
Delta - Intuition

Underlying moves from level at last fixing $S_0$ to $S$

$$V^{BS}(I(S), \sigma^{eff}(S), K, T)$$

- The value of $I$ changes in proportion to current weight.
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Effective volatility $\sigma^{\text{eff}}$ is even in log$(S)$.

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial \sigma^{\text{target}}} \frac{\sigma^{\text{target}}}{\sigma}$$
Gamma - Result

Gamma convergence 6m 10% 30day VT v 10% call

- Red: Gamma VT
- Blue: Gamma vanilla

Bump size vs Gamma

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Gamma - Result

6m 10% 30 day VT v 10% vanilla

- MonteCarlo VT
- Hedged vanilla
- Hedged VT

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Jump sensitivity - How big is a jump?

Gamma P&L hedged 1y atm 15% vol target
As $\Delta$ increases

- **Fair Value** decreases because less discretisation effect.
- **Vega** increases because takes time for historical volatility to change.
- **Skew** increases because historical vol contains values from today.
- **Jump sensitivity** decreases.
As $\Delta$ increases

- **Fair Value** decreases because less discretisation effect.
- **Vega** increases because it takes time for historical volatility to change.
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As $\Delta$ increases

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Summary - $\sigma^{hist}$ measurement period

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Summary - Largest pricing effects

- Jumps
- Stochastic rates
- Discretisation
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Summary - Largest pricing effects

- Jumps
- Stochastic rates
- Discretisation
Some simulation studies I

Steven Morrison, Laura Tadrowski
Guarantees and target volatility funds.
*Barrie and Hibbert research paper* Sept 2013.

Maximilian Nelte, Peter Roche
Controlling volatility to reduce uncertainty.