REAL TIME COUNTERPARTY CREDIT RISK MANAGEMENT WITH ADJOINT ALGORITHMIC DIFFERENTIATION (AAD)

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Outline

- The Problem:
  Real Time Counterparty Credit Risk Management

- The Solution:
  Adjoint Algorithmic Differentiation (AAD)
  - Pathwise Derivative Method
  - Algebraic Adjoint Approaches
  - Adjoint Algorithmic Differentiation (AAD)
  - AAD and the Pathwise Derivative Method: Adjoints made easy

- The Benefits
  - All the risk you want in 4x the cost of computing the portfolio
  - 100x speed ups

- Conclusions
Real Time Counterparty Risk Management in Monte Carlo

What is hard about it:

- **CVA/DVA are exotic derivative-like exposures**
  - Non Linear and Path Dependent Problem

- **Portfolio Problem: extreme high dimensionality**
  - All the trades facing a given counterparty that can be netted against each other (netting sets) need to be valued simultaneously.
  - Netting sets typically include trades from different asset classes with hundreds of Risk Factors.

- **Monte Carlo Problem**
  - Computing risk by means of traditional methods is challenging to do even overnight.
  - Pre Trade assessment of CVA/DVA is vital to run businesses efficiently
The Solution: Adjoint Algorithmic Differentiation (AAD)

- AAD is the only numerical technique that allows us to cope with the huge amount of sensitivities that arise in portfolio problems.

- All sensitivities are available at little more cost than computing the portfolio PV: a nearly impossible Risk calculation task become achievable.

- 2012: AAD goes mainstream:

  See e.g.

Setting up a CCRM engine:

- **Usual time discretization**:
  
  \[
  V_{\text{CVA}} \simeq \sum_{i=1}^{N_O} \mathbb{E} \left[ \mathbb{I}(T_{i-1} < \tau_c \leq T_i) D(0, T_i) \right. \\
  \times \left. L_{\text{GD}}(T_i) \left( NPV(T_i) - C \left( R(T_i^-) \right) \right)^+ \right]
  \]

- **Problem formulation**:
  
  \[
  V = \mathbb{E}_Q \left[ P(R, X) \right]
  \]

- **Payout**:
  
  \[
  P = \sum_{i=1}^{N_O} P \left( T_i, R(T_i), X(T_i) \right)
  \]
Quantitative Strategies

§ Monte Carlo Expectation Values

\[ V(\theta) = \mathbb{E}_Q \left[ P(X(T_1), \ldots, X(T_M)) \right] \]

... and sensitivities

\[ \frac{\partial V(\theta)}{\partial \theta_k} = \mathbb{E}_Q \left[ \frac{\partial P(X(\theta))}{\partial \theta_k} \right] \]

§ Pathwise Derivative Estimator

\[ \bar{\theta}_k \equiv \frac{\partial P(X(\theta))}{\partial \theta_k} = \sum_{j=1}^{N \times M} \frac{\partial P(X)}{\partial X_j} \times \frac{\partial X_j(\theta)}{\partial \theta_k} \]

Payout Derivatives

Tangent Process

Lipschitz

Chain Rule

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Pathwise Derivative Method: Challenges

\[ \bar{\theta}_k \equiv \frac{\partial P(X(\theta))}{\partial \theta_k} = \sum_{j=1}^{N \times M} \frac{\partial P(X)}{\partial X_j} \times \frac{\partial X_j(\theta)}{\partial \theta_k} \]

Since the variance of the estimator is comparable to the one of finite differences, all this is worth the hassle if the resulting computational time is significantly lower than the one of Bumping.

We need an efficient way to calculate:

1. Simulation of the Tangent Process
2. Derivatives of the Payout
“Algebraic” Adjoint Methods

Giles and Glasserman’s `Smoking Adjoints’, Risk Magazine 2006
Leclerc et al., Risk Magazine 2009
Joshi et al., several preprints

Libor Market Model & Swaptions
Concentrate on the efficient Simulation of the Tangent Process

In a nutshell:

1. Formulate the propagation of the Tangent process in terms of Linear Algebra Ops
2. Optimize the computation time by rearranging the order of the computations
3. Implement the rearranged sequence of operations
Algebraic Adjoint Methods: Libor Market Model

Log Euler scheme:

\[
\frac{L_i(n+1)}{L_i(n)} = \exp \left[ (\mu_i(L(n)) - \|\sigma_i(n)\|^2/2) h_e + \sigma_i^T(n)Z(n+1)\sqrt{h_e} \right]
\]

\[
\mu_i(L(t)) = \sum_{j=\eta(t)}^{i} \frac{\sigma_i^T \sigma_j hL_j(t)}{1 + hL_j(t)}
\]

Delta tangent process:

\[
\Delta_{i,k}(t) = \frac{\partial L_i(t)}{\partial L_k(0)}
\]

\[
\Delta_{i,k}(n+1) = \Delta_{i,k}(n) \frac{L_i(n+1)}{L_i(n)} + L_i(n+1) \sum_{j=1}^{N} \frac{\partial \mu_i(n)}{\partial L_j(n)} \Delta_{j,k}(n) h
\]

Matrix Recursion:

\[
\Delta(n+1) = D(n)\Delta(n) \quad \Delta(0) = I
\]

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Algebraic Adjoint Methods

\[ \bar{\theta} = \frac{\partial P(L(N))}{\partial L(N)}^T \Delta(N) \]

Tangent Process

Matrix Recursion

\[ \Delta(n + 1) = D(n)\Delta(n) \quad \Delta(0) = I \]

Matrix Matrix Forward Recursion

\[ O(N^3) \]

\[ \bar{\theta} = \frac{\partial P(L(N))}{\partial L(N)}^T D(N - 1) \ldots D(0)\Delta(0) \]

Matrix Vector Backward Recursion

\[ O(N^2) \]

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Algebraic Adjoint Methods

Arbitrary number of sensitivities at a **fixed small cost**

Giles and Glasserman, Risk Magazine 2006

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Limitations of Algebraic Adjoint Methods

- LMM is bit of an ad-hoc application ...
  - Difficult to generalize to Path Dependent Options
  - The required Algebraic Analysis is in general cumbersome
  - Not general enough for all the applications in Finance
  - The derivatives required are often not available in closed form
  - What about the derivatives of the Payout?
Algorithmic Adjoint Approaches: AAD

- Adjoint implementations can be seen as instances of a programming technique known as Adjoint Algorithmic Differentiation (AAD).

- In general, AAD allows the calculation of the gradient of an algorithm at a cost that is a small constant (~4) times the cost of evaluating the function itself, independent of the number of input variables.

The Payoff estimator is a mapping of the form:

$$\theta \rightarrow P(X(\theta))$$

AAD gives all the Risk estimators for a small fixed cost:

$$\bar{\theta}_k \equiv \frac{\partial P(X(\theta))}{\partial \theta_k}$$
How does AAD work anyway?

$$Y = \text{FUNCTION}(X)$$

$$X \rightarrow \ldots \rightarrow U \rightarrow V \rightarrow \ldots \rightarrow Y$$

Adjoints

$$\bar{V}_k = \sum_{j=1}^{m} \bar{Y}_j \frac{\partial Y_j}{\partial V_k}$$

Propagation Rule

$$\bar{U}_i = \sum_k \bar{V}_k \frac{\partial V_k}{\partial U_i}$$

$$\bar{X} \leftarrow \ldots \leftarrow \bar{U} \leftarrow \bar{V} \leftarrow \ldots \leftarrow \bar{Y}$$

$$\bar{X} = \text{FUNCTION}_B(X, \bar{Y})$$

Main Result

$$\bar{X}_i = \sum_{j=1}^{m} \bar{Y}_j \frac{\partial Y_j}{\partial X_i}$$

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Computational Cost:

Main Result of AAD:

\[ Y = \text{FUNCTION}(X) \]

\[ \bar{X} = \text{FUNCTION}_B(X, \bar{Y}) \]

\[ \bar{X}_i = \sum_{j=1}^{m} \bar{Y}_j \frac{\partial Y_j}{\partial X_i} \]

\[ \frac{\text{Cost}[\text{FUNCTION}_B]}{\text{Cost}[\text{FUNCTION}]} \leq \omega_A \]

\[ \omega_A \sim 4 \]
Simple Example:

\[
(P) = \text{payout} (r, X[N]) \{ \\
B = 0.0; \\
\text{for} (i = 1 \text{ to } N) \\
\quad B += w[i] \times X[i]; \\
x = B - K; \\
D = \exp(-r \times T); \\
P = D \times \max(x, 0.0); \\
\}
\]

\[
B = \sum_{i=1}^{N} w_i X_i \\
x = B - K \\
D = \exp(-rT) \\
P = D \max(x, 0)
\]

\[
P = \exp(-rT) \max \left( \sum_{i=1}^{N} w_i X_i - K, 0 \right)
\]

\[
(P, r_b, X_b[N]) = \text{payout}_b(r, X[N], P_b)\{

    B = 0.0;
    \text{for (} i = 0 \text{ to } N \text{)}
    \quad B += w[i] \times X[i];

    x = B - K;
    D = \exp(-r \times T);
    P = D \times \max(x, 0.0);

    D_b = \max(x, 0.0) \times P_b;

    x_b = 0.0;
    \text{if (} x > 0 \text{)}
    \quad x_b = D \times P_b;

    r_b = -D \times T \times D_b;
    B_b = x_b;

    \text{for (} i = 0 \text{ to } N \text{)}
    \quad X_b[i] = w[i] \times B_b;
\}

// Forward sweep
\[
B = \sum_{i=1}^{N} w_i X_i
\]
\[
x = B - K
\]
\[
D = \exp(-r T)
\]
\[
P = D \max(x, 0)
\]

// Backward sweep
\[
\bar{D} = \bar{P} \partial P / \partial D = \bar{P} \max(x, 0)
\]
\[
\bar{x} = \bar{P} \partial P / \partial x = \bar{P} D \theta(x)
\]
\[
\bar{r} = \bar{D} \partial D / \partial r = \bar{D}(-D T)
\]
\[
\bar{B} = \bar{x} \partial x / \partial B = \bar{x}
\]
\[
\bar{X}_i = \bar{B} \partial B / \partial X_i = \bar{B} w_i
\]

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
AAD as a Design Paradigm

- AAD can be used as a **design paradigm** even for large inhomogeneous algorithms

- Addresses both aspects of the implementation of the Pathwise Derivative Method

\[
\frac{\partial P(X)}{\partial X_j} \quad \frac{\partial X_j(\theta)}{\partial \theta_k}
\]


\[
\bar{\theta}_k \equiv \frac{\partial P(X(\theta))}{\partial \theta_k} = \sum_{j=1}^{N \times M} \frac{\partial P(X)}{\partial X_j} \times \frac{\partial X_j(\theta)}{\partial \theta_k}
\]

- Linear combination of the rows of the Jacobian

- All the Greeks at a cost that is a small (~4) multiple of the PV estimator

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Diffusive Setting

\[ P_0(X) \]

\[ \theta \]

\[ X(T_1) \ldots X(T_M) \]

\[ X(t_{N_s}) \]

\[ \theta \]

\[ \{X(t_m)\}_{m \leq N_s - 1} \]

\[ \ldots \]

\[ X(t_2) \]

\[ \theta \]

\[ \text{PROP}_{N_s - 1} \]

\[ \theta \]

\[ X(t_1) \]

\[ \theta \]

\[ \text{PROP}_1 \]

\[ \theta \]

\[ \text{PROP}_C \]

\[ \theta \]

\[ \text{Forward} \]

\[ \theta \]

\[ \text{Backward} \]
Lognormal Example

\[ X(t_{n+1}) = \text{PROP}_n(X(t_n), \theta) \]

- **Step 1**
  \[ \mu = rX(t_n) \]

- **Step 2**
  \[ \Sigma = \sigma X(t_n) \]

- **Step 3**
  \[ X(t_{n+1}) = X(t_n) + \mu \Delta t + \Sigma \sqrt{\Delta t} \]

\[ (\bar{X}(t_n), \bar{\theta}) = \text{PROP}_n(., \bar{X}(t_{n+1})) \]

- **Step \bar{1}**
  \[ \bar{X}(t_n) + = \bar{\mu} r \quad \bar{\theta}_r + = \bar{\mu} X(t_n) \]

- **Step \bar{2}**
  \[ \bar{X}(t_n) + = \bar{\Sigma} \sigma \quad \bar{\theta}_\sigma + = \bar{\Sigma} X(t_n) \]

- **Step \bar{3}**
  \[ \bar{\mu} = \bar{X}(t_{n+1}) \Delta t \]
  \[ \bar{\Sigma} = \bar{X}(t_{n+1}) \sqrt{\Delta t} Z \]
  \[ \bar{X}(t_n) = \bar{X}(t_{n+1}) \]

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Best of Asian Option

- Full Delta and Vega calculation for just twice the cost to calculate the PV

L.C. & Mike Giles, Risk (2012)
Back to the LMM test ground

- Full Delta and Vega calculation for just twice the cost to calculate the PV
- Similar results for both Euler and predictor corrector discretizations
- 100x savings in typical applications

L.C. & Mike Giles, Risk (2012)

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Real Time Counterparty Risk Management in Monte Carlo

- **CVA Problem:**

  \[
  V_{CVA} = \mathbb{E}\left[ \mathbb{I}(\tau_c \leq T) D(0, \tau_c) \times \frac{L_{GD}(\tau_c)}{\tau_c} \right. \\
  \left. \left( \text{NPV}(\tau_c) - C(R(\tau_c^-)) \right)^+ \right]
  \]

- Risk manage CVA/DVA is challenging because all the trades facing the same counterparty must be valued at the same time, typically with Monte Carlo.

- AAD is naturally suited for this task.

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Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
A new challenge: Rating Dependent Payoffs

\[ P(T_i, R(T_i), X(T_i)) = \sum_{r=0}^{N_R} \tilde{P}_i(X(T_i); r) \delta_{r, R(T_i)} \]

Rating Transition Markov Chain model (Jarrow, Lando and Turnbull ‘97)

\[ R(T_i) = \sum_{r=1}^{N_R} \mathbb{I}(\tilde{Z}_i^R > Q(T_i, r)) \]

The Rating state space is discrete (hence the Payoff is non Lipschitz)

The Pathwise Derivative method gives only part of the Risk
Real Time Counterparty Risk Management in Monte Carlo

- Payout rewrite:

\[ P(T_i, R(T_i), X(T_i)) = \sum_{r=0}^{N_R} \tilde{P}_i(X(T_i); r) \delta_{r,R(T_i)} \]

\[ P(T_i, \tilde{Z}_i^R, X(T_i)) = \tilde{P}_i(X(T_i); 0) + \sum_{r=1}^{N_R} \left( \tilde{P}_i(X(T_i); r) - \tilde{P}_i(X(T_i); r-1) \right) I \left( \tilde{Z}_i^R > Q(T_i, r; \theta) \right) \]

**Singular Contribution:**

\[ \partial_{\theta_k} P(T_i, \tilde{Z}_i, X(T_i)) = - \sum_{r=1}^{N_R} \left( \tilde{P}_i(X(T_i); r) - \tilde{P}_i(X(T_i); r-1) \right) \delta \left( \tilde{Z}_i^R = Q(T_i, r; \theta) \right) \partial_{\theta_k} Q(T_i, r; \theta) \]

This cannot be sampled by MC
Real Time Counterparty Risk Management in Monte Carlo

Singular Contribution:

\[ \partial_{\theta_k} P(T_i, \tilde{Z}_i, X(T_i)) = - \sum_{r=1}^{N_R} \left( \tilde{P}_i(\tilde{Z}_i; r) - \tilde{P}_i(\tilde{Z}_i; r - 1) \right) \delta \left( \tilde{Z}_i^R = Q(T_i, r; \theta) \right) \partial_{\theta_k} Q(T_i, r; \theta) \]

Can be integrated out analytically

\[ \bar{\partial}_k = - \sum_{r=1}^{N_R} \frac{\phi(Z^*, Z_i^X, \rho_i)}{\sqrt{i} \phi(Z_i^X, \rho_i^X)} \partial_{\theta_k} Q(T_i, r; \theta) \times \left( \tilde{P}_i(X(T_i); r) - \tilde{P}_i(X(T_i); r - 1) \right) \]

Variance Reduction vs. Bumping:

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>VR[Q(1,1)]</th>
<th>VR[Q(1,2)]</th>
<th>VR[Q(1,3)]</th>
</tr>
</thead>
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<tr>
<td>0.1</td>
<td>24</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
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<td>245</td>
<td>165</td>
<td>125</td>
</tr>
<tr>
<td>0.001</td>
<td>2490</td>
<td>1640</td>
<td>1350</td>
</tr>
</tbody>
</table>

Real time counterparty credit risk management with Adjoint Algorithmic Differentiation (AAD)
Real Time Counterparty Risk Management in Monte Carlo

- Calculation of risk for the CVA of a portfolio of commodity swaps

\[
\frac{dF_T(t)}{F_T(t)} = \sigma_T \exp(-\beta(T - t))dW_t
\]

1-factor lognormal model of the Futures curve

Conditional value of the commodity swap:

\[
NPV(t) = \sum_{j=1}^{N_e} D(t, t_j) \left( F_{t_j}(t) - K \right)
\]
Real Time Counterparty Risk Management in Monte Carlo

- **Forward Propagation:**

  \[ F_T(T_i) = F_T(T_{i-1}) \exp \left( \sigma_i \sqrt{\Delta T_i} Z - \frac{1}{2} \sigma_i^2 \Delta T_i \right) \]

  \[ \sigma_i^2 = \frac{\sigma_T^2}{2\beta \Delta T_i} e^{-2\beta T} \left( e^{2\beta T_i} - e^{2\beta T_{i-1}} \right) \]

- **Adjoint Propagation:**

  \[ \bar{F}_T(T_{i-1}) = \bar{F}_T(T_i) \exp \left( \sigma_i \sqrt{\Delta T_i} Z - \frac{1}{2} \sigma_i^2 \Delta T_i \right) \]

  \[ \bar{\sigma}_i = \bar{F}_T(T_i) F(T_i) \left( \sqrt{\Delta T_i} Z - \sigma_i \Delta T_i \right) \]

  \[ \bar{\sigma}_T = \frac{\bar{\sigma}_i}{\sqrt{2\beta \Delta T_i}} \sqrt{e^{-2\beta T} \left( e^{2\beta T_i} - e^{2\beta T_{i-1}} \right)} \]
Real Time Counterparty Risk Management in Monte Carlo

- Test Application: Calculation of risk for the CVA of a portfolio of 5 commodity swaps over a 5 years horizon (over 600 risks)

- Bumping: \(~ 1h 40\) min
- AAD: \(~ 10\) sec

L.C., J. Lee and M. Peacock, Risk (2011)
Conclusions

▪ Algebraic Adjoint approaches can be seen as specific instances of a more general paradigm: Adjoint Algorithmic Differentiation (AAD)

▪ AAD can be employed to evaluate efficiently option sensitivities for virtually any model and financial security encountered in practice

▪ AAD allows the calculation of the Greeks in at most 4 times the time necessary for the calculation of the P&L of the portfolio

▪ Risk is calculated orders of magnitude faster than standard bumping, thus producing a significant reduction in infrastructure costs, and allowing “real time” monitoring of Risk and more effective hedging strategies

▪ Real Time CCRM:
  ▪ Analytical Integration of the Rating Singular Contribution
  ▪ Additional significant Speed Up coming from Variance Reduction
References


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