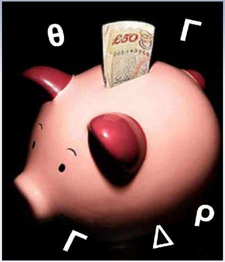
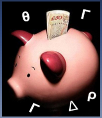


HPCFinance: New Thinking in Finance

Calculating Variable Annuity Liability ‘Greeks’ Using Monte Carlo Simulation

Dr. Mark Cathcart, Standard Life

February 14, 2014



Outline of Presentation

Overview of Variable Annuity contracts

Variable Annuities

- What are Variable Annuity (VA) Products?
- Types of VA
  - GMDB,GMAB,GMIB,GMWB
- Nature of Liabilities of Guarantee
  - complex
  - multi-dimensional
  - path-dependent

- Even valuing the liabilities is fairly involved - simulation. Can we construct more sophisticated estimators of the VA Greeks to aid insurer's hedging strategies?

Estimating ‘Greeks’ by MC simulation

- **There are three main approaches...**

1. Bump and Revalue

2. Pathwise Approach

3. Likelihood Ratio Method

Estimating ‘Greeks’ by MC simulation

1. Bump and Revalue

- Idea is to simulate for the price of the option under some base scenarios and then again under some ‘bumped’ scenarios
- e.g., delta: perturb the initial stock price by $\Delta S(0)$:

$$\frac{C(S(0)+\Delta S(0), K, \sigma, r, T) - C(S(0), K, \sigma, r, T)}{\Delta S(0)}$$

- i.e., a Forward Difference estimate

Common Random Numbers

Good practice to use same random number generator with same initial seed in base and bumped scenarios

Estimating ‘Greeks’ by MC simulation

2. Pathwise Approach

- Let $\alpha(\theta) = \mathbb{E}[Y(\theta)]$ then we can estimate the derivative of $\alpha(\theta)$ using:

$$Y'(\theta) = \lim_{h \to 0} \frac{Y(\theta + h) - Y(\theta)}{h}$$

- This estimator has expectation $\mathbb{E}[Y'(\theta)]$ and is an **unbiased** estimator of $\alpha'(\theta)$ if interchanging differentiation and taking expectations is justified, i.e.,

$$\boxed{\mathbb{E}\left[\frac{d}{d\theta} Y(\theta)\right] = \frac{d}{d\theta}\mathbb{E}[Y(\theta)]}$$

Estimating ‘Greeks’ by MC simulation

2. Pathwise Approach

Let us consider a simple, illustrative example...

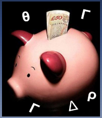
European option under B-S model: Delta

$$Y = e^{-rT}\max(S(T)-K,0) \quad \textbf{Payoff}$$

$$S(T) = S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} \quad \textbf{Dynamics}$$

Applying the chain rule gives...

$$\frac{dY}{dS(0)} = \frac{dY}{dS(T)}\frac{dS(T)}{dS(0)}$$



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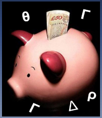

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Applying the chain rule gives...

$$
\frac{dY}{dS(0)} = e^{-rT}\mathbb{I}\{S(T)>K\}\frac{dS(T)}{dS(0)}
$$

Estimating ‘Greeks’ by MC simulation

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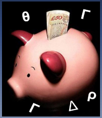

Estimating ‘Greeks’ by MC simulation

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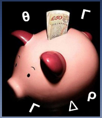

Estimating ‘Greeks’ by MC simulation

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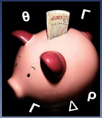
$$

Applying the chain rule gives...

$$\frac{dY}{dS(0)} = e^{-rT}\mathbb{I}\{S(T)>K\}\frac{S(T)}{S(0)}$$



Estimating 'Greeks' by MC simulation

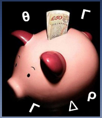
2. Pathwise Approach

Let us consider a simple, illustrative example...

European option under B-S model: Delta

Thus, the Pathwise Estimator for a European call option (under the B-S model underlying dynamics) is given as:

$$\frac{dY}{dS(0)} = e^{-rT}\mathbb{I}\{S(T) > K\}\frac{S(T)}{S(0)}$$

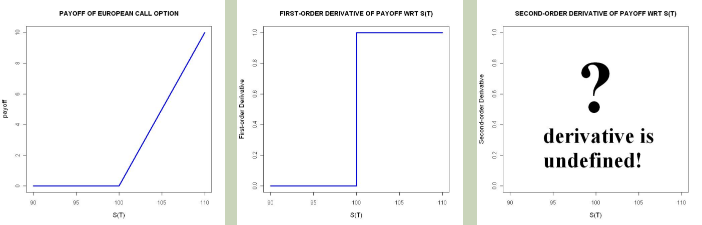


Estimating ‘Greeks’ by MC simulation

2. Pathwise Approach

What about estimating the **Gamma** option ‘Greek’?

The Pathwise is inapplicable in this case!



$\searrow \frac{d}{dS(T)} \nearrow$ $\searrow \frac{d}{dS(T)} \nearrow$

Estimating ‘Greeks’ by MC simulation

2. Pathwise Approach

What about estimating the **Gamma** option ‘Greek’?

The Pathwise is inapplicable in this case!

More formally, the requirement for using the PW method

$$\boxed{\mathbb{E}\left[\frac{d}{d\theta}Y(\theta)\right] = \frac{d}{d\theta}\mathbb{E}[Y(\theta)]}$$

is not satisfied in this instance. Indeed,

$$0 = \mathbb{E}\left[\frac{dY}{dS(0)}\right] \neq \frac{d}{dS(0)}\mathbb{E}[Y]$$

Estimating 'Greeks' by MC simulation

3. Likelihood Ratio Method

- Relies on differentiating prob.density rather than payoff funct., thus does not require smoothness in the payoff function
- Suppose we have a discounted payoff Y expressed as a function $f(\mathbf{X})$, where $\mathbf{X}$ is a m-dimensional vector of different asset prices (or alternatively, one asset price at multiple valuation dates).
- Then assuming that $\mathbf{X}$ has a probability density g with parameter θ, and that $\mathbb{E}_\theta$ denotes an expectation taken with respect to the density g_θ, taking the expected discounted payoff with respect to this density gives

$$\mathbb{E}_\theta[Y] = \mathbb{E}_\theta[f(X_1, \ldots, X_m)] = \int_{\mathbb{R}^m} f(\mathbf{X}) g_\theta(\mathbf{X}) d\mathbf{X}$$

Estimating ‘Greeks’ by MC simulation

3. Likelihood Ratio Method

- Now, similarly to the pathwise derivative approach, we assume the order of differentiation and integration can be interchanged. Here, however, this is not so strong an assumption, as typically densities are smooth functions, whereas payoff functions are not. This gives

$$\begin{aligned}\frac{d}{d\theta}\mathbb{E}_\theta[Y] &= \int_{\mathbb{R}^m} f(\mathbf{X})\frac{d}{d\theta}g_\theta(\mathbf{X})d\mathbf{X}\\ &= \int_{\mathbb{R}^m} f(\mathbf{X})\frac{\frac{d}{d\theta}g_\theta(\mathbf{X})}{g_\theta(\mathbf{X})}g_\theta(\mathbf{X})d\mathbf{X}\\ &= \mathbb{E}_\theta\left[f(\mathbf{X})\frac{\frac{d}{d\theta}g_\theta(\mathbf{X})}{g_\theta(\mathbf{X})}\right]\end{aligned}$$

Estimating ‘Greeks’ by MC simulation

3. Likelihood Ratio Method

- Then $f(\mathbf{X})\frac{\frac{d}{d\theta}g_\theta(\mathbf{X})}{g_\theta(\mathbf{X})}$ gives the Likelihood-Ratio estimator for the sensitivity with respect to the parameter θ, and this estimator is unbiased
- The term $\frac{\frac{d}{d\theta}g_\theta(\mathbf{X})}{g_\theta(\mathbf{X})}$ is often known in the statistics literature as the “score function”
- The Likelihood-Ratio method will still be applicable and robust in the case of options with discontinuous payoff functions (and estimating second-order sensitivities)

Estimating ‘Greeks’ by MC simulation

3. Likelihood Ratio Method

Let us again consider our simple, illustrative example...

European option under B-S model: Delta

The log-normal density used to calculate $S(T)$ is given by

$$g(x) = \frac{1}{x\sigma\sqrt{T}}\phi\left(\frac{\ln\left(\frac{x}{S(0)}\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

where $\phi(\cdot)$ represents the standard normal density function. In this case the score function is:

$$\frac{\frac{dg(x)}{dS(0)}}{g(x)} = \frac{\ln\left(\frac{x}{S(0)}\right) - (r - \frac{1}{2}\sigma^2)T}{S(0)\sigma^2 T}$$

Estimating 'Greeks' by MC simulation

3. Likelihood Ratio Method

Let us again consider our simple, illustrative example...

European option under B-S model: Delta

Evaluating this at $S(T)$ and multiplying by the option payoff gives the unbiased estimator of the Black-Scholes delta as:

$$e^{-rT} \max(S(T) - K, 0) \frac{\ln\left(S(T)/S(0)\right) - (r - \sigma^2/2)T}{S(0)\sigma^2 T}$$

Using: $S(T) = S(0)e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z}$, where $Z \sim \mathrm{N}(0,1)$...

Estimating 'Greeks' by MC simulation

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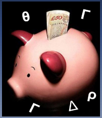

Estimating ‘Greeks’ by MC simulation

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European option under B-S model: Delta

Evaluating this at $S(T)$ and multiplying by the option payoff gives the unbiased estimator of the Black-Scholes delta as:

$$e^{-rT}\max(S(T)-K,0)\frac{Z}{S(0)\sigma\sqrt{T}}$$

Using: $S(T) = S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}$, where $Z \sim \mathrm{N}(0,1)\ldots$

Estimating ‘Greeks’ by MC simulation

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Giving the simplified unbiased LRM estimate of Delta, for a European call option under the B-S model for the underlying

Estimating 'Greeks' by MC simulation

3. Likelihood Ratio Method

And what about the case of discontinuous payoff functions...

Binary (cash-or-nothing) option under B-S model

$$\text{Payoff} = \begin{cases} 1 & \text{if } S(T) \geq K \\ 0 & \text{if } S(T) < K \end{cases} = \mathbb{I}\{S(T) > K\}$$

$$\text{Digital call:} \qquad e^{-rT}\,\mathbb{I}\{S(T) > K\}\frac{Z}{S(0)\sigma\sqrt{T}}$$

$$\left(\text{European call:}\;\; e^{-rT}\max(S(T) - K, 0)\frac{Z}{S(0)\sigma\sqrt{T}}\right)$$

Estimating 'Greeks' by MC simulation

3. Likelihood Ratio Method

And finally what about estimating the gamma sensitivity...

European option under B-S model: Gamma

As the gamma is the second-order sensitivity of the payoff wrt to S_0 the likelihood ratio weight will be different to that calculated for the delta. However, it is easy to show that the LRM estimator for the gamma is given by:

$$e^{-rT}\max(S(T)-K,0)\frac{Z^2 - Z\sigma\sqrt{T}-1}{S(0)^2\sigma^2 T}$$

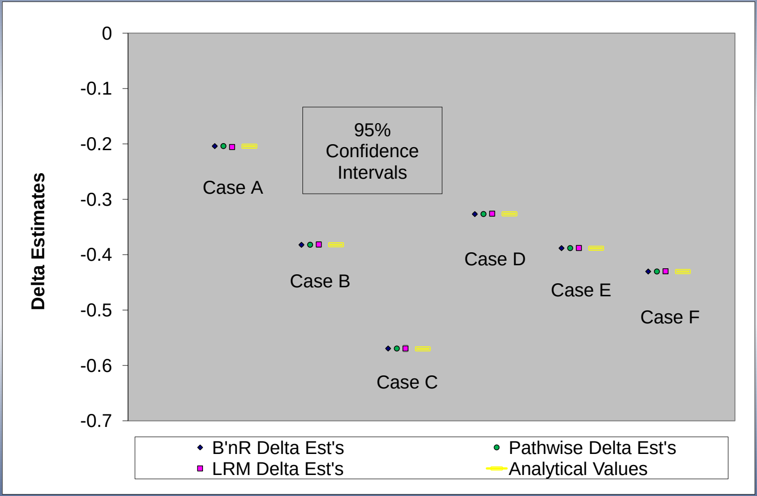
B-S Model Test Cases for analysis

Let us look at applying the three approaches for the Delta and Gamma sensitivities of a simple European put option

Case	S	K	σ	r	T
A	100	90	20%	4%	1
B	100	100	20%	4%	1
C	100	110	20%	4%	1
D	100	100	10%	4%	1
E	100	100	30%	4%	1
F	100	100	20%	1.5%	1

Table: Different put option set-ups considered in the tests. In all the above cases 100,000 paths and $S(0)$ perturbation of 0.2% used.

European put option delta estimators



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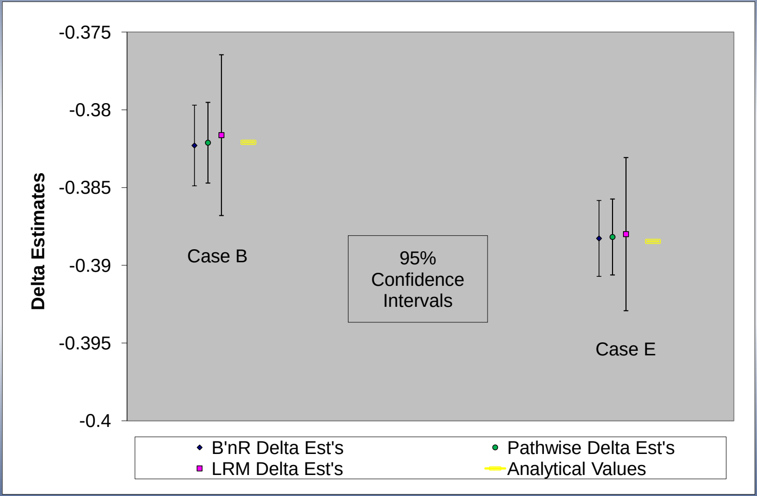
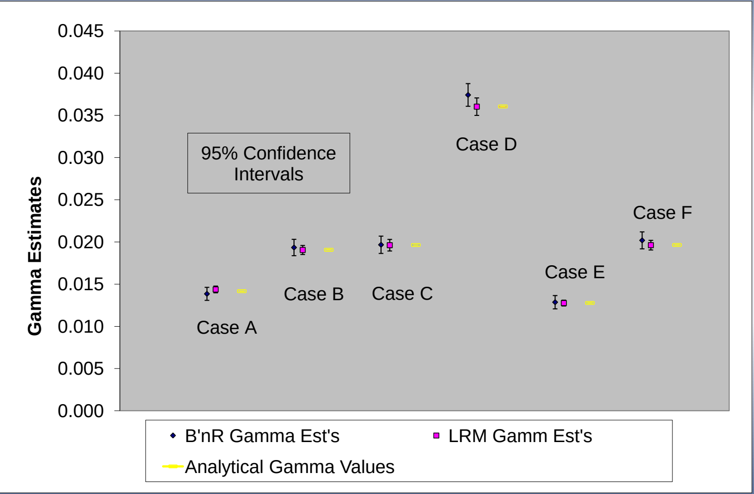
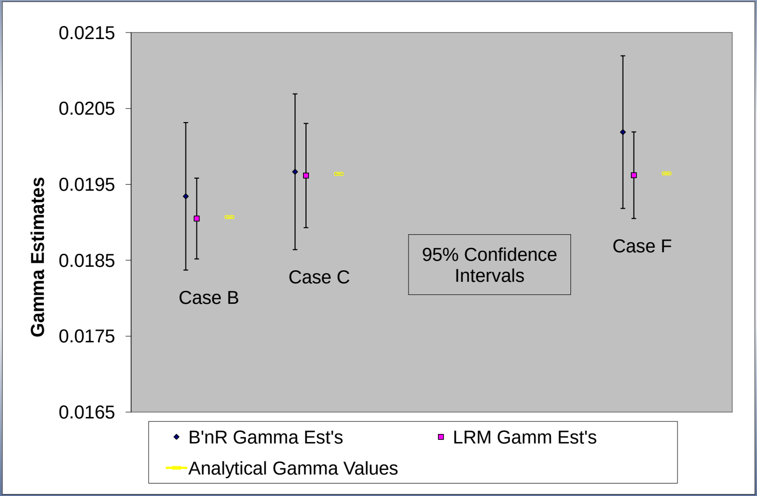


European put option gamma estimators



European put option gamma estimators



Conclusions from B-S analysis

1. B&R and PW give very similar estimates and standard errors

> This is expected as the Pathwise Estimator is the small perturbation limit of the Bump and Revalue method

2. When the Pathwise estimator is available it generally gives estimators with smaller standard errors than the LRM estimator

> This is because Pathwise estimator is specific to a given payoff function, whereas the LRM determines a weight for a given density which can then be multiplied by **any** payoff function

Conclusions from B-S analysis

3. The LRM provides an estimate of Gamma with smaller standard error than the corresponding Bump and Revalue estimate.

> If the Pathwise method is inapplicable, then, essentially, the Bump&Revalue approach cannot converge to the true value

Note: Although these observations have been inferred from analysis on simple European options, they generally hold for other options and models (see, e.g., Glasserman).

Estimating 'Greeks' by MC simulation

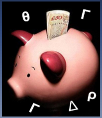
Mixed estimator for second-order Greeks

Let us again consider our simple, illustrative example...

European option under B-S model: Gamma

Can form a hybrid estimator for call option Gamma by applying the PW method to the LRM estimator for Delta...

$$\frac{d}{dS(0)}\left(e^{-rT}\max(S(T)-K,0)\frac{Z}{S(0)\sigma\sqrt{T}}\right)$$



Estimating 'Greeks' by MC simulation

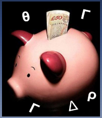
Mixed estimator for second-order Greeks

Let us again consider our simple, illustrative example...

European option under B-S model: Gamma

. . . resulting in the **LR-PW** mixed estimator for Gamma:

$$e^{-rT}\frac{Z}{S(0)^2\sigma\sqrt{T}}\mathbb{I}\{S(T) > K\}K$$



Estimating 'Greeks' by MC simulation

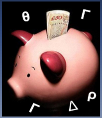
Mixed estimator for second-order Greeks

Let us again consider our simple, illustrative example...

European option under B-S model: Gamma

Can also form a **PW-LR** hybrid estimator for Gamma...

$$e^{-rT}\left(\mathbb{I}\{S(T)>K\}\frac{S(T)}{S(0)}\cdot\frac{Z}{S(0)\sigma\sqrt{T}}\right.$$
$$\left.+\mathbb{I}\{S(T)>K\}S(T)\frac{d}{dS(0)}\frac{1}{S(0)}\right)$$



Estimating ‘Greeks’ by MC simulation

Mixed estimator for second-order Greeks

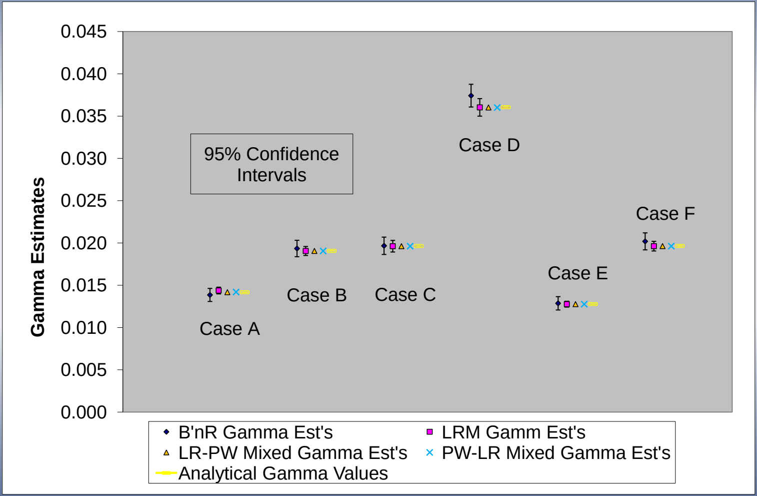
Let us again consider our simple, illustrative example...

European option under B-S model: Gamma

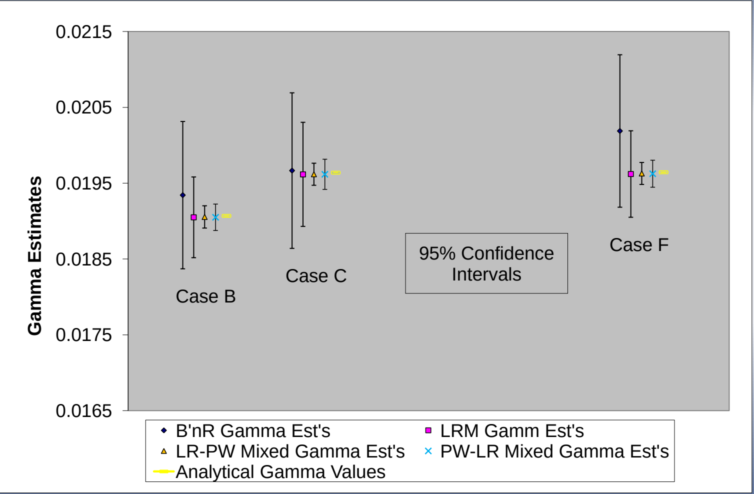
. . . and after a couple of lines of maths, the **PW-LR** hybrid estimator for the call option Gamma is given as:

$$e^{-rT}\mathbb{I}\{S(T) > K\}\left(\frac{S(T)}{S(0)^2}\right)\left(\frac{Z}{\sigma\sqrt{T}} - 1\right)$$

European put gamma mixed estimators



European put gamma mixed estimators



Conclusions of mixed estimator analysis

1. Both mixed estimators provide estimates with significantly lower standard errors than the LRM estimator (which gave lower standard errors than the Bump&Revalue)

2. Mixed estimators seem to utilise the respective advantages of the PW and LRM estimators

3. Given that they are constructed from the PW and LRM approaches, the mixed estimators are also unbiased

Estimating 'Greeks' by MC simulation

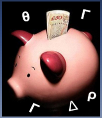
Estimators for Greeks on path-dependent payoffs

Path-dependent payoff under B-S: LRM Estimator

We can see that $S(0)$ is a parameter of the first factor, $g_1(x_1|S(0))$, only. Then the score function is:

$$\begin{aligned}\frac{\partial \log g(S(t_1), \ldots, S(t_m))}{\partial S(0)} &= \frac{\partial \log g_1(S(t_1)|S(0))}{\partial S(0)} \\ &= \frac{\zeta_1(S(t_1)|S(0))}{S(0)\sigma\sqrt{t_1}} = \frac{Z_1}{S(0)\sigma\sqrt{t_1}}\end{aligned}$$

where $Z_1 = \zeta_1(S(t_1)|S(0))$ is the Gaussian increment which takes us from time zero to time t_1, the first monitor point.



Accommodating more advanced models

The problems with Black-Scholes

- Black-Scholes model presents many problems for use with risk-management of Variable Annuity products:
    - Underestimates the probability of extremely low equity returns
    - Assumes volatility and risk-free rate are constant in time
    - Assumes stock price movements are always continuous
- However, B-S model tractable and widely used/assumed
- Thus we wish to address the problems without losing too much of the B-S structure...

Accommodating more advanced models

Alternative models considered

1. Heston Stochastic Volatility model (**Hest**):

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^S \\
dV_t &= \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V
\end{aligned}
$$

where $\kappa > 0$ represents the mean reversion speed, $\theta > 0$ denotes the mean reversion level and $\sigma_V > 0$ is the volatility of the variance process.
Brownian motions have correlation ρ, i.e., $dW_t^S dW_t^V = \rho dt$

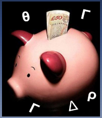
Accommodating more advanced models

Alternative models considered

2. Heston SV with C-I-R stochastic rates (**Hest-CIR**):

$$
\begin{aligned}
dS_t &= r_t S_t dt + \sqrt{V_t} S_t dW_t^S \\
dV_t &= \kappa_V(\theta_V - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V \\
dr_t &= \kappa_r(\theta_r - r_t)dt + \sigma_r \sqrt{r_t} dW_t^r
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{corr}(W_t^S, W_t^V) &= \rho_{S,V}, \qquad \text{i.e.,} \qquad dW_t^S dW_t^V = \rho_{S,V} dt \\
\mathrm{corr}(W_t^S, W_t^r) &= \rho_{S,r}, \qquad \text{i.e.,} \qquad dW_t^S dW_t^r = \rho_{S,r} dt \\
\mathrm{corr}(W_t^V, W_t^r) &= \rho_{V,r}, \qquad \text{i.e.,} \qquad dW_t^V dW_t^r = \rho_{V,r} dt
\end{aligned}
$$



Accommodating more advanced models

How do the three approaches change?

- With the B&R and PW approaches, the estimators just require the generated equity returns (mechanics same if SV or S-IR)
- On the other hand, LRM uses the score function in determining estimate and for the B-S model this score function can be found
- With **Hest** or **Hest-CIR** the returns are no longer lognormal and thus the score function cannot be determined
- However, if we condition on a realisation of the variance (and interest-rate) process, the returns are (conditionally) lognormal, allowing the LRM to be applied

How does the LRM change with SV?

Building in equity-volatility correlation

Cholesky Decomposition

Brownian motions for the asset and variance processes have correlation ρ. This can be achieved by setting:

$$dW_t^S = \rho dW_t^V + \sqrt{1-\rho^2} dW_t^{Ind}$$

Get expression for asset price process (with corr. structure):

$$dS_t = rS_t dt + \sqrt{V_t} S_t \left(\rho dW_t^V + \sqrt{1-\rho^2} dW_t^{Ind}\right)$$

How does the LRM change with SV?

Itô's calculus and Geometric Brownian Motion

Itô's Lemma:

Given a r.v y satisfying the s.d.e

$$dy = a(y,t)dt + b(y,t)dW_t$$

where W_t is a Brownian motion and a funct. $f(y,t)$ differentiable wrt t and twice so wrt y, then f itself satisfies:

$$df = \left(\frac{\partial f}{\partial t} + a(y,t)\frac{\partial f}{\partial y} + \frac{1}{2}b(y,t)^2\frac{\partial^2 f}{\partial y^2}\right)dt + b(y,t)\frac{\partial f}{\partial y}dW_t$$

How does the LRM change with SV?

Itô's calculus and Geometric Brownian Motion

Geometric Brownian motion:

$$dy = dS_t = S_t(rdt + \sigma dW_t) \qquad \textbf{Black-Scholes}$$

Then, $f(y,t) = \ln y \implies \frac{\partial f}{\partial t} = 0,\ \frac{\partial f}{\partial y} = \frac{1}{y},\ \frac{\partial^2 f}{\partial y^2} = \frac{-1}{y^2}$ and

$$\begin{aligned} d\ln S_t = dX_t &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial y} + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dy^2) \\ &= 0 + dS_t/S_t + (1/2)(-1/S_t^2)(dS_t^2) \\ (dW)^2 = dt \rightarrow \quad &= rdt + \sigma dW_t + (1/2)(-1/S_t^2)S_t^2\sigma^2 dt \\ &= (r - \sigma^2/2)dt + \sigma dW_t \end{aligned}$$

How does the LRM change with SV?

Itô's calculus and Geometric Brownian Motion

Geometric Brownian motion:

$$d \ln S_t = (r - \sigma^2/2)dt + \sigma dW_t$$

$$\Downarrow$$

$$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma dW_T \right)$$

i.e., **lognormal** returns in the **Black-Scholes model**

Same approach used to obtain the conditional lognormal returns from conditioning on the Heston & CIR processes

How does LRM change with SV & S-IR?

Heston-CIR Model:

$$\frac{dS_t}{S_t} = r_t dt + \sqrt{V_t}\left(C_1 dW_t^V + C_2 dW_t^r + C_3 dW_t^{\text{Ind}} \right)$$

where

$$C_1 = \rho_{S,V}$$

$$C_2 = \frac{\rho_{S,r} - \rho_{S,V}\rho_{V,r}}{\sqrt{1-\rho_{V,r}^2}}$$

$$C_3 = \sqrt{1-\rho_{S,V}^2 - \frac{(\rho_{S,r} - \rho_{S,V}\rho_{V,r})^2}{1-\rho_{V,r}^2}}$$

Applying Itô's Formula and collecting terms gives...

How does LRM change with SV & S-IR?

Heston-CIR Model:

$$
\begin{aligned}
S_T &= S_0 \exp(Y_T) \exp\left(\frac{1}{T} \int_0^T r_t dt - \frac{1}{2} C_3^2 \frac{1}{T} \int_0^T V_t dt \right. \\
&\qquad \left. + C_3 \sqrt{\frac{1}{T} \int_0^T V_t dW_t^{\mathrm{Ind}}} \right)
\end{aligned}
$$

where

$$
\begin{aligned}
Y_T &= -\frac{1}{2} C_1^2 \int_0^T V_t dt - \frac{1}{2} C_2^2 \int_0^T V_t dt \\
&\quad + C_1 \int_0^T \sqrt{V_t} dW_t^V + C_2 \int_0^T \sqrt{V_t} dW_t^r
\end{aligned}
$$

How does LRM change with SV & S-IR?

Heston-CIR Model:

$$\begin{aligned} S_T &= S_0 \exp(Y_T) \exp\left(\frac{1}{T}\int_0^T r_t dt - \frac{1}{2} C_3^2 \frac{1}{T}\int_0^T V_t dt \right. \\ &\qquad \left. + C_3 \sqrt{\frac{1}{T}\int_0^T V_t dW_t^{\text{Ind}}} \right) \end{aligned}$$

Therefore, the equity returns are **conditionally lognormal**:

$$\begin{aligned} S_0 &\rightarrow S_0 e^{Y_T} \\ \sigma^2 &\rightarrow C_3^2 \frac{1}{T}\int_0^T V_t dt \end{aligned}$$

Hest-CIR Test Cases for analysis

Case	κ_V	θ_V	ϵ_V	κ_{IR}	θ_{IR}	ϵ_{IR}	$\rho_{S\text{-}V}$	$\rho_{S\text{-}IR}$	$\rho_{V\text{-}IR}$
A	2	0.04	0.15	4	0.04	0.06	-0.7	-0.6	0.7
B	1	0.04	0.3	4	0.04	0.06	-0.7	-0.6	0.7
C	2	0.04	0.15	2	0.04	0.2	-0.7	-0.6	0.7
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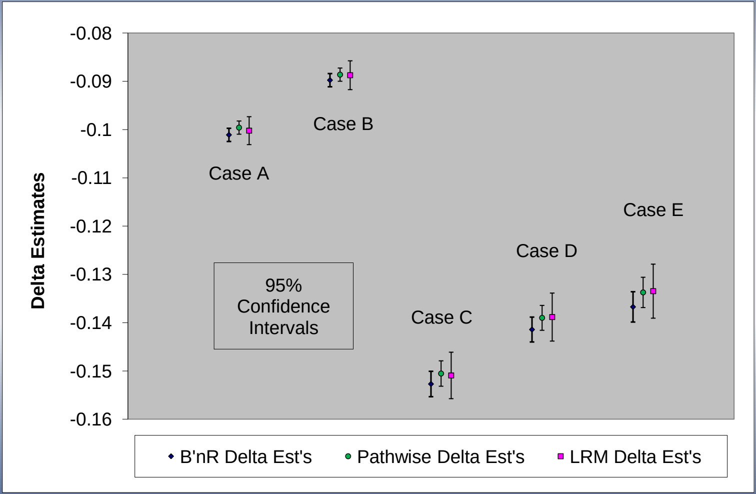
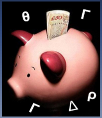
E	2	0.04	0.15	4	0.04	0.06	-0.9	-0.6	0.7

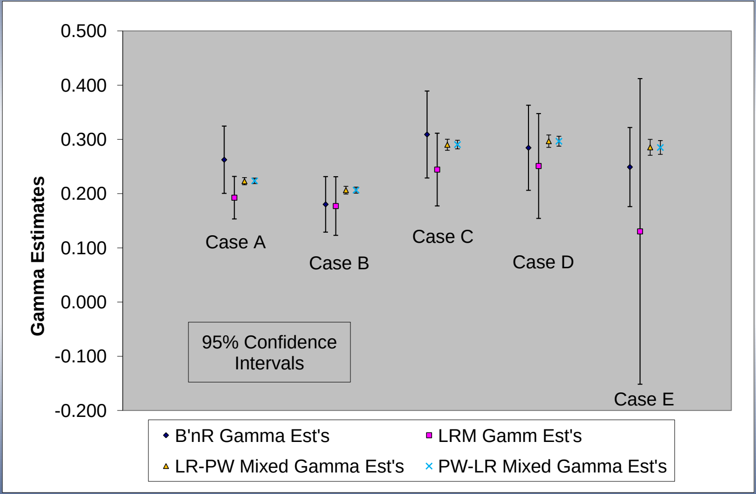
Table: Different model settings considered in tests. Heston SV parameters are denoted by a V subscript and the CIR parameters are given by an IR subscript. The correlations denoted by obvious subscripts. **Asian put option**: $S_0 = 1$, $K = 1.25$, $T = 30$. Annual monitor points.

Hest-CIR Asian Option Delta





Hest-CIR Asian Option Gamma



Conclusions from analysis

1. The Likelihood Ratio Method can be extended to price Asian options under the **Hest-CIR** model

2. For the delta sensitivity, the PW/B&R method produced estimates with smaller standard errors than the CLRM.

3. For the Gamma sensitivity, a mixed estimator will generally be preferable to the B&R and CLRM for path-dependent payoff functions. In certain cases the CLRM estimator can become very inefficient.

Variable Annuity Example

Variable Annuity Assumptions

- We now introduce a fairly simple VA contract to demonstrate that the three MC Greek approaches can be extended to estimate sensitivities of unit-linked insurance products
- Assumptions in VA Example:
  - Policyholder is male and 65 years of age at annuitisation (and static mortality rates assumed)
  - The policyholder has saved an amount $\pounds P$ which constitutes the initial premium of the VA contract
  - Policyholder will take (full) withdrawals from end of year one
  - Policyholder lapsation behaviour is ‘static’

Variable Annuity Example

VA Product Structure

- **Fund/Account Value** begins at the level of the policyholder premium and grows/falls at the same rate as the equity index with which the product is linked (at annual rebalancing dates)
- At each rebalancing date the policyholder withdraws some amount of **Income** from this Fund Value
- This Income amount can grow from one year to the next if the FV grows above its 'high watermark level'. This maximum lookback level will be denoted the **Guarantee Base**
- Yearly withdrawal level given as fixed % of this GB level

Variable Annuity Example

VA Product Structure: Mathematically

- Fund Value t years after annuitisation $\rightarrow F_t$
- Guarantee Base t years after annuitisation $\rightarrow G_t$
- Income Level t years after annuitisation $\rightarrow I_t$
- Fixed percentage of GB taken as Income $\rightarrow w$
- Equity Index, return from year $t-1$ to $t \rightarrow R_t$ $[= S_t / S_{t-1}]$

$$\boxed{F_t = \max\{(F_{t-1} - I_{t-1})(1 + R_t), 0\}} \text{ with } \boxed{F_0 = P \cdot S_0}$$

$$\boxed{G_t = \max\{F_t, G_{t-1}\}} \quad \text{and} \quad \boxed{I_t = wG_t}$$

Variable Annuity Example

VA Product Structure: Mathematically

- Fund Value t years after annuitisation $\rightarrow F_t$
- Guarantee Base t years after annuitisation $\rightarrow G_t$
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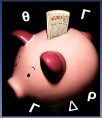
- Fixed percentage of GB taken as Income $\rightarrow w$
- Equity Index, return from year $t-1$ to t $\rightarrow R_t$ $[= S_t / S_{t-1}]$

Then Value of Liability of this particular VA contract is:

$$L = \mathbb{E}\left[\sum_{t=1}^{T} D_t p_t^{surv} \max(I_t - F_t, 0)\right]$$



Variable Annuity Example

VA Product: Pathwise Estimator

- Now a Pathwise method for this VA product will be developed
- To demonstrate this consider the problem of estimating the Delta Greek

$$\Delta_{\text{PW}} = \frac{\partial L}{\partial S_0} = \frac{\partial}{\partial S_0}\mathbb{E}\left[\sum_{t=0}^{T} D_t p_t^{surv} \max(I_t - F_t, 0)\right]$$

- With static policyholder behaviour, the derivative will only act on third term above...

Variable Annuity Example

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- With static policyholder behaviour, the derivative will only act on third term above...

Variable Annuity Example

VA Product: Pathwise Estimator

- This derivative is given by...

$$\frac{\partial}{\partial S_0}\max(I_t - F_t, 0) = \mathbb{I}\{I_t > F_t\} \cdot \left(\frac{\partial I_t}{\partial S_0} - \frac{\partial F_t}{\partial S_0}\right)$$

- ...but how are $\frac{\partial I_t}{\partial S_0}$ and $\frac{\partial F_t}{\partial S_0}$ calculated?

Variable Annuity Example

VA Product: Pathwise Estimator

These are given recursively by:

$$\frac{\partial F_t}{\partial S_0} = \left(\frac{\partial F_{t-1}}{\partial S_0} - \frac{\partial I_{t-1}}{\partial S_0}\right)(1 + R_t) \qquad \boxed{\text{with } \frac{\partial F_0}{\partial S_0} = P}$$

$$\frac{\partial I_t}{\partial S_0} = w \cdot \frac{\partial G_t}{\partial S_0} \quad = \quad w \cdot \frac{\partial}{\partial S_0} \max(F_t, G_{t-1})$$

$$\boxed{\text{with } \frac{\partial I_0}{\partial S_0} = 0} \quad = \quad w \cdot \left[\mathbb{I}_{\{F_t > G_{t-1}\}} \frac{\partial F_t}{\partial S_0} + \mathbb{I}_{\{F_t \le G_{t-1}\}} \frac{\partial G_{t-1}}{\partial S_0}\right]$$

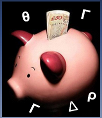
Variable Annuity Example

VA Product: Likelihood Ratio Method Estimator

- Generality of LRM means it can be applied to VA product liability Greeks without much extra effort
- Similar structure to Asian option est'r. Uses implied shock out to first valuation date, in our case the end of year 1

$$Z^{\text{Imp.}} = \frac{\log(S_1/\bar{\xi}_1 S_0) - (\bar{r}_1 - \bar{\sigma}_1^2/2) * 1}{\bar{\sigma}_1 \sqrt{1}}$$

$$\Delta^{\text{LRM}} = \mathbb{E}\left[\mathbb{E}\left[\left(\frac{Z^{\text{Imp.}}}{S_0 \bar{\sigma}_1 * 1}\right) \sum_{t=0}^{T} D_t p_t^{surv} \max(I_t - F_t, 0) \middle| v_1, r_1\right]\right]$$



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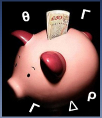
$$\Delta^{\text{LRM}} = \mathbb{E}\left[\mathbb{E}\left[\left(\frac{Z^{\text{Imp.}}}{S_0 \bar{\sigma}_1 * 1}\right) \times \text{Liability} \middle| v_1, r_1\right]\right]$$

Variable Annuity Example

VA Product: Gamma Mixed PW-LR Estimator

- The Pathwise and CLRM can be combined to give a PW-LR estimator for the Gamma Greek. Along each path:

$$
\begin{aligned}
\Gamma^{\text{LR-PW}} &= \frac{\partial}{\partial S_0} \Delta^{\text{LRM}} \\
&= \frac{\partial}{\partial S_0} \left(\left[\frac{Z^{\text{Imp.}}}{S_0 \bar{\sigma}_1 \sqrt{1}} \right] \sum_{t=1}^{T} D_t p_t^{surv} \max(I_t - F_t, 0) \right) \\
&= \left[\frac{Z^{\text{Imp.}}}{S_0 \bar{\sigma}_1 \sqrt{1}} \right] \sum_{t=1}^{T} D_t p_t^{surv} \frac{\partial}{\partial S_0} \max(I_t - F_t, 0) \\
&\quad - \left[\frac{Z^{\text{Imp.}}}{S_0^2 \bar{\sigma}_1 \sqrt{1}} \right] \sum_{t=1}^{T} D_t p_t^{surv} \max(I_t - F_t, 0)
\end{aligned}
$$



Variable Annuity Example

VA Product: Gamma Mixed PW-LR Estimator

$$\Gamma^{\text{LR-PW}} = \left[\frac{Z^{\text{Imp.}}}{S_0\bar{\sigma}_1\sqrt{1}}\right]\sum_{t=1}^{T} D_t p_t^{surv}\frac{\partial}{\partial S_0}\max(I_t - F_t, 0)$$
$$-\left[\frac{Z^{\text{Imp.}}}{S_0^2\bar{\sigma}_1\sqrt{1}}\right]\sum_{t=1}^{T} D_t p_t^{surv}\max(I_t - F_t, 0)$$

- The two sums are already calculated in the intermediate steps to find the Pathwise Delta estimator
- Thus we just take the result of these sums along each path and multiply by the respective factors above
- A PW-LR estimator can be constructed in a similar way

Hest-CIR Test Cases for analysis

Case	κ_V	θ_V	ϵ_V	κ_{IR}	θ_{IR}	ϵ_{IR}	$\rho_{S\text{-}V}$	$\rho_{S\text{-}IR}$	$\rho_{V\text{-}IR}$
A	2	0.04	0.15	4	0.04	0.06	-0.7	-0.6	0.7
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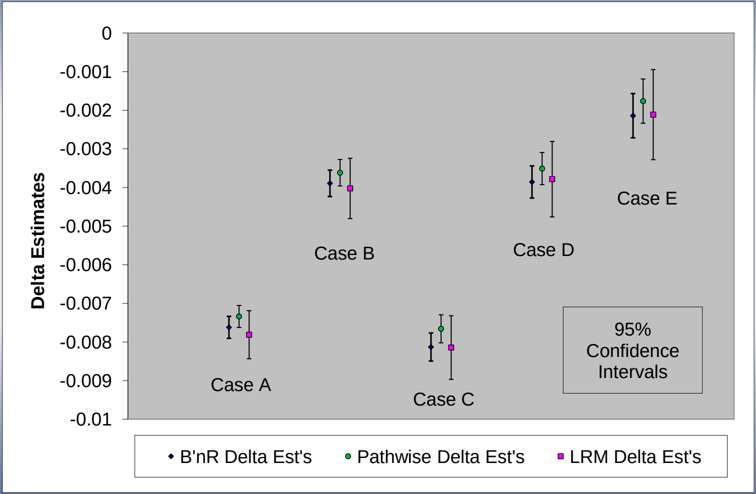
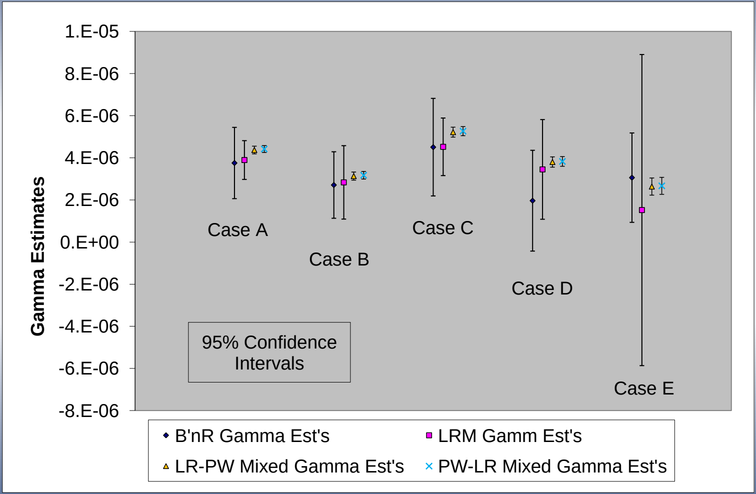
Table: Different model settings considered in tests. Heston SV parameters are denoted by a V subscript and the CIR parameters are given by an IR subscript. The correlations denoted by obvious subscripts. $V_0 = \theta_V$, $r_0 = \theta_{IR}$. VA contract has $T = 30$ and Ratchet Term $= 10$.

Variable Annuity Delta Estimators



Variable Annuity Gamma Estimators



Conclusions from VA analysis

1. PW est'r has lower st.err. than CLRM est'r for VA Delta

> This is expected, as the series of options in time which make up the liability has a simple, continuous payoff

2. B&R & the CLRM estimators are both relatively inefficient estimators for the VA Gamma. However, mixed estimators have much lower variance than both of these estimators.

> Similarly to Asian option, the CLRM estimator for Gamma does not give as low st.err's as for European options. (Uses limited part of path in shock.) However, mixed estimator again seems to provide the benefits from both approaches

Conclusions/Further Research

Conclusions from research to date

VA Greek estimators constructed are successful, in particular the mixed second-order sensitivity estimators

Future Research

- **Variance Reduction.** Control Variate, Importance Sampling, Low-Discrepancy MC
- **Interest-rate sensitivities.** wrt CIR param's
- **More complex VA product.** Two Equity Indices, Equity/Bond mix, dynamic policyholder behaviour
- **Computational framework.** GPU, Automatic Diff. ?

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