MARKET-CONSISTENT VALUATION OF PENSION LIABILITIES

Abstract

Life insurance companies and pension funds have liabilities on their books with very long-dated maturities. Most people start at age 25, expected to live to age 85, and at he longest living to age 115. Hence, life business facing contractual obligations that can easily last 60 years, and sometimes even 80 or 90 years into the future. To give a feel for the size of the problem: for life-insurance and pension products roughly a portion of 20% of the NPV of the cash-flows is located in the ``tail'' of 30+ years. While most of these very long-dated contracts are not tradable, the supervisor's requirement for ``Market-Consistent'' valuation makes the pricing and risk-management of such liabilities very important. To obtain the market-consistent price, we combine the hedgeable financial risk with an unhedgeable insurance risk in the market and we price the general payoff that depends on both risks, using a pricing procedure called ``Two-step market evaluation''. The valuation process of the portfolio will be implicitly consist of the no-arbitrage price of the pure financial risk, value of the partially hedged insurance risk due to its correlation with financial risk, and finally the value of pure insurance risk via well-known actuarial premium principles. We implement two-step valuation together with backward iteration method to simultaneously achieve Time-Consistency property of the price.

1. Introduction

Life insurance companies and pension funds have liabilities on their books with very long-dated maturities. Most people start saving for their pension from age 25, and people are expected to live to age 85, with the longest people living to age 115. Hence, pension funds and life insurance companies are facing contractual obligations that can last easily last 60 years, and sometimes even 80 or 90 years into the future. The valuation and risk-management of these very long-dated contracts is therefore an important problem. To give a feel for the size of the problem: for life-insurance and pension products roughly a portion of 20% of the NPV of the cash-flows is located in the "tail" of 30+ years. While most of these very long-dated contracts are not tradable, Solvency II requirements to ``Market-Consistent'' valuation of them makes the pricing and risk-management of such liabilities very important.

Usually actuarial pricing (by premium principles as the price operators) ignore the mechanism of the financial markets and hedging strategies, while the classical financial pricing typically ignores unhedgeable risks available in the payoff or portfolio. On the other hand, the actuarial pricing methods usually valuate the risk in a ``static'' way by a one-period pricing procedure ignoring the mid-time evolution of the insurance risk drivers. But then, financial pricing operates in a ``dynamic'' way to take the evolution of the risk over time to maturity, into account. This is needed to capture the ``path-dependent'' nature of the payoff in wide range of the traded financial contracts. By the market-consistent actuarial valuation, we integrate the financial and actuarial pricing methods while the setup allows us to consider both hedgeable and unhedgeable risks including their interactions, and price a combined payoff in a dynamic way.

We investigate what happens to the pricing formula when we consider an environment where we have both financial risk that can be traded and hedged in a market, and unhedgeable insurance risk. This connects us to the literature in which the risk measures/valuations in a so called market-consistent setting are studied. This started by the pricing of contracts in an incomplete market setting, where one seeks to extend the arbitrage-free pricing operators (which are only defined in a complete market setting) to the larger space of (partially) unhedgeable contracts. The paper by Hodges and Neuberger [14] is often cited as the root-idea of the utility-indifference pricing literature mentioned above. A related branch of literature extends the arbitrage-free pricing operators using (local) risk-minimization techniques and the related notion of minimal martingale measures, see Föllmer and Schweizer [10], Schweizer [30], and Delbaen and Schachermayer [9]. A rich duality theory has been developed that makes deep connections between utility maximization and minimization over martingale measures, see Cvitanic and Karatzas [7], Kramkov and Schachermayer [17], for a very elegant summary we refer to Rogers [28].

Using utility-indifference (and duality) methods, the market-consistency of pricing operators is automatically induced. However, an explicit formal definition of market-consistent pricing operators has only begun to emerge recently, see Kupper et al. [18] and Malamud et al. [22].

To implement the market-consistent valuation, Pelsser and Stadje [26] introduced a method called ``Two-step Market Evaluation'' with an axiomatic characterization for that including ``market local property''. The importance of the study was that, in light of some theoretical results and under certain assumptions (which are normally available for the standard actuarial premium principles), they proved that the two-step market evaluation can turn any conditional valuation operator into a market-consistent version. In general what two-step evaluation does is as follow:

- **First step**: Conditioning the general payoff/position on the financial risk driver and apply the actuarial pricing operator, which turns the payoff into a function of only financial risk and by the structure perfectly hedgeable,

- **Second step**: Applying conditional expectation under the unique equivalent martingale measure $\mathbb{Q}$ which reflects the no-arbitrage argument for the hedgeable part of the general position.

Before this, a fundamentally similar evaluation was used by Møller [23] to valuate a standard deviation actuarial principle. Musiela and Zariphopoulou [24] also used a similar method to calculate the indifference premium via exponential utility function in incomplete market by first conditioning the premium on financial trading.

Insurance companies and pension funds need to revaluate their liabilities and assets in a certain periods for different reasons such as asset liability management and supervision purposes. The importance of the problem is higher when the insurance company have very long dated liabilities such as annuities and pension products in their portfolio. Hence, the pricing method should be adjusted so that such a revaluation process is taken into account. Although the real revaluation of the liabilities and future payoffs takes place in a forward way, the compatible pricing method can react to that in a backward way starting from maturity time. So essentially we need to price the payoffs and contracts by revaluating the value of the payoff in the middle times in a backward way and reach the price at present time. Pricing under this criterion is called in the literature ``Time-Consistent'' valuation which requires a dynamic setting for price operators. Pelsser and Stadje [26] also showed that, under certain assumptions, by use of Time-consistency property, market-consistency and market local property also hold in a dynamic setting. Time-consistency implies that the value order of different positions measured via a dynamic price operator in the future time t is consistent with their order at any time prior $s < t$. Construction of the time-consistent risk measures have been under investigation recently by a number of researchers. See Cheridito et al. [6], Rosazza Gianin [29], Artzner et al. [1] for general ideas about dynamic risk measures and Peng [27], Frittelli and Gianin [11], Maccheroni et al. [21], Bion-Nadal [3] and Barrieu and El Karoui [2] for time-consistency in continuous-time.

More specifically Jobert and Rogers [15] showed that time-consistent price operators can be constructed via a backward iteration procedure to stick shorter one-period static price operators together to find the price over a longer period. In a recent work, Pelsser and Salahnejhad [25] used Jobert and Rogers [15]'s method to find the continuous-time limit of the well-known actuarial premium principles under time-consistent setting. In addition, Pelsser and Stadje [26] proved that, for finitely many stopping times $\tau \in [0, T]$ of the underlying insurance process, time-consistency and market-consistency implies that every evaluations (including actuarial premium principles) have to admit a representation of the two-step market evaluation. Note that the result is stable even when the insurance process follows a jump diffusion process, and jumps happen only at finitely many predictable times.

The main contribution of this paper is twofold. First, we provide an easy implementation of the two-step valuation method. Second, we apply this implementation to price a stylized pension / insurance contract. We show to practitioners that the methods outlined to calculate the market-consistent value are easy to implement for practical problems, such as pension and insurance liabilities. We first apply the method in a one-period valuation setting and give some insights about how the market-consistent price is different than the usual one. This will be done via both analytical and numerical representation of the price for a simple product. We will then turn to a multi-period valuation and use a dynamic setting with use of the time-consistent actuarial premium principles and we will, in practice, employ the two-step market evaluation in a backward iteration procedure.

We work with two main product that have enough flexibility to present a realistic picture of the method. We start with a simple unit-linked contract without guarantee and apply the two-step actuarial valuation in one-period and multi-period setting. We also provide a numerical scheme to construct the two-step actuarial valuation with dependent financial and actuarial risks. We also provide the valuation method for a typical simple pension contract discussed by Grosen and Jorgensen [13] that comes with some numerical results for its valuation in multi-period setting.

Both contracts have two main risk drivers; market valua of the assets/equity as the financial risk and longevity risk of an individual or a cohort entered in the contract as the actuarial risk. The market value of the asset or equity is modeled by a geometric Brownian Motion (GBM) to be priced in a complete market with no-arbitrage argument. The longevity risk is modeled via the famous Lee-Carter model introduced by Lee and Carter [19]. Both models are constructed in a diffusion setting without jump.

2. Market-Consistent Valuation

This section provides a general framework for market-consistent valuation of life insurance and pension liabilities. We consider a class of contracts whose payoffs are contingent on hedgeable financial risk and unhedgeable (or partially hedgeable) longevity risk. We first introduce the appropriate market-consistent valuation operator in a one-period setting and then we develop the framework to a multi-period setting where we construct the operator based on the time-consistency property.

2.1. Setup and Assumption

We consider a European contract that has a payoff at maturity T contingent on the evolution of the financial market and the mortality pattern of the portfolio. The goal is to determine a market-consistent price of the contract at time $t \in [0, T)$,. Due to the dual nature of the payoff, we separate the information flow. We denote the financial information that the insurance company or the pension fund has available at time t by $\mathcal{F}_t^S$ and the information on the observed mortality by $\mathcal{G}_t^A$. This is formalized by introducing an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtrations $(\mathcal{F}_t^S)_{t\geq 0}$ and $(\mathcal{G}_t^A)_{t\geq 0}$.

We assume that the financial market is complete, which means that all financial risk can be hedged (by trading in continuous-time) and we assume the financial risk can priced it under the no-arbitrage argument. It is well-known that under these assumptions prices of financial derivatives can be calculated using "risk-neutral pricing". The Fundamental Theorems of Asset Pricing (see, e.g., Delbaen and Schachermayer [8]) implies that there exists a so-called risk-neutral measure $\mathbb{Q}$ under which the financial price operator is always a conditional expectation.

On the other hand, we assume that the actuarial risk is not (completely) hedgeable and also rarely tradable. Thus, we cannot exploit a replicating portfolio argument for insurance derivatives. This turns to the fact that we can not use an expectation operator to value the risk and secure arbitrage free price. Of course, in the context of pure actuarial risks, many pricing methods called ``actuarial premium principles'' have been proposed. The following display recalls the three most well-known actuarial principles to price the discounted loss H at maturity; see, e.g., Kaas et al. [16] for a motivation and details.

$$\text{Variance Principle:} \qquad \Pi[H] = \mathbb{E}[H] + \frac{1}{2}\alpha \mathbb{V}\mathrm{ar}[H], \qquad \alpha \geq 0 \tag{2.1a}$$

$$\text{Standard-Deviation Principle:} \qquad \Pi[H] = \mathbb{E}[H] + \beta\sqrt{\mathbb{V}\mathrm{ar}[H]}, \qquad \beta \geq 0 \tag{2.1b}$$

$$\text{Cost-of-Capital Principle:} \qquad \Pi[H] = \mathbb{E}[H] + \delta \mathrm{VaR}_q\left[H - \mathbb{E}[H]\right], \qquad \delta \geq 0 \tag{2.1c}$$

Most of the actuarial premium principles including the ones above, are a nonlinear function of the discounted loss and impose an extra risk premium (also called ``Risk Loading'') to the "Best-Estimate" of the future insurance loss. Both best-estimate and risk loading are calculated under the real world measure $\mathbb{P}$. This postulates that the actuarial pricing is rather an economic decision under uncertainty, while financial pricing is normally based on the risk-neutral valuation built by an equivalent martingale measure that uses conditional expected value under a risk adjusted underlying process. Goovaerts and Laeven [12] discussed the no-arbitrage argument for the actuarial principles for pure insurance risks and securitized version of the insurance products.

The risk loading by itself is another risk measure that plays the role of a buffer to cover the possible deviation of the future losses from what is expected by the best-estimate of the losses. In the Variance principle, the loading is a positive proportion of the variance risk measure that applies to the discounted loss, while in the Standard-Deviation Principle, the standard deviation of the discounted loss is the measures the risk premium. Bühlmann [4] discussed the properties of these two premium pricniples in details. In the Cost-of-Capital principle, the risk loading is measured by Value-at-Risk with probability threshold q (in solvency II equal to 0.995), where the unexpected loss is calculated as $\mathrm{VaR}_q[H - E(H)]$. There is always a small probability $1 - q$ that the risk loading (capital buffer) is needed to compensate a real unexpected risk. Therefore, if the buffer is borrowed (e.g. from the shareholders of the insurance company), the lender will require a cost of capital δ. The insurer includes this cost of capital as the risk loading in the price of the insurance risk to be paid by the policyholder.

We assume we have only two stochastic processes x_t as a hedgeable pure financial risk and y_t as a non-traded unhedgeable insurance risk. The general payoff $G(x_T, y_T)$ is a European derivative of the financial and insurance risks in the incomplete market. Intuitively, the price $\Pi_{\mathcal{G}^S}(t, x)$ of a pure financial claim $G^S(x_T)$ should be equal to the arbitrage-free price which is a conditional expectation conditional of the financial information at time t (i.e. $\mathcal{F}_t^S$). Furthermore, if we add a pure financial claim to a given general payoff $G(x_T, y_T)$, then we should have that the pure financial part of the portfolio be priced consistently with arbitrage-free pricing. Note that, the payoffs replicable by perfectly liquid assets, don't carry any risk more than market risk. We can formalize this in the following definition.

Definition 2.1 *An actuarial pricing operator $\Pi_{\mathcal{G}^A}$ conditional on the actuarial information $\mathcal{G}^A$ is market-consistent, if for any financial derivative $G^S(x_T)$ and any general claim $G(T, x_T, y_T)$ we have*

$$\Pi^{\mathbb{P}}_{\mathcal{G}^A}(G + G^S) = \Pi^{\mathbb{P}}_{\mathcal{G}^A}[G] + \mathbb{E}^{\mathbb{Q}}_{\mathcal{G}^A}\left[G^S\right]. \tag{2.2}$$

Market consistency definition postulates that given actuarial information, the actuarial price of a general payoff plus pure financial payoff, is equal to actuarial price of the general payoff plus the arbitrage free price of the pure financial payoff. This establishes a generalized notion of ``translation invariance'' for the $\mathcal{G}^A-$conditional [1] valuation operator with respect to the pure financial risk. This implies that if there is any risk hedgeable (even in the payoff G), it must be hedged via market-consistent valuation. Hence, market-consistent valuation cannot be improved by hedging. The similar representation of the market consistency can be found in Kupper et al. [18], Malamud et al. [22] or Pelsser and Stadje [26].

We also assume x_t and y_t are adapted respectively to the corresponding information flows $\mathcal{F}_t^S$ and $\mathcal{G}_t^A$ and have Markov property so that they reflect all the information at time $t \leq T$. Therefore, to exhibit conditioning at any time $0 \leq t \leq T$, we will use x_t and y_t respectively instead of $\mathcal{F}_t^S$ and $\mathcal{G}_t^A$.

We model the financial and actuarial risks with a simple diffusion stochastic process containing the drift and diffusion terms. Hence,

$$\text{Financial Risk:} \qquad dx_t = \mu(t, x_t)dt + \sigma(t, x_t)dW_f(t) \tag{2.3a}$$

$$\text{Actuarial Risk:} \qquad dy_t = a(t, y_t)dt + b(t, y_t)dW_a(t) \tag{2.3b}$$

where $W_f(t)$ and $W_a(t)$ are independent standard Brownian motions and μ, σ, a and b are adapted processes with respect to information set at time t. In application, x_t can be a model for stock price with growth rate μ and volatility σ and y_t be a mortality trend with drift a and standard deviation b.

[1] In market-consistent actuarial valuation, by default, the main pricing operator is the actuarial premium principle, while we adjust it under market-consistency to hedge the mixed insurance and financial position, as much as possible. As we should use the actuarial information $\mathcal{G}^A$ at time t to value a payoff at maturity T, the possible price/valuation operator is constructed given those information and is called ``$\mathcal{G}^A$-conditional''.

Later, we will use Geometric Brownian Motion for stock price S_t and a Brownian motion with constant drift and diffusion to model longevity trend κ_t, as a specified representation of the (2.3).

For discounting we will assume a constant interest rate over the valuation period. Stochastic interest rate can be involved in the market-consistent valuation as an extra financial risk to be prices in complete market setting.

2.2. Two-step Actuarial Valuation in One-period Setting

Based on the paper by Pelsser and Stadje [26], market-consistent value of the contingent payoff can be constructed by the ``Two-step market evaluation'' method without loss of generality. By the two-step market evaluation method, the market-consistent actuarial price is built by splitting the no-arbitrage financial price operator, $\mathbb{E}^{\mathbb{Q}}$, (which prices hedgeable pure financial payoff) and actuarial premium principle, $\Pi^{\mathbb{P}}$ (which prices general (partially) unhedgeable payoff). From now on, we call this operator a ``Two-step Actuarial Operator''.

In a one-period valuation setting, to calculate the two-step actuarial value of the general payoff $G(x_T, y_T)$ at time t, for each time-step of the form (t, T),

- In the first (inner) step we assume that we know all financial information up to and including time T. but our knowledge about actuarial information is up to time $t < T$ (i.e. we know y_t). We calculate the actuarial value of the payoff $G(x_T, y_T)$ under the real-world measure $\mathbb{P}$ given (x_T, y_t).
$$\Pi^{\mathbb{P}}\Big[G(x_T, y_T) \mid (y_t, x_T)\Big]$$
The result of this step, turns the general payoff G into a function exhibited by $G^S(x_T)$.

- As $G^S(x_T)$ depends only on the financial risk can be perfectly hedged due to the completeness of the financial market and no-arbitrage argument. Hence, the second (outer) step can be performed by the conditional expectation under the risk adjusted measure $\mathbb{Q}$ while we condition on x_t.
$$\mathbb{E}^{\mathbb{Q}}\left[G^S(x_T) \,\middle|\, (y_t, x_t)\right]$$
Note that, the second step is still implemented given the actuarial information via y_t.

This shows that by the two-step actuarial valuation, we use both sources of information we have in hand for the risk processes x_T and y_T to value the position over the period (t, T).

A simplified form of the two-step actuarial valuation for an actuarial premium principle Π can be defined as follows:

Definition 2.2 *A $\mathcal{G}^A$-conditional premium principle $\Pi_{\mathcal{G}^A}$ is a two-step actuarial valuation if*

$$\Pi_{\mathcal{G}_t^A}\left[G(x_T, y_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[G(x_T, y_T) \,\middle|\, (y_t, x_T)\right] \middle| \, (y_t, x_t)\right]. \tag{2.4}$$

The structure of the above two-step operator is well-defined. In the inner step under the actuarial operator Π, conditional on x_T and y_t, the only randomness is via the actuarial risk process at time T, y_T. Then, applying the outer conditional Expectation $\mathbb{E}^{\mathbb{Q}}$ conditional on x_t, the only source of randomness is x_T [2]. We emphasize that at each time-step or sub-interval, two valuation operator

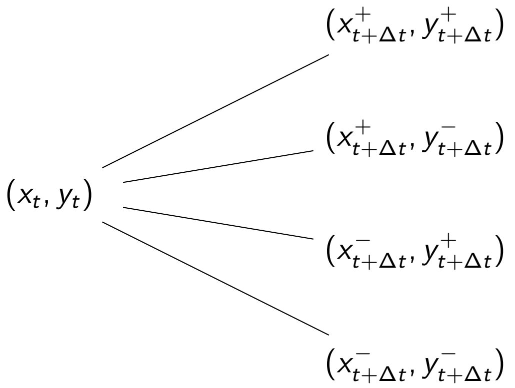
[2] The More technical representation of the two-step actuarial valuation at time t is:

$$\Pi_{\mathcal{G}_t^A}\left[G(x_T, y_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[G(x_T, y_T) \,\middle|\, \sigma\left(\mathcal{G}_t^A, \mathcal{F}_T^S\right)\right] \middle| \, \sigma\left(\mathcal{G}_t^A, \mathcal{F}_t^S\right)\right]. \tag{2.5}$$

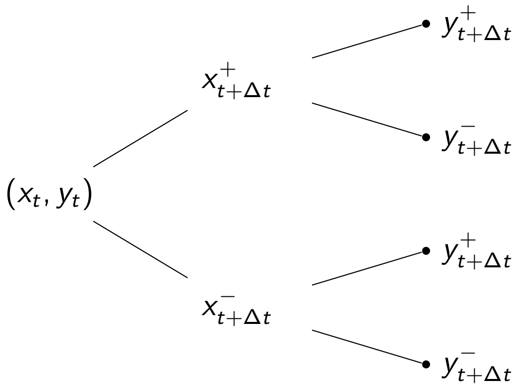
where $\sigma\left(\mathcal{G}^A, \mathcal{F}^S\right)$ is the information flow of the both risk processes. The equation (2.4) is equivalent to this operator under the assumption of markov property for the stochastic process (x_t, y_t)

needs to be applied.

A simplified exhibition of the two-step method, can be provided via a binomial discretization of the two state variables x and y. At a typical time-step $(t, t + \Delta t)$, every state (x_t, y_t) of the payoff at time t will develop to four different states of the world at time $t + \Delta t$ as below,



We pretend that first, x_t evolves ending to two different states at $t + \Delta t$. Only then, given each state of $x_{t+\Delta t}$, the process y_t moves. This leads to the following pattern



In a two-step valuation, given each state of the $x_{t+\Delta t}$ (i.e. we know whether it is x^+ or x^-) in the inner step (actuarial valuation step), we perform the actuarial valuation for nodes $y_{t+\Delta t}^{+}$ and $y_{t+\Delta t}^{-}$. Then in the outer step (financial valuation step), we have two state of the world only depending on $x_{t+\Delta t}$ where we compute the price under the binomial risk-neutral probabilities.

2.3. Two-step Actuarial Valuation in Multi-period Setting

Insurance companies and pensions fund, due to regulatory requirements or reporting purposes, need to ``re-valuate'' their liabilities at regular intervals. Suppose we stand at time zero and want to value a contingent payoff at time T. At a middle time $0 < s < T$, an economic shock, non-economic decision or evolutionary finding, can totally change the state and trend of the financial and actuarial risk drivers. we believe that such condition has a significant effect on the re-valuation of the liabilities and must be taken into account at time zero (present time) in valuation method. Our aim is to achieve this requirement in construction of the market consistent actuarial pricing operator.

Such a re-valuation requirement is consistent with the concept of time-consistency property of the valuation operator. As a result, the value of the time-T liability at time zero, should be equal to the price obtained as if we value the liability in a middle time $0 < s < T$ and then value the time-s value of liability at time zero. This can be translated to the following definition:

Definition 2.3 *A conditional valuation operator (Π_t) is Time-Consistent if and only if, for all*

$0 \le t < s \le T$ *and the non-negative payoff* $f(y(t))$,

$$\Pi_t\,[\,f(y(T))\,] = \Pi_t\,(\,\Pi_{t+s}[\,f(y(T))\,]\,)\,. \tag{2.6}$$

This is a ``recursive'' form of the time-consistency definition that we will use to construct the time-consistent actuarial operator in multi-period setting. Formal definition of time consistency can be found in **?**], **?**], and **?**].

We apply the time-consistency property to the two-step actuarial valuation to expand market-consistency, at least in a finite number of points, over the whole period $[0,\ T]$ in a dynamic setting. In fact, we are interested to preserve market-consistency for all possible middle points in valuation period. This can be achieved by applying ``Backward Iteration'' method proposed by Jobert and Rogers [15] to above two-step operator. If the valuation period $[0,\ T]$ is divided into a set of sub-intervals, the backward iteration construct the time-consistent valuation by connecting and re-valuating the one-period valuation (in our case, the two-step actuarial valuation) over the sub-intervals in a backward way starting from T. See for example, Pelsser and Salahnejhad [25] to have an overview about how time consistency can be obtained for well-known actuarial premium principles by the backward iteration method. For the two-step actuarial valuation, the idea is applied in the following general representation:

Suppose we would like to value a time-T payoff at time zero. The representation in equation (2.4) does not necessarily imply that the two-step actuarial valuation must be applied in a one-period valuation setting over $(0,\ T)$. We divide the time interval $[0,\ T]$ by a discrete set of points $\{0, \Delta t, 2\Delta t, ..., T-\Delta t,\ T\}$ into a number $n = \frac{T}{\Delta t}$ of the sub-intervals of the form $(t,\ t+\Delta t)$. The backward iteration procedure starts from time T over the last sub-interval $(T-\Delta t,\ T)$ to value the payoff $G(T, x_T, y_T)$ at time $T-\Delta t$. In the backward iteration method, we calculate the price process of a contract at time $0 \le t < T$ where at any t, the price at $t+\Delta t$ is considered as a new payoff. Let the one-period $\mathcal{G}^A$-conditional actuarial price at time $T-\Delta t$ be denoted by $\pi(T-\Delta t, x, y)$. By the two-step actuarial valuation in equation (2.4) we have,

$$\pi_{\mathcal{G}^A}\left(T-\Delta t, x_{T-\Delta t}, y_{T-\Delta t}\right) = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[G\left(T, x_T, y_T\right)\,\middle|\, y_{T-\Delta t}, x_T\right]\middle|\, y_{T-\Delta t}, x_{T-\Delta t}\right]. \tag{2.7}$$

where x and y are the shorter notation of the financial and actuarial processes $x_{T-\Delta t}$ and $y_{T-\Delta t}$, respectively.

Then we move one step backward in time and over the sub-interval $(T-2\Delta t,\ T-\Delta t)$ we assume $\pi_{\mathcal{G}^A}(T-\Delta t, x, y)$ as a new payoff where we apply (2.4) again as

$$\pi_{\mathcal{G}^A}(T-2\Delta t, x_{T-2\Delta t}, y_{T-2\Delta t}) = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left(\pi\left(T-\Delta t, x_{T-\Delta t}, y_{T-\Delta t}\right)\,\middle|\, y_{T-2\Delta t}, x_{T-\Delta t}\right)\,\middle|\, y_{T-2\Delta t}, x_{T-2\Delta t}\right].$$

We continue this backward iteration of the one-period valuation for all sub-intervals $(t,\ t+\Delta t)$ to finally reach the time step $(0, \Delta t)$ where we obtain the price of the contract at time zero. Again note that, by the two-step valuation, in each time step $(t,\ t+\Delta t)$, we perform two valuation steps: First step for the payoff conditional on the financial risk which can be considered as a pure insurance risk, in the real world measure $\mathbb{P}$, and the Second step is an arbitrage-free valuation on the financial derivative, in the equivalent martingale measure $\mathbb{Q}$.

The general time-consistent two-step valuation operator for the typical time step $(t,\ t+\Delta t)$ will be,

$$\pi\Big(t, x_t, y_t\Big) = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[\pi\Big(t+\Delta t, x_{t+\Delta t}, y_{t+\Delta t}\Big)\,\Big|\, y_t, x_{t+\Delta t}\right]\,\Big|\, y_t, x_t\right]$$

with terminal condition,

$$\pi(T, x_T, y_T) = G(T, x_T, y_T). \tag{2.8}$$

This is a general representation of the simultaneously ``*Time-consistent and Market-consistent*'' actuarial price when, in the general payoff, there is only one univariate state variable for each financial and actuarial risks. To derive the price in a theoretical setting, one can increase the number of intervals of the form $(t, t+\Delta t)$ in the limit by taking the limit of this valuation equation when $\Delta t \to 0$. As a result, the price will have time consistency and market consistency simultaneously all over the valuation period $[0, T]$.

The important point in application is that, as we mentioned in the introduction, based on [26] market-consistency resulted by applying the two-step valuation, holds even if we apply the backward iteration in a finite number of the predictable middle points of time over $[0, T]$, without taking the limit for Δt. Hence, for example if we are interested to calculate market-consistent price of a 20-year unit-linked contract, dividing the period $[0, 20]$ into any number of sub-intervals ($n = \frac{T}{\Delta t} = 1, 2, 3, ...$), and applying two-step valuation, will preserve the result as a market-consistent price. Note that, the price produced from a finite number of sub-intervals under this framework will be an approximation of the continuous-time limit of Time-Consistent and Market-Consistent price, mentioned above.

2.3.1. Time-Consistency Risk Premium

Different operators may offer different prices for the same contingent payoff. Suppose we exhibit the one-period actuarial value of the risk at time t by Π_t, and the time-consistent actuarial valuation driven by the multi-period valuation by Π_t^{TC}. We assume that both operators are constructed based on the same actuarial premium principle and all other parameters are equal. The possible fundamental difference between these two prices is only due to enforcing the time-consistency property mentioned in definition 2.3. We call this price difference as ``Time-Consistency Risk Premium (TCRP)'' and show it as

$$\text{Time-Consistency Risk Premium:} \qquad k^{\mathrm{TC}}(t, T) = \Pi_t^{\mathrm{TC}} - \Pi_t \tag{2.9}$$

In case there exist an analytical solution for the time-consistent actuarial valuation (resulted by calculating the continuous-time limit of the time-consistent operator when $\Delta t \to 0$), k^{TC} also will be obtained analytically. In application, practitioners use approximation of the time-consistent price (e.g. working with $\Delta t =$ one year) which upon use will result in the approximation of the TCRP: $\widehat{k}^{\mathrm{TC}}(t, T) = \widehat{\Pi_t^{\mathrm{TC}}} - \Pi_t$.

clearly, if we calculate the one-period and time-consistent actuarial values under the two-step actuarial valuation operator, then k^{TC} will be obtained for market-consistent price.

Interesting: It will be interesting to calculate and compare the time-consistency risk premium for different actuarial premium principles as well as market consistent and non-market consistent valuations.

2.4. Two-step Actuarial Valuation under EIOPA Standard

As we mentioned before, for the long-term liabilities, the one-period setting is not flexible enough to reflect the possible middle time changes of underlying risk drivers conditional on the new information. It normally construct the price by projecting the risk drivers with information at start time and accordingly with determined initial values of parameters. In the price/valuation operators such as Cost-of-Capital principle that include the risk measures with a one-year scale such as VaR, this problems gets high importance. To improve such a shortcoming in the Cost-of-Capital principle, EIOPA standard offers an adjusted version of the operator as follows

$$\Pi_t^{EIOPA}\left[f(y_T)\right] = \mathbb{E}\left[f(y_T) \mid y_t\right] + \delta \sum_{k=1}^{T-t} \mathrm{VaR}_q\left[f(y_{t+k}) - \mathbb{E}\big[f(y_{t+k}) \mid y_t\big] \,\Big|\, y_{t+k-1}\right]. \tag{2.10}$$

where Π_t is the conditional Cost-of-Capital operator with available information at time $t < T$ and $k = 1, 2, ..., T - t$. The base value of the price is still the one-period best-estimate of the payoff at time T computed by the conditional expectation $\mathbb{E}\left[f(y_T) \mid y_t\right]$. Comparing to the one-period setting, the loading part of the operator is adjusted so that the conditional VaR value of the payoff at maturity T given the available information at time t is broken down into $T - t$ one-year VaR values each conditional on the information available one year earlier. This formulates the risk measurement as follows: over the valuation period $[t, T]$ for each point of time $t + k \leq T$, based on the information available at time t, we make a projection for the future development of the payoff $f(y_{t+k})$. Walking along the best-estimate of the payoff for each point of time k, we measure the deviation of the possible projection and the best-estimate payoff by the one-year conditional value at risk; $\mathrm{VaR}_q\left[f(y_{t+k}) - \mathbb{E}\big[f(y_{t+k}) \mid y_t\big] \,\big|\, y_{t+k-1}\right]$ which is calculated given the information available one year earlier at $t + k - 1$. Finally the summation of the $T - t$ one-year VaR values are presented as the risk loading for the actuarial premium over $[t, T]$. This improved version of the operator in equation (2.10) is suggested by the directives provided by EIOPA[3] and is widely used by the practitioners in industry to valuate the long-term liabilities. We call the operator in equation (2.10) operator as ``EIOPA Cost-of-Capital actuarial operator''.

We apply the two-step actuarial valuation in equation (2.4) to EIOPA Cost-of-Capital operator Π_t^{EIOPA} in equation (2.10). The resulted market-consistent actuarial price for a general payoff $G(x_T, y_T)$ will have the following form:

$$\Pi_{\mathcal{G}_t^A}\left[G(x_T, y_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\Pi_t^{EIOPA}\left[G(x_T, y_T) \,\Big|\, (y_t, x_T)\right] \,\Big|\, (y_t, x_t)\right] \tag{2.11}$$

where

$$\begin{aligned}\Pi_t^{EIOPA,CoC}\left[G(x_T, y_T) \mid (y_t, x_T)\right] &= \mathbb{E}^{\mathbb{P}}\left[G(x_T, y_T) \,\big|\, (y_t, x_T)\right] \\ &+ \delta \sum_{k=1}^{T-t} \mathrm{VaR}_q^{\mathbb{P}}\left[G(x_T, y_{t+k}) - \mathbb{E}^{\mathbb{P}}\left[G(x_T, y_{t+k}) \,\big|\, (y_t, x_T)\right] \mid (y_{t+k-1}, x_T)\right]. \end{aligned} \tag{2.12}$$

In the inner step, we calculate the best-estimate value of the risk given the value of financial risk x_T and actuarial risk y_t. We add the summation of the one-year VaR values for each year k while each of them are conditional on the known financial state variable x_T and the insurance process one year earlier, y_{k-1}. The output of the inner step again by structure is a payoff depending only on x_T, that

[3] European Insurance and Occupational Pension Authority

can be valuated by the conditional expectation under the risk-adjusted measure $\mathbb{Q}$, given x_t. Note that, conditional on the pair (y_{t+k-1}, x_T), the conditional expectation $\mathbb{E}^{\mathbb{P}}\left[G(x_T, y_{t+k}) \mid (y_t, x_T)\right]$ can come out of the VaR operator due to the translation invariance property.[4]

The above mechanism can be applied with the Standard-Deviation premium principle via the summation of the one-year standard deviation of the risk conditional on the actuarial information one year earlier as below,

$$\Pi_t^{EIOPA,Std}\left[G(x_T, y_T) \mid (y_t, x_T)\right] = \mathbb{E}^{\mathbb{P}}\left[G(x_T, y_T) \mid (y_t, x_T)\right] + \beta \sum_{k=1}^{T-t} \mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[G(x_T, y_{t+k}) \mid (y_{t+k-1}, x_T)\right]. \quad (2.13)$$

[4]Technically saying $\mathbb{E}^{\mathbb{P}}\left[G(x_T, y_{t+k}) \mid (y_t, x_T)\right]$ is not $\sigma\left(\mathcal{G}^A_{t+k-1}, \mathcal{F}^S_T\right)$-measurable.

3. Market-Consistent Valuation of Unit-linked Contract

We start with a simple unit-linked contract for a policyholder of age x which participates in the contract by buying one unit of it at start time $t = 0$. We assume the contract has no guarantee. The payoff is defined as a combination of the financial and actuarial risk,

$$G(S_T, T_x) = S_T \times \mathbb{1}_{\{T_x > T\}} \tag{3.1}$$

where T is the maturity time, S_T is the market value of the investment made by the premiums, and T_x is the remaining life time random variable for the policyholder x. Normally in unit-linked contracts T is the retirement age minus age. For this case we assume $T = 70 - x$.

3.1. Risk Drivers

In order to model the payoff and implement the two-step actuarial valuation, we need to model both financial risk S_t and actuarial risks T_x.

We assume that S_T as a financial risky asset follows a Geometric Brownian Motion (GBM)

$$dS_t = \mu_S \, S_t \, dt + \sigma_S \, S_t \, dW_t^S. \tag{3.2}$$

Here μ_S and σ_S are constant and W_t^S is a standar Brownian motion defined on filtration $\mathcal{F}^S(t)$ on the time interval $[0, T]$ under the real-world measure $\mathbb{P}$. For the assumed complete financial market, we will work with the unique equivalent martingale measure $\mathbb{Q}$. Hence, for a constant continuously compounded interest rate r, we have

$$dS_t = r \, S_t \, \mathrm{d}t + \sigma_S \, S_t \, \mathrm{d}\tilde{W}_t^S. \tag{3.3}$$

with the solution

$$S_t = S_0 \exp\left((r - \frac{1}{2}\sigma_S^2)t + \sigma_S \tilde{W}_t^S \right). \tag{3.4}$$

where S_0 is the initial value of the investment and $\tilde{W}_t^S$ is $\mathbb{Q}$-standard Brownian motion.

In the context of actuarial risk, we start with defining some notations. we denote the survival and death probability of (x) by ${}_tp_x = \mathbb{P}(T_x > t)$ and ${}_tq_x = 1 - {}_tp_x$, respectively, and the force-of-mortality of the same individual at time t is defined as $\mu(x+t) = -\frac{d}{dt} \ln \left({}_tp_x\right)$ [5].

By dynamic mortality, we assume that ${}_tp_x$ is random in any future time t and effectively $\mu(x+t)$ as an index of the mortality risk follows a stochastic process. To model the evolution of the survival probability and consider projection of the future mortality/longevity index, we use the model introduced by Lee and Carter [19]. In the Lee-Carter model, for an integer age x and calendar year t, the force-of-mortality (intentionally denoted by the different notation compared to static force-of-mortality), $\mu_x(t)$, is assumed to be given by

$$\ln \mu_x(t) = \alpha_x + \beta_x \kappa(t) \tag{3.5}$$

where $\kappa(t)$ is the general level of mortality, α_x is the average age-specific mortality, β_x is the age-specific sensitivity of the mortality to change of $\kappa(t)$. With an alternative point of view, $\kappa(t)$ can be interpreted as a latent process to model the longevity trend, specified by $\kappa(0) = 0$ and

$$\mathrm{d}\kappa_t = \mu_\kappa \mathrm{d}t + \sigma_\kappa \mathrm{d}W_t^\kappa, \tag{3.6}$$

[5] Considering the stochastic evolution of the mortality risk through time, the more precise concept will be ``the remaining life time at the beginning of the calendar year t'' for which the notation is $T_x(t)$.

with the solution $\kappa_t = \kappa_0 + \mu_\kappa t + \sigma_\kappa W_t^\kappa$ and W^κ a standard Brownian motion under measure $\mathbb{P}$.[6] The best estimate for κ_t process at time zero is,

$$\mathbb{E}(\kappa_t) = \kappa_0 + \mu_\kappa t.$$

The general procedure of using Lee-Carter model starts by choosing the realized mortality rates of the past calendar years and continues by calibrating the parameters of stochastic mortality model, α_x and β_x. It is noteworthy to mention that in the notation, if we assume the present $t = 0$, the notation t is the representative of the past ``calendar times'' $\{t_0, t_0 + 1, \dots, 0\}$ in the model. Once the age-specified Lee-Carter parameters are estimated, the future force-of-mortality can be projected for individual (x) and in that sense $t > 0$ is the notation of future time. In this paper, we have the estimated parameters of the Lee-Carter model in hand via the most recent realizations of the mortality rates in the Netherlands and we concentrate on simulation of the future mortality rates.

Conditional on $\kappa(t)$, the remaining lifetimes of the policyholders are assumed to be independent. Moreover, the force-of-mortality of and individual that has age $x + t$ at future time t, is given by

$$\mu_x(t) = \exp\left(\alpha(x+t) + \beta(x+t)\,\kappa_t\right) \tag{3.8a}$$
$$= \exp\left(\alpha(x+t) + \beta(x+t)\mu_\kappa t + \beta(x+t)\sigma_\kappa W_t^\kappa\right). \tag{3.8b}$$

Note that $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ are assumed to be piecewise constant on the intervals $[t, t+1)$. This means that for all $u \in [0,1)$, $\mu_{x+u}(t+u) = \mu_x(t)$.

As we may valuate the price of a product for a group of participants, let $N_x(t)$ denote the cohort of policyholders of age $x + t$ at time t. We assume the death event of the group members are conditionally independent. Then $N_\ell(t+1)$ has, conditional on (κ_t) and $N_x(t)$, a binomial distribution with parameters $N_x(t)$ and success probability $\exp(-\alpha(x+\ell) - \beta(x+\ell)(\kappa_{t+1} - \kappa_t))$.

Note that κ_t is the underlying process of the remaining life time random variable T_x. Moreover, consistent with Lee-carter model, we define the actuarial filtration as $\mathcal{G}_t^A = \sigma\left(\{T_x \le s\}, \kappa_s, s \le t\right)$ where κ_s is part of the information set. Note that, we only focus on the systemic risk where the idiosyncratic mortality risk should be a second-order effect compared to the effect of the longevity trend. This assumption could be motivated by observing realized mortality tables mentioned above. We therefore can rephrase the unit-linked payoff based on the underlying risk drivers

$$G(S_T, \kappa_T) = S_T \times \mathbb{1}_{\{\kappa_T : T_x > T\}}. \tag{3.9}$$

3.2. One-period Market-Consistent Valuation

Suppose that we want to calculate the market-consistent actuarial value of the payoff (3.9) at time $t < T$ by a two-step valuation in equation (2.4). Let $\Pi_t^{\mathcal{G}^A}$ be the Cost-of-Capital premium principle and the interest rate r be constant over the period. The two-step Cost-of-Capital actuarial value at time t will be:

$$\Pi_t^{\mathcal{G}^A}\left[S_T \mathbb{1}_{\{T_x>T\}}\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S_T \mathbb{1}_{\{T_x>T\}} \,\middle|\, \kappa_t, S_T\right] + \delta\sqrt{T-t}\,\mathrm{VaR}_q^{\mathbb{P}}\left[S_T \mathbb{1}_{\{T_x>T\}} - \mathbb{E}^{\mathbb{P}}\left[S_T \mathbb{1}_{\{T_x>T\}} \mid \kappa_t, S_T\right] \,\middle|\, \kappa_t, S_T\right] \,\middle|\, \kappa_t, S_t\right]. \tag{3.10}$$

where q is the confidence level of the VaR and δ is the annual cost of capital to borrow the capital buffer for a certain period. Since VaR measures the risk over the period of one-year, for a

[6] Note that κ can also be modeled as below

$$\kappa_{t_k} = \mu_\kappa + \kappa_{t_k} + \varepsilon_{t_k:t_{k+1}}, \tag{3.7}$$

with $\varepsilon_{t_k:t_{k+1}}$ i.i.d. $N(0, \sigma_\kappa^2)$, which is the time series model used in Lee and Carter (1992).

shorter/longer period $T-t$ we need to adjust it by dividing the VaR operator over the period $[t, T]$ by $\sqrt{T-t}$. On the other hand, as δ operates like an interest rate, for a period of length $T-t$, the insurance company will have to pay $\delta\sqrt{T-t}$ per € to the lender of the capital buffer. In sum up, to adjust the time scale for VaR and δ, we adjust δ by $\delta \times (T-t)/\sqrt{T-t} = \delta\sqrt{T-t}$ and multiply that to VaR operator. For more details on dimensionality problems, see Pelsser and Salahnejhad [25].

What happens in practice is that, we fix S_T and calculate the Cost-of-Capital premium for the remaining part. This produces a payoff that is a function of S_T. Then, in a Black-Scholes setting we take the expectation of the S_T-payoff under measure $\mathbb{Q}$ and discount it to time t.

Note that the expectation under VaR operator can be factorized by its translation invariance property and (3.10) can be represented by a shorter version as follows:

$$\Pi_t^{\mathcal{G}^A}\left[S_T \mathbb{1}_{\{T_x>T\}}\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(1-\beta\sqrt{T-t}\right)\mathbb{E}^{\mathbb{P}}\left[S_T\mathbb{1}_{\{T_x>T\}}\,\middle|\,\kappa_t, S_T\right] + \delta\sqrt{T-t}\,\text{VaR}_q^{\mathbb{P}}\left[S_T\mathbb{1}_{\{T_x>T\}} \mid \kappa_t, S_T\right]\,\middle|\,\kappa_t, S_t\right]. \tag{3.11}$$

3.2.1. Analytical Solution under Independence

We are interested to obtain an analytical solution for the two-step actuarial value of the unit-linked contract. This helps up to have a reliable benchmark about our numerical implementation. For simplicity we assume that actuarial risk is independent of the financial risk, while in reality this is a rational assumption as we don't expect people die or live longer by the trend of the stock market. By independence assumption and due to the ``factorisation structure'' of unit-linked payoff at (3.9), the two-step actuarial evaluation can be written as follows,

$$\begin{aligned}\Pi_{\mathcal{G}_t^A}\left[S_T\mathbb{1}_{\{T_x>T\}}\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[S_T\mathbb{1}_{\{T_x\geq T\}}\,\middle|\,\kappa_t, S_T\right]\,\middle|\,\kappa_t, S_t\right]\\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mid S_t\right]\times \Pi^{\mathbb{P}}\left(\mathbb{1}_{\{T_x\geq T\}} \mid \kappa_t\right)\\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mid S_t\right]\times\left(\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{T_x\geq T\}}\,\middle|\,\kappa_t\right] + \delta\sqrt{T-t}\,\text{VaR}_q^{\mathbb{P}}\left[\mathbb{1}_{\{T_x\geq T\}} - \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{T_x\geq T\}}\,\middle|\,\kappa_t\right]\,\middle|\,\kappa_t\right]\right)\end{aligned} \tag{3.12}$$

where on the actuarial operator conditioning on S_T is omitted due to independence. Similar representation can be found at Pelsser and Stadje [26] (Remark 3.8) and Wüthrich et al. [31].

By Black-Scholes formula we know that, $e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mid S_t\right] = S_0$. Moreover, $\mathbb{1}_{\{T_x\geq T\}}$ has a Bernoulli distribution with survival probability $p = Pr(T_x \geq T) = {}_Tp_x$ as parameter. Therefore, the value of $\text{VaR}_q^{\mathbb{P}}\left[\mathbb{1}_{\{T_x\geq T\}}\right]$ function is,

$$\text{VaR}_q^{\mathbb{P}}\left(\mathbb{1}_{\{T_x\geq T\}}\right) = \begin{cases} 1, & 1-q < {}_Tp_x;\\ 0.5, & 1-q = {}_Tp_x;\\ 0, & 1-q > {}_Tp_x.\end{cases} \tag{3.13}$$

Note that $\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{T_x\geq T\}}\,\middle|\,\kappa_t\right]$ is $\mathcal{G}_t^A$-measurable and as a non-random value by ``translation invariance'' property of VaR, we can take it out and simplified the calculation.

Hence, under independence assumption, we can obtain the analytical solution of the one-period market-consistent actuarial price of the unit-linked product in (3.12) at time $t=0$ as below,

$$\pi_0 = \Pi_{\mathcal{G}_0^A}\left[S_T\mathbb{1}_{\{T_x>T\}}\right] = S_0\left(\left(1-\delta\sqrt{T}\right){}_Tp_x + \delta\sqrt{T}\begin{cases} 1, & 1-q < {}_Tp_x;\\ 0.5, & 1-q = {}_Tp_x;\\ 0, & 1-q > {}_Tp_x.\end{cases}\right) \tag{3.14}$$

In a real world case (e.g Solvency II directives), $q = 99.5\%$ and for most of the ages the survival probability will be greater than that; $1-q < {}_Tp_x$. Thus, with such an assumption the analytical solution of the one-period two-step valuation of the unit-linked contract at time zero is,

$$\pi_0 = \Pi_{\mathcal{G}_0^A}\left[S_T \mathbb{1}_{\{T_x>T\}}\right] = S_0\left({}_Tp_x + \delta\sqrt{T}\left(1 - {}_Tp_x\right)\right). \tag{3.15}$$

Note that the ``Best Estimate'' value of the unit-linked payoff in (3.9) in the one-period two-step evaluation setting is,

$$\pi_t^{BE}\left(S_T \mathbb{1}_{\{T_x>T\}}\right) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[S_T \mathbb{1}_{\{T_x\geq T\}} \,\middle|\, \mathcal{G}_t^A, \mathcal{F}_T^S\right] \,\middle|\, \mathcal{G}_t^A, \mathcal{F}_t^S\right] \tag{3.16}$$

where by considering independence assumption, the best-estimate price at time $t = 0$ will be,

$$\pi_0^{BE} = \mathbb{E}_{\mathcal{G}_0^A}\left[S_T \mathbb{1}_{\{T_x>T\}}\right] = S_0\,{}_Tp_x \tag{3.17}$$

The best estimate value is basically the conditional expectation of the payoff and it does not consider the fact that the longevity risk is unhedgeable. On the other hand, the actuarial value in equation (3.15) adds a risk premium (also called ``risk loading'') $\beta\sqrt{T}(1 - {}_Tp_x)$ to compensate for the penalty needed for the cost of hedge. In equation (3.15) the maturity time T has direct relationship with the actuarial risk premium while the survival probability ${}_Tp_x$ has negative effect on it. This implies that longer period contracts will result in relatively higher price. Also, if the survival probability is lower (i.e. the policyholder is old or the pricing period is long), the price will be adjusted by adding a higher value of $(1 - {}_Tp_x)$ to the expected value part of the price which is lower. On the other hand, if the survival probability is high (i.e. the policyholder is young or the period is short), the major part of the price is reflected in the expected value that makes higher confidence for the insurer who will require in return less risk premium.

3.2.2. Numerical Implementation

We setup a numerical implementation of the two-step actuarial valuation for unit-linked product with dependent risks, where the same procedure can be generalized for other products even with more risk factors. For simplicity we generate an equal number of scenarios, let us say N, for both risks so that at maturity T, we have N, pair of $(S_T, N_x(T))$. Note that when $N_x = 1$, then $N_x(T) = \mathbb{1}_{\{T_x>T\}}$. Fix the needed parameters such as the present time $t = 0$, maturity T, age x (accordingly the parameters of the Lee-Carter model $\alpha(x)$ and $\beta(x)$), starting cohort N_x, and drift and diffusion of the models. A simple numerical implementation is as follows:

1. Simulate N scenarios of the S_T and κ_T and use κ_T vector to compute $\mu_x(T)$ at time $x + T$ via equation (3.8a).

2. Use $\mu_x(T)$ to compute the survival probability ${}_Tp_x$ and simulate N scenarios of the survivals at time T, $N_x(T)$, using Binomial distribution with parameters N_x and ${}_Tp_x$.

3. Fix S_T and calculate the actuarial value of the payoff $S_T \times N_x(T)$ via an empirical analogue of the actuarial premium principle.

4. Calculate the mean over the resulting N values of financial payoff and discount it.

5. Repeat the steps 1-6 n times and compute the mean over the all simulated prices.

Table 1: Analytical and numerical values of the unit-linked contract with a one-period pricing operator for different maturities

		Analytical Price			Numerical Price	
Age (Year)	Duration (Year)	Best Estimate	Cost of Capital	Extra Loading	Best Estimate	Cost of Capital
69	1	98.26	98.37	0.11 (0.1%)	98.23 (0.0%)	98.33 (0.0%)
60	10	89.71	91.66	1.95 (2.2%)	89.52 (0.2%)	91.50 (0.2%)
50	20	87.23	90.65	3.42 (3.9%)	86.82 (0.5%)	90.30 (0.4%)

In this paper, we use the mortality data aggregated for ``men and women'' of the Netherlands during the calendar years 1960-2006 (where $t_0 = 1960$), to estimate the parameters of Lee-carter model. Then after for pricing issue, we use 2006 as ``$t = 0$''. This is always the time when the individual is/was at the age x. Based on the estimation output of the above mortality tables the diffusion process to model the mortality trend κ_t in equation (3.6) is estimated as

$$\kappa_t = -0.8089dt + 1.4734dW_t^{\kappa} \tag{3.18}$$

where $\kappa_0 = -24.5637$ is the initial value of κ_t process.

To benchmark the above numerical procedure, we calculate both analytical and numerical price of the unit-linked contract for an individual in a one-period setting. We valuate the contract for which its financial and actuarial parameters are $S_0 = 100$, $r = 0.04$, $\sigma_S = 0.15$. We use the Cost-of-Capital principle with $\delta = 0.06$ and $1 - q = 0.995$ and apply the equation (3.12) to valuate the contract. We repeated the simulation $n = 100$ times where each time, S_T and κ_T are simulated with $N = 3000$ scenarios and are assumed to be independent.

Similar to (3.9), the payoff is

$$G(S_T, \kappa_T) = S_T \times \mathbb{1}_{\{\kappa_T : T_x > T\}}.$$

In the calculation, the best-estimate value of the survival probability ${}_Tp_x$ is calculated via the equation

$${}_Tp_x = exp(-\mu_x(T))$$

where the best-estimate $\mu_x(T)$ can be obtained by replacing the best-estimate κ_T in equation (3.8a) as well as the other generated scenarios. For example, the best-estimate values for an individual of age $x = 60$ with $T = 10$ years maturity is,

$$BE(\kappa_{10}) = -24.5637 - 0.8089 \times (10) = -32.6526$$

$$BE(\mu_{69}(1)) = exp(\alpha(70) + \beta(70) \times BE(\kappa_{10}) = exp(-3.6715 + 0.0147 \times (-32.6526)) = 0.0157$$

$${}_{10}p_{60} = exp(-\sum_{i=0}^{9} BE(\mu_{60+i}(1))) = 0.8971.$$

Table 1 represents the result of pricing for three starting age x from the set $\{50, 60, 69\}$ where for the retirement age of 70 we will have three maturities T in $\{20, 10, 1\}$. First of all, it can be seen that the numerical algorithm converges to the analytical solution so that the relative difference between the two prices are less than 0.5%. Note that, for the shorter period the difference is very trivial while for the longer periods, we expect that generating more scenarios will result in more accurate numerical value.

Also comparing the best-estimate value $S_0 \times {}_Tp_x$ with Cost-of-Capital premium, confirms that for the longer period contracts more loading is needed. For a one-year period in the example, it is around 0.1% while for a 20-year contract it reached to 3.9%. This is a natural expectation from the price operator to reflect the fact that the longer maturity T, will bring the higher volatility and uncertainty of the the both financial and actuarial risk processes for the insurer and this must be taken into account in the price.

3.3. Multi-Period Market-Consistent Valuation

Now we turn to time-consistent and market-consistent valuation of the simple unit-linked contract. For every time step $(t, t + \Delta t)$, we perform the backward iteration of the valuation in equation (3.10) or alternatively equation (3.12) under independence assumption, while the procedure starts from maturity time T. Although in applied situations it would be more practical to take $\Delta t = 1$ (1-year period), to check the result of the numerical procedure, we need a benchmark that can be obtained by the analytical solution of the price in dynamic setting.

3.3.1. Dynamic Analytical Solution under Independence

The analytical solution will be a continuous-time limit of the time-consistent two-step actuarial valuation in (2.8) when $\Delta t \to 0$. We continue by independent S_t and κ_t, where we use equation (3.12) to apply the operator $\pi_{T-\Delta t} = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}[S_T \mid S_{T-\Delta t}] \times \Pi^{\mathbb{P}}[f(\kappa_T) \mid \kappa_{T-\Delta t}]$ and obtain the two-step actuarial value of the unit-linked contract at time $T - \Delta t$. Then, for the period $(T - 2\Delta t, T - \Delta t)$ we apply the time-consistent and market-consistent operator in (2.8) as follows,

$$
\begin{aligned}
\pi_{T-2\Delta t} &= e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[\pi_{T-\Delta t} \mid \kappa_{T-2\Delta t}, S_{T-\Delta t}\right] \;\middle|\; S_{T-2\Delta t}\right] \\
&= e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}[S_T \mid S_{T-\Delta t}] \times \Pi^{\mathbb{P}}[f(\kappa_T) \mid \kappa_{T-\Delta t}] \;\middle|\; \kappa_{T-2\Delta t}, S_{T-\Delta t}\right] \;\middle|\; S_{T-2\Delta t}\right] \\
&= e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}[S_T \mid S_{T-\Delta t}] \times \Pi^{\mathbb{P}}\left[\Pi^{\mathbb{P}}[f(\kappa_T) \mid \kappa_{T-\Delta t}] \;\middle|\; \kappa_{T-2\Delta t}\right] \;\middle|\; S_{T-2\Delta t}\right] \\
&= e^{-2r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[S_T \mid S_{T-\Delta t}] \;\middle|\; S_{T-2\Delta t}\right] \times \Pi^{\mathbb{P}}\left[\Pi^{\mathbb{P}}[f(\kappa_T) \mid \kappa_{T-\Delta t}] \;\middle|\; \kappa_{T-2\Delta t}\right] \\
&= S_{T-2\Delta t}\Pi^{\mathbb{P}}\left[\Pi^{\mathbb{P}}[f(\kappa_T) \mid \kappa_{T-\Delta t}] \;\middle|\; \kappa_{T-2\Delta t}\right]
\end{aligned}
\tag{3.19}
$$

where the third equality is obtained as $\mathbb{E}^{\mathbb{Q}}[S_T \mid S_{T-\Delta t}]$ is $S_{T-\Delta t}$-measurable, forth equality is provided due to independence and the last one is clear by tower property and martingale property of S_t under risk-neutral measure $\mathbb{Q}$.

If we repeat the above valuation procedure in a backward way for all time intervals $(t, t + \Delta t)$ in $[0, T]$ ending by $t = 0$, and take the limit when $\Delta t \to 0$, we can obtain the analytical solution of the time-consistent and market-consistent actuarial valuation in (3.19). Pelsser and Salahnejhad [25] showed that in continuous-time when $\Delta t \to 0$, for some well-known actuarial premium principles Π, the value of the nested actuarial operator

$$
\pi(t, \kappa(t)) = \Pi^{\mathbb{P}}\left[\Pi^{\mathbb{P}}[f(\kappa_{t+2\Delta t}) \mid \kappa_{t+\Delta t}] \;\middle|\; \kappa_t\right]
$$

has an analytical solution. They showed that for the Cost-of-Capital price, $\Pi^c = \mathbb{E}[f(\kappa_T)|\kappa_t] + \delta\sqrt{T-t}\text{VaR}_q\left[f(\kappa_T) - \mathbb{E}[f(\kappa_T)]|\kappa_t\right]$, the analytical solution of the $\pi(t, \kappa_t)$ is as follows,

$$\pi^c(t, \kappa(t)) = \mathbb{E}^{\mathbb{C}}\left[f(\kappa_T)|\kappa_t\right], \tag{3.20}$$

where $\mathbb{E}^{\mathbb{C}}[.|\kappa_t]$ denotes the conditional expectation at time t with respect to the ``risk-adjusted'' process $\kappa^{\mathbb{C}}$ defined as

$$d\kappa_t^{\mathbb{C}} = \left(\mu_\kappa \pm \delta k \sigma_\kappa\right) dt + \sigma_\kappa \, dW_t^{\mathbb{C}}. \tag{3.21}$$

and $k = \Phi^{-1}(1-q)$ for standard normal distribution function Φ.

Having this theoretical result in hand, the continuous time limit of the two-step Cost-of-Capital price in equation (3.19) at time zero, π_0^c, for the payoff function $S_T f(\kappa_T) = S_T \mathbb{1}_{\{T_x > T\}}$ will be,

$$\pi_0^c = S_0 \mathbb{E}^{\mathbb{C}}\left[\mathbb{1}_{\{T_x > T\}}\middle|\kappa_0\right] = S_0 \, {}_T p_x{}^{\mathbb{C}} \tag{3.22}$$

where ${}_T p_x{}^{\mathbb{C}}$ is survival probability of an individual age x for T years later, under the risk-adjusted $\kappa^{\mathbb{C}}$ process in (3.21).

π^{CoC} is, by construction, a time-consistent and market-consistent analogue of the one-period market-consistent Cost-of-Capital price π_0 of the simple unit-linked contract, in equation (3.14) and (3.15).

Moreover, Pelsser and Salahnejhad [25] proved that if instead of the Cost-of-capital price, we use the Standard-Deviation actuarial principle of the form

$$\Pi^s = \mathbb{E}[f(\kappa_T)|\kappa_t] + \beta\sqrt{T-t}\sqrt{\mathbb{V}\text{ar}\left[f(\kappa_T)|\kappa_t\right]}$$

the time-consistent and market-consistent Standard-deviation price π^s will converge to the same limit as of the Cost-of-Capital price π^c in (3.22),

$$\pi_0^s = S_0 \, {}_T p_x{}^{\mathbb{S}} \tag{3.23}$$

while an slightly different risk-adjusted process for the underlying risk driver $\kappa_t^{\mathbb{S}}$ is as follows:

$$d\kappa_t^{\mathbb{S}} = \left(\mu_\kappa \pm \beta \sigma_\kappa\right) dt + \sigma_\kappa \, dW_t^{\mathbb{S}}. \tag{3.24}$$

This implies that for independent actuarial and financial risks, we can use Standard-Deviation price instead of Cost-of-Capital price for dynamic market-consistent valuation of the insurance products which their payoff has a factorization structure similar to the simple unit-linked contract.

The most impressive result is that, the time-consistent and market-consistent actuarial price achieved by both of the Cost-of-Capital ad Standard-deviation premium principles in (3.22) and (3.23) corresponds to the ``Best Estimate'' price of the unit-linked contract in (3.17), where the underlying actuarial risk process, $\kappa_t$, needs to be adjusted through (3.21) or (3.24). The risk-adjustment of the longevity trend $\kappa_t$ intuitively implies using the concept of ``actuarial prudence'' for pricing where the premium offered by the insurer uses an adjusted drift to make a risk loading on top of the expected loss as a more conservative assessment of risk. According to the risk-adjusted process in (3.24), when the payoff $f(\kappa)$ is monotonically increasing at κ, the drift rate is adjusted upwards ($\mu_\kappa + \beta\sigma_\kappa$). For a negative μ_κ, this will offer a more expensive price for the products containing a survival benefit. The opposite happens when $f(\kappa)$ monotonically decreases at κ. This makes the drift adjusted downward ($\mu_\kappa - \beta\sigma_\kappa$) and accordingly offers a higher price for the payoff promising to pay on the event of death. So, the risk is always adjusted in the upward direction, providing the market-consistent price higher than the real-world expected value $\mathbb{E}^{\mathbb{P}}[f(\kappa)]$.

3.3.2. Numerical Method for Multi-period

As by backward iteration method we have to repeat the valuation process in a multi-period setting for every time step, opting an efficient numerical method is highly important. On the other hand, in the two-step actuarial valuation we apply conditional operators at each time step given the state of the underlying processes at previous step. Suppose we price a unit-linked product with 30 years maturity where we repeat the two-step valuation on an annual basis and we start with only 3 scenario for each of the underlying processes κ_t and S_t. If naively we apply a nested simulation of the underlying processes at each time step the calculation will be explode. After 30 step we will have $3^{30} = 2.059 \times 10^{14}$ scenarios for each process to start the calculation. Note that if S_T and κ_T are dependent, in the first step we will have to make $(3^{30})^2 = 4.24 \times 10^{28}$ calculation. Note that, this number is subject to increase by adding the number of calculation when we repeat valuation in the backward iteration procedure.

A better technique to deal with the dynamic two-step actuarial valuation is constructing a Markov grid via the method called ``finite difference interval''. An efficient special case of this method can be implemented by construction of a ``recombining trinomial tree''. We have to discretize $S - T$ and κ_T each of which into a finite number of states where they are both located in their special boundary region. Each state will be connected to three nodes Suppose we divide each underlying process into 1000 states (i.e. scenarios) at time T. Applying two-step valuation for the first time step $(T-1, T)$, for each state of S_T in outer step (i.e. assuming S_T known), we have 1000 trinomial computation of the inner actuarial operator. Hence, moving backward over 30 years, for each time step we repeat 10^6 computation which gathers to 3×10^7. Perhaps in a low dimension such as two-factor model the finite difference method is , but in higher dimension where in application we have to work with a portfolio of assets and liabilities, this is neither promising efficiency.

The more efficient method is Least Square Monte Carlo (LSMC) method also call ``regression now'' as it uses regression method to value . LSMC was proposed and used by Carriere [5] and Longstaff and Schwartz [20] to price different types of American options. The method is then widely used in dynamic valuation of contingent claims and payoffs with path-dependent risk drivers. LSMC postulates that the conditional expectation of payoffs can be calculated by using the cross-sectional information of the underlying risk drivers (i.e. state variables).

To price the unit-linked contract by LSMC, we expect that the conditional expectation of any general payoff $G(T, S_T, \kappa_T)$, can be obtained by a series of basis function of S_T and κ_T as,

$$\mathbb{E}\Big[G\big(T, S_T, \kappa_T\big) \mid \kappa_t, S_t\Big] = f(S_T, \kappa_T) = \sum_{k=0}^{\infty} a_k\, e_k\left(\kappa_t, S_t\right) \tag{3.25}$$

where $e_k(x)$ denotes different types of the basis functions such as,

- Polynomials: $1, x, x^2, \ldots$
- Fourier basis: $1, cos(x), cos(2x), \ldots.$

The target function $f(S_T, \kappa_T)$ at (3.25) can be approximated by a finite number of terms in the series that turns it to a regression line

$$f(S_T, \kappa_T) \approx f_k(S_T, \kappa_T) = \sum_{k=0}^{K-1} a_k\, e_k\left(\kappa_t, S_t\right). \tag{3.26}$$

The fitted value is an estimation of the conditional expectation and can be used to accurately estimate conditional expectations for different forms of G.

We are interested to calculate the two-step valuation of the unit-linked contract above in a multi-period setting and we will repeat this valuation over each year. We use the Standard-Deviation premium principle as the inner actuarial operator in the equation (2.8). To implement the LSMC for two-step actuarial valuation of the unit-linked contract the procedure can be as follows,

1. Simulate N scenarios of the pair (S_t, κ_t) over the finite number of points for $t = 1, 2, ..., T$ and at time T calculate the payoff $S_T \times \mathbb{1}_{\{T_x > T\}}$ (or alternatively $S_T \times N_x(T)$). This can be done by steps 1-4 of the numerical procedure in Subsection 3.2.2.

2. choose the form of e_k and number of terms in the series.

3. Starting by the time step $(T-1, T)$ apply the **regressions for the first/inner step** as below

$$\widehat{\mathbb{E}^{\mathbb{P}}}\left[S_T \mathbb{1}_{\{T_x > T\}} \mid S_T, \kappa_{T-\Delta t}\right] = \sum_{k=0}^{K-1} a_k^{(1,T)} e_k(S_T, \kappa_{T-\Delta t}) \tag{3.27a}$$

$$\widehat{\mathbb{E}^{\mathbb{P}}}\left[(S_T \mathbb{1}_{\{T_x > T\}})^2 \mid S_T, \kappa_{T-\Delta t}\right] = \sum_{k=0}^{K-1} a_k^{(2,T)} e_k(S_T, \kappa_{T-\Delta t}) \tag{3.27b}$$

and calculate the conditional premium $\pi^s(S_T\,,\ \kappa_{T-\Delta t})$ that depends on $\kappa_{T-\Delta t}$ and S_T as below,

$$\pi^s(S_T\,,\ \kappa_{T-\Delta t}) = e^{-r\Delta t}\left[\widehat{\mathbb{E}^{\mathbb{P}}}\left[S_T \mathbb{1}_{\{T_x > T\}}\right] + \beta\sqrt{\Delta t}\sqrt{\widehat{\mathbb{E}^{\mathbb{P}}}\left[(S_T \mathbb{1}_{\{T_x > T\}})^2\right] - \left(\widehat{\mathbb{E}^{\mathbb{P}}}\left[S_T \mathbb{1}_{\{T_x > T\}}\right]\right)^2}\,\right].$$

4. Apply the **regression for the second/outer step**

$$\pi^s(S_{T-\Delta t}\,,\ \kappa_{T-\Delta t}) = \widehat{\mathbb{E}^{\mathbb{Q}}}\left[\pi^s(S_T\,,\ \kappa_{T-\Delta t}) \mid S_{T-\Delta t}, \kappa_{T-\Delta t}\right] = \sum_{k=0}^{K-1} b_k^T e_{\pi^s}(S_{T-\Delta t}\,,\ \kappa_{T-\Delta t}) \tag{3.28}$$

where $\pi^s(S_{T-\Delta t}\,,\ \kappa_{T-\Delta t})$ is a new vector of payoff at time $T - \Delta t$ to be used to perform the similar two-step valuation, one step backward.

5. Repeat steps 3-4 for all time steps of the form $(t, t + \Delta t)$ in a backward way to reach time zero. Technically, in the last period instead of the regression in equation (3.28) the price will be only a mean over the vector of $\pi^s(S_{\Delta t}\,,\ \kappa_0)$.

   Note that, if you are willing to have a standard error and confidence interval for the price at time zero, meaning over the final vector is not the proper result as it is a single value. Hence, a numerical trick is needed: Instead of starting from age x at time zero, start from one age earlier $x - 1$ (time -1 insted of zero) and generate the scenarios. Repeat steps 3-4 in a same number of backward iterations as number (4) till the time zero (i.e. age x) and calculate the mean and standard error of the vector to build the confidence interval!

If we assume 1000 scenarios for S_T and κ_T, we have to execute 2×1000 regression in the inner step for the two moments in (3.27a) and (3.27b) to estimate the actuarial premium, and another 1000 regression in the outer step to obtain the no-arbitrage price in (3.28), ending up to 3000 calculation step. In case we combine the inner step into one regression, the total number of calculation steps will even decrease to 2000. This is significantly more efficient than the other two methods mentioned above.

For independent financial and actuarial risks the numerical implementation can be shortened by using some analytical results in hand. For example, the value of S_T at time t can be factorized from the two-step valuation and then the regression is only needed for actuarial risk in the inner step.

We don't provide the result of the numerical work for multi-period market-consistent valuation by LSMC here, as we will perform a very similar task in the next step for a pension contract that its payoff is similar to the unit-linked contract.

3.4. Market-Consistent Valuation under EIOPA Standard

We construct the two-step actuarial valuation by the adjusted actuarial operator suggested by EIOPA standard for the simple unit-linked contract. Using the two-step EIOPA Cost-of-Capital operator in equation (2.12) for the individual unit-linked payoff at time T in equation (3.9), the market-consistent Cost-of-Capital price under EIOPA standard at time $t < T$, when the policyholder is at age x, will be as follows:

$$\Pi^{EIOPA}_{\mathcal{G}^A_t}\left[S_T \mathbb{1}_{T_x>T}\right] = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{EIOPA}_t\left[S_T \mathbb{1}_{\{T_x>T\}} \,\middle|\, \kappa_t, S_T\right] \,\middle|\, \kappa_t, S_t\right] \tag{3.29a}$$

where

$$\begin{aligned}\Pi^{EIOPA,CoC}_t\left[S_T \mathbb{1}_{\{T_x>T\}} \mid (\kappa_t, S_T)\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}\left[S_T \mathbb{1}_{\{T_x>T\}} \,\middle|\, (\kappa_t, S_T)\right] \\ &+ \delta \sum_{k=1}^{T-t} e^{-rk}\mathrm{VaR}^{\mathbb{P}}_q\left[S_T \mathbb{1}_{\{T_x>k\}} - \mathbb{E}^{\mathbb{P}}\left[S_T \mathbb{1}_{\{T_x>k\}} \,\middle|\, (\kappa_t, S_T)\right] \,\middle|\, (\kappa_{t+k-1}, S_T)\right].\end{aligned} \tag{3.29b}$$

where e^{-rk} is the discount factor for maturity k given the constant annual rate of discount r. In applied view, κ_{t+k-1} is the best-estimate value of κ at time $t+k-1$ and the formula assert the fact that to valuate any risk measure depending on the survival event at time $t+k$, we have full information about the longevity trend till one step earlier at time $t+k-1$.

An alternative representation for equation (3.29b) can be constructed for two-step Valuation of the unit-linked contract under the Standard-Deviation principle by equation (2.13) as follows,

$$\begin{aligned}\Pi^{EIOPA,Std}_t\left[S_T \mathbb{1}_{\{T_x>T\}} \mid (\kappa_t, S_T)\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}\left[S_T \mathbb{1}_{\{T_x>T\}} \,\middle|\, (\kappa_t, S_T)\right] \\ &+ \beta \sum_{k=1}^{T-t} e^{-rk}\sqrt{\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[S_T \mathbb{1}_{\{T_x>k\}} \,\middle|\, (\kappa_{t+k-1}, S_T)\right]}.\end{aligned} \tag{3.30}$$

3.4.1. Analytical Solution

Suppose at time $t = 0$ (when the policyholder is at age x), we would like to price the contract with maturity T. With independence assumption in hand, the operator in equation (3.29) can be factorized for the financial risk driver S_T and simplified as follows:

$$\begin{aligned}
\Pi^{EIOPA,CoC}_{\mathcal{G}^A_0}\left[S_T \mathbb{1}_{T_x>T}\right] &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_T \mid S_0\right] \times \\
&\left[\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{T_x>T\}} \,\middle|\, \kappa_0\right] + \delta \sum_{k=1}^{T} e^{r(T-k)} \mathrm{VaR}^{\mathbb{P}}_q \left[\mathbb{1}_{\{T_x>k\}} - \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{T_x>k\}} \,\middle|\, \kappa_0\right] \,\middle|\, \kappa_{k-1}\right]\right] \\
&= S_0 \times \left[{}_Tp_x + \delta \sum_{k=1}^{T} e^{r(T-k)} \mathrm{VaR}^{\mathbb{P}}_q \left[\mathbb{1}_{\{T_x>k\}} - {}_kp_x \,\middle|\, \kappa_{k-1}\right]\right] \\
&= S_0 \times \left[{}_Tp_x + \delta \left(\sum_{k=1}^{T} e^{r(T-k)} \mathrm{VaR}^{\mathbb{P}}_q \left[\mathbb{1}_{\{T_x>k\}} \mid \kappa_{k-1}\right] - \sum_{k=1}^{T} {}_kp_x\right)\right] \\
&= S_0 \times \left[{}_Tp_x + \delta \left(\sum_{k=1}^{T} e^{r(T-k)} - \sum_{k=1}^{T} e^{r(T-k)} {}_kp_x\right)\right].
\end{aligned} \tag{3.31}$$

The second equation is the result of the martingale property for S_T under the risk neutral measure $\mathbb{Q}$ while ${}_kp_x$ is the survival probability of the policyholder at age x given the initial value of the longevity trend κ_0. The 3rd equality is obtained due to translation invariance property of VaR for ${}_kp_x$ conditional on the best-estimate longevity trend κ_{k-1}. The last equality is the result of the underlying (and in the same time ``realistic'') assumption for which, for a high value of q (e.g. in Solvency II, 99.5%), since $1-q < \mathbb{P}(T_x > k \mid \kappa_{k-1})$, therefore $\mathrm{VaR}^{\mathbb{P}}_q\left[\mathbb{1}_{\{T_x>k\}} \mid \kappa_{k-1}\right] = 1$.

For $k = 1, 2, ..., T$, the summation $\sum_{k=1}^{T} e^{r(T-k)}$ is the accumulated value of a T-year deterministic immediate annuity (with payments at the end of the period) exhibited by the notation $s_{\overline{T|}r}$ and the summation $\sum_{k=1}^{T} e^{r(T-k)} {}_kp_x$ is the future value of a T-year life annuity shown by the actuarial notation $s_{x:\overline{T|}r}$, both with the constant interest rate r. Hence, substituting for the annuities in equation (3.31) by actiarial notations, the market-consistent Cost-of-Capital price under EIOPA standard will be rephrased as

$$\Pi^{EIOPA,CoC}_{\mathcal{G}^A_0}\left[S_T \mathbb{1}_{T_x>T}\right] = S_0 \times \left[{}_Tp_x + \delta\left(s_{\overline{T|}r} - s_{x:\overline{T|}r}\right)\right]. \tag{3.32}$$

In the above equation, price under EIOPA standard offers a proportion of the difference between the future value of the T-year deterministic annuity and life annuity as the risk loading for the actuarial price, where the proportion coefficient is the annual cost of capital rate, δ. The risk loading in the actuarial price under EIOPA standard is constructed under a hypothetical mechanism where an imaginary annuity is payable by the insurer to policyholders during T years. For a cohort of independent individuals N_x, in the worst case scenario, at maturity T, nobody will die and the future value of the imaginary annuity $s_{\overline{T|}r}$ is payable to all of the policyholders while the insurer expects that on average $N_x - N_x(T)$ people will die during the annuity payment period and as a result the average liability is $s_{x:\overline{T|}r}$.

The Cost-of-Capital premium principle under EIOPA thus implies that the insurer basically considers the worst case scenario of longevity risk as the unexpected loss and covers the difference between this risk and expected liability. In case of real event of unexpected loss, it will be compensated by the money borrowed from shareholders at rate of δ. Note that the above result is obtained under the realistic assumption of $1-q < \mathbb{P}(T_x > k \mid \kappa_{k-1})$. Also note that the whole liability is discounted to present time zero by discount factor hidden in $S_0 = e^{-rT} S_T$.

The analytical solution provided in equation (3.32) for the two-step Cost-of-Capital operator under EIOPA is an alternative for the analytical solution of the one-period two-step Cost-of-Capital actuarial value obtained earlier in equation (3.15) and the time-consistent Cost-of-Capital two-step actuarial value obtained in equation (3.22).

With a similar method, the analytical solution for the two-step Standard-Deviation operator under EIOPA directives can be obtained in a similar way as follows,

$$\begin{aligned}\Pi^{EIOPA,Std}_{\mathcal{G}^A_0}\left[S_T \mathbb{1}_{T_x>T}\right] &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_T \mid S_0\right] \times \\ &\left[\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{T_x>T\}} \mid \kappa_0\right] + \beta\sum_{k=1}^{T} e^{r(T-k)}\sqrt{\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\mathbb{1}_{\{T_x>k\}} \mid \kappa_{k-1}\right]}\right] \\ &= S_0 \times \left[{}_Tp_x + \beta\sum_{k=1}^{T} e^{r(T-k)}\sqrt{p_{x+k-1}(1-p_{x+k-1})}\right].\end{aligned} \quad (3.33)$$

where p_{x+k-1} is the survival probability of an individual at age "$x+k-1$" for one year later with longevity information available on year "$k-1$" in a T-year contract.

4. Market-Consistent Pension Valuation

In this section we provide the market-consistent value of a typical policy called ``participating (i.e. with profit) contract''. This contract is a practically familiar form of the pension contracts that has most of the main attributes of the pension policies including: periodical crediting of a policy interest with a guaranteed rate, its link to financial market. The payoff will be a mixture of the pension reserve and longevity risk that we aim to price it in a market-consistent way.

4.1. Payoff and Dynamics of Pension Contract

We chose a well structured version of the product discussed by Grosen and Jorgensen [13]. The contract is, by construction of the payoff and crediting mechanism, linked to Dutch pension payoff. The financial part of the contract works under two main dynamics consist of assets and liabilities. At time $t = 0$ a policyholder buys one unit of the contract with nominal value P_0 for a single price V_0. The insurance company invests the whole money in the financial market and at the end of every year t is committed to credit policy interest rate $r_P(t)$ that can not be less than a guaranteed rate r_G. At maturity time T, the policy ends by paying a single value P_T to the policyholder. Let A_t be the market value of the invested money and P_t be the Policy reserve or part of insurer's liability to the policyholder. The difference $B_t = A_t - P_t$ is called Bonus reserve. The value of the policy reserve is then,

$$P_t = \Big(1 + r_P(t)\Big) \times P_{t-1} \qquad , \qquad t = 1, 2, ..., T \tag{4.1}$$

which is equivalent to calculate it by P_0 as $P_t = P_0 \prod_{i=1}^{t}(1 + r_P(t))$.

Crediting mechanism of the policy interest rate at each year t, is a function of A_t, r_G and the management decision on the buffer ratio B_t/P_t. If the management determines a ``target buffer ratio'' γ, in case $B_t/P_t > \gamma$, the fund will distribute a positive fraction α of the excessive amount of buffer ratio on top of G in form of the policy interest rate.[7] Considering the fact that normally at any year t the fund management decides on $r_P(t)$ based on the state of the fund at previous year (i.e. P_{t-1} and B_{t-1}), the analytical formula for $r_P(t)$ can be [8]

$$r_P(t) = max\left\{ r_G, \alpha \left(\frac{B_{t-1}}{P_{t-1}} - \gamma \right) \right\}. \tag{4.2}$$

The equation (4.1) will be updated as

$$\begin{aligned} P_t &= P_{t-1}\left(1 + max\left\{ r_G, \alpha \left(\frac{B_{t-1}}{P_{t-1}} - \gamma \right) \right\}\right) \\ &= P_{t-1}\left(1 + r_G + max\left\{ 0, \alpha \left(\frac{A_{t-1} - P_{t-1}}{P_{t-1}} - \gamma \right) - r_G \right\}\right) \end{aligned} \tag{4.3}$$

This mechanism features that the bonus interest rate is an option element with strike value r_G whereas P_t is path-dependent process with respect to A_t, we can not find an analytical solution for the value.

The above mechanism is pretty similar to the typical Dutch pension contract where by rewriting (4.3) as

$$P_t = P_{t-1}\left(1 + max\left\{ r_G, \alpha \left(\frac{A_{t-1}}{P_{t-1}} - (1 + \gamma) \right) \right\}\right) \tag{4.4}$$

[7] α is called the ``distribution ratio'' where by Grosen and Jorgensen [13] its realistic value is around 20-30%. They referred to the usual value of γ to be around 10-15%.

[8] By crediting $r_P(t)$ at year t, we mean that the interest rate is credited for the period of $(t-1, t)$, is determined at time $t-1$.

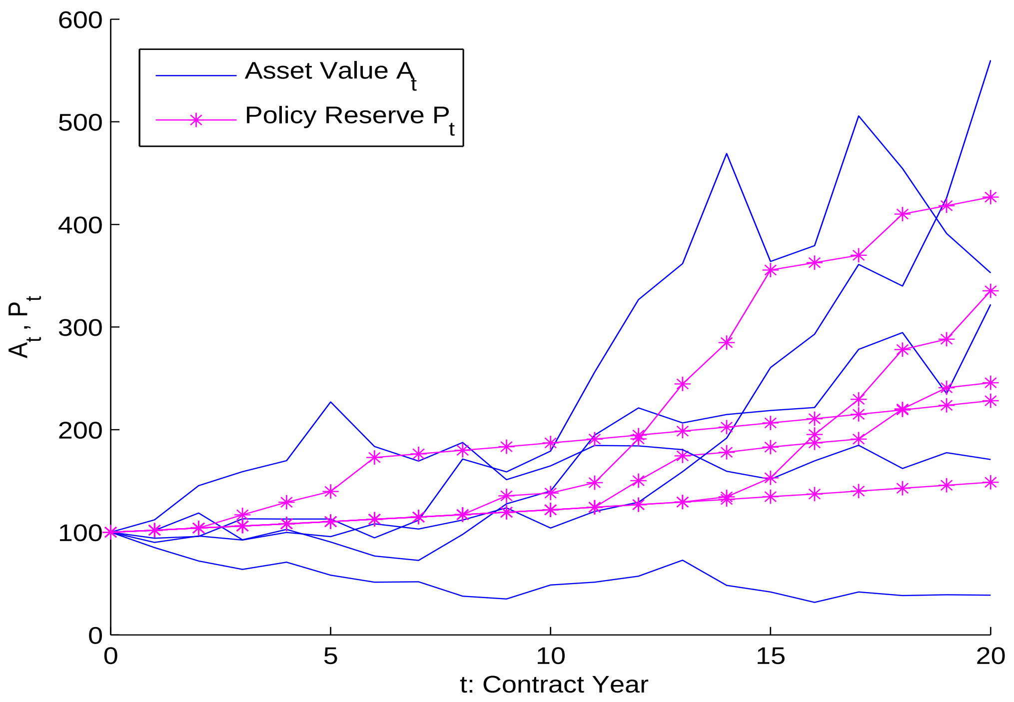


Figure 1: Simulation of the Asset Value A_t and Policy Reserve P_t of the Pension Contract. Parameter set: $S_0 = P_0 = 100$, $r = 4\%$, $\sigma_S = 0.15$, $r_G = 2\%$.

the index A_{t-1}/P_{t-1} is the well-known ``Funding Ratio'' of the pension fund and accordingly $1+\gamma$ can be called the target funding ratio. Furthermore, the crediting mechanism and evolution of the fund value in (4.4) is a (very stylised) example of an ``Indexation Ladder'' used in practice by Dutch pension funds.

We assume that A_t, similar to S_t in Section 3, follows the GBM process under Black-Scholes setting and risk neutral probability measure $\mathbb{Q}$.

Figure 1 illustrate a simulation of five paths of market value of invested assets A_t and policy reserve for a 20-years pension contract with guaranteed interest rate $r_G = 2\%$. It is clearly observable that the above mechanism smooths the policy interest rate in the sense of having lower volatility.

We add the actuarial risk of mortality/longevity to the above financial setting to get a mixed general payoff at time T that requires the policyholder to be alive to get the policy reserve P_T. We assume the longevity risk is modeled via Lee-carter model described in Subsection 3 with underlying process κ_t. The general payoff G is a function of A_T, P_T and κ_T. Note that at maturity T, the value of P_T will be known by A_{T-1} and P_{T-1}. Therefore, we denote the payoff in a shorter notation only as a function of P_T and κ_T as below:

$$G(P_T, \kappa_T) = P_T \times \mathbb{1}_{\{T_x > T\}}. \tag{4.5}$$

The above payoff represents a European style contract defined based on the survival event of an individual aged x only at maturity. This is chosen for the sake of simplicity to show the implementation. Of course, it is possible to extend the method for the contract with surrender option before maturity. Moreover, if instead of an individual, the fund gets started by a cohort N_x of people aged x, the final payoff will be

$$G(P_T, \kappa_T) = P_T \times N_x(T). \tag{4.6}$$

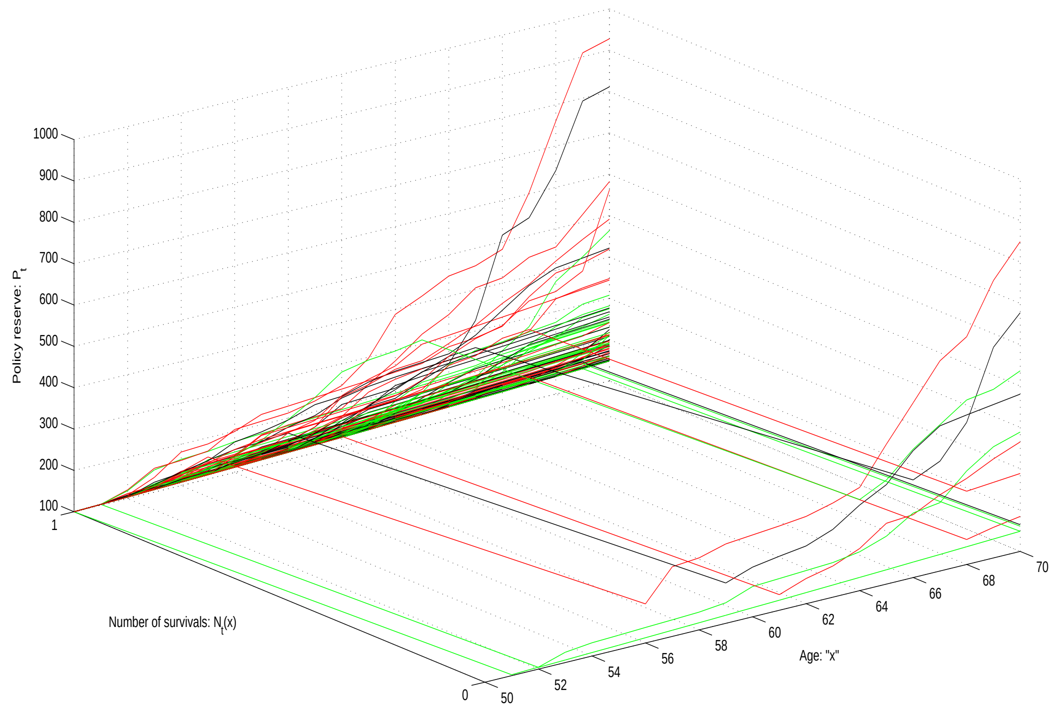


Figure 2: Simulation of the Policy reserve P_t and Survival event $\mathbb{1}_{\{T_x > T\}}$ for an individual of age 50 for maturity $T = 20$. Other Parameters: $P_0 = 100$, $r = 4\%$, $\sigma_S = 0.15$, $r_G = 2\%$.

Figure 2 exhibits a simulation of simultaneous policy reserve and survival event $\mathbb{1}_{\{T_x > T\}}$ of an individual with age $x = 50$, up to retirement age 70 where $T = 20$ and with guaranteed interest rate $r_G = 2\%$. The simulation is performed for 100 scenarios where in some of the scenarios the death event shifts the evolution of $P - t$ from the left side of the graph to right side for which the payoff will be zero at age 70.

Although the above participating pension payoffs are by construction different than the unit-linked contract discussed earlier, their form is pretty similar. Hence, with a slight adjustment, we can implement the mechanism of the two-step actuarial valuation to obtain the market-consistent value for the payoff in equation (4.5) for different settings:

- **One-period setting**: The one-period value of the contract at time $0 \le t < T$ can be represented as

$$\pi_t^{\text{1-period}}(P_T, \kappa_T) = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[P_T \times N_x(T) \mid \kappa_t, P_T\right] \,\middle|\, \kappa_t, P_t\right]. \tag{4.7}$$

  Let V_t be the value of the financial part of the pension contract. By using the risk-neutral valuation of the contingent payoff under $\mathbb{Q}$, the value of the European participating contract (See [13]) at $t < T$ is,

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} P_T | P_t] \tag{4.8}$$

  Note that, the policy reserve P_t is a path-dependent process. Hence, it is not possible to provide an analytical solution for V_t. Accordingly, even if regardless of the practical feasibility, we provide the continuous-time limit of the price as the analytical solution for the pure actuarial part, it will not be possible to give that for pension contract with mixed payoff in (4.5) or (4.6).

The discounted mean of the m generated scenarios of $P_i(T)$ is suggested as the Monte Carlo estimation of the price at time zero as follows,

$$\widehat{V}_0 = \frac{e^{-rT}}{m} \sum_{i=1}^{m} P_i(T). \tag{4.9}$$

Besides, by independence assumption, with a similar mechanism established for the unit-linked contract in equation (3.12), the two-step actuarial valuation for the participating pension contracts for different settings can be factorized as

$$\pi_t(P_T, \kappa_T) = V_t \times \Pi^{\mathbb{P}}[\mathbb{1}_{\{T_x > T\}} \mid \kappa_t] \tag{4.10}$$

where Π is the actuarial operator. If we use the Standard-Deviation principle Π^s to value the actuarial part, then under the independence assumption in (4.10), the price of the participating contract at time zero will be,

$$\pi_0^{\text{1-period}}(P_T, \kappa_T) = V_0 \left({}_Tp_x + \beta\sqrt{T}\sqrt{{}_Tp_x(1 - {}_Tp_x)} \right) \tag{4.11}$$

- **Time-Consistent (Multi-period) setting**: In a multi-period setting, we apply the backward iteration of the two-step actuarial valuation and we repeat the valuation on an annual basis where $\Delta t = 1$. Hence the price is time-consistent on a finite predictable points of time $t \in \{0, 1, 2, ..., T-1\}$. Based on equation (2.8) the general formulation of the time-consistent value of the contract at any time t is,

$$\pi_t^{\text{TC}}(P_T, \kappa_T) = \mathbb{E}^{\mathbb{Q}} \left[\Pi^{\mathbb{P}} \left[\pi\left(P_{t+1}, \kappa_{t+1}\right) \middle| \kappa_t, P_{t+1} \right] \middle| \kappa_t, P_t \right]$$

with terminal condition,

$$\pi(P_T, \kappa_T) = P_T \times \mathbb{1}_{\{T_x > T\}}. \tag{4.12}$$

Under the independence assumption in equation (4.10) we calculate V_0 and time-consistent Π^S separately, where the later is obtained via the backward iteration of the one-period Standard-Deviation of the pure longevity risk over $[0, T]$. With annual time-steps $Dt = 1$, we can not provide any analytical solution for time-consistent actuarial value. However, out of any practical feasibility, if $\Delta t \to 0$, then based on the results of Pelsser and Salahnejhad [25] such analytical solution was provided in Subsection 3.3.1 in equation (3.23) where, $\Pi^{s,\text{TC}}[\mathbb{1}_{\{T_x > T\}} \mid \kappa_0] = {}_Tp_x^{\ \mathbb{S}}$ with adjusted κ_t process in equation (3.24).

- **Setting under EIOPA directives:** By equation (3.29a), the market consistent value of the participating pension contract is

$$\pi_t^{\text{EIOPA}}(P_T, \kappa_T) = \mathbb{E}^{\mathbb{Q}} \left[\Pi_t^{EIOPA} \left[P_T \times \mathbb{1}_{\{T_x > T\}} \middle| \kappa_t, P_T \right] \middle| \kappa_t, P_t \right] \tag{4.13}$$

where once again by the independence assumption in (4.10) and using the result of the EIOPA Standard-Deviation operator in equation (3.33), we have,

$$\pi_0^{\text{EIOPA}}(P_T, \kappa_T) = V_0 \left[{}_Tp_x + \beta \sum_{k=1}^{T} e^{r(T-k)} \sqrt{p_{x+k-1}(1 - p_{x+k-1})} \right] \tag{4.14}$$

Note that the above results can be generalized to a cohort of N_x instead of an individual, by using ${}_tp_x$ and N_x as the parameters of the binomial distribution for N_{x+t}, for $t \in \{1, 2, ..., T\}$.

4.2. Numerical Implementation

We present the result of the numerical implementation to compute the value of the European (without surrounding option) participating pension contract with payoff in equation (4.5). To do so, we use slightly the similar numerical scheme explained in Subsection 3.2.2 for the one-period and EIOPA setting and LSMC scheme in Subsection3.3.2 for time-consistent setting, to implement the simulation.

We deliver the actuarial Standard-Deviaition prices for the all three settings mentioned in previous subsection as well as the conditional expected value of the contract. First, in a one-period setting, we compare the actuarial value of the contract with the expected value to see the effect of the risk loading when the maturity of the contract increases. Then, we are interested to compare the time-consistent and EIOPA actuarial value with the one-period price of the contract. We expect that both comparisons show how the time-consistent and EIOPA price capture the effect of the possible changes of the underlying risk drivers during the long-term valuation period. As we are not able to provide an analytical price for this contract, the numerical comparison is needed to capture such an effect and obtain a realistic inference about the excess price (premium) necessary to fulfill the so-called ``re-valuation'' requirement of the regulator. We will calculate the prices for different maturities to study how the excess time-consistency premium of the price is affected by the duration of the contract.

4.2.1. Time-Consistent Pension Value

For the backward iteration of the of the LSMC method in multi-period setting, we apply the inner and outer steps of the regression model in the equation (3.27) and (3.28) for P_T instead of S_T. There exist numerous possibilities to choose the regressors and basis functions in LSMC that can result in the reasonable valuation. To be efficient on the computation speed, for $T = 1, 2, ..., 30$ in both inner and outer steps, we reduce the number of the variables in the polynomial by the below transformation:

$$\text{Inner step:} \qquad e_{k,l}(P_T, \kappa_{T-1}) = P_T^k \times [exp(-exp(\kappa_{T-1}))]^l \tag{4.15a}$$

$$\text{Outer step:} \qquad e_{k,l}(P_{T-1}, \kappa_{T-1}) = P_{T-1}^k \times [exp(-exp(\kappa_{T-1}))]^l \tag{4.15b}$$

where the polynomial basis is a creative mixture of of $K = 3$ degree for P_T and $L = 1$ degree for κ_T for $k = 0, 1, 2, 3$ and $l = 0, 1$.

To obtain a better regression fit, we used the payoff made of the cohort $N_x = 1000$ in equation (4.6) so that the response variable in the regressions (consist of the number of survivals) will not be degenerated on the value zero, whereas using (4.5) will result in such a deficiency. We generate $n = 2000$ scenario of the asset value A_T and κ_T. For the policy reserve, we use the recursive scheme in equation (4.3) ending up with 2000 scenarios of P_T for all values of T, including the final maturity $T = 30$. For each maturity, we iterate each numerical scheme for $N = 100$ times and report the mean as the point estimation of price and use standard error to build a 95% confidence interval for that.

Figure 3 illustrates the numerical results of the market-consistent Standard-Deviation value obtained for an individual with age $x = 40$ who may buy the participating contract with different maturities $T = 1, 2, ..., 30$. Note that for each maturity, the result is obtained out of an annual backward iteration of the two-step valuation along the valuation period. We set the target buffer ratio as $\gamma = 15\%$ (i.e. target funding ratio, $1 + \gamma = 115\%$) and the distribution ratio $\alpha = 50\%$. The rest of the parameter set is as follows: $A_0 = 100$, $r = 4\%$, $\sigma_A = 0.15$, $P_0 = 100$, $B_0 = 0$, $r_G = 2\%$, $\beta = 0.03$.

The value of the contract can be affected by the policy of the pension fund on distribution of the bonus interest rate (and accordingly the accredition mechanism) in the indexation ladder. We capture

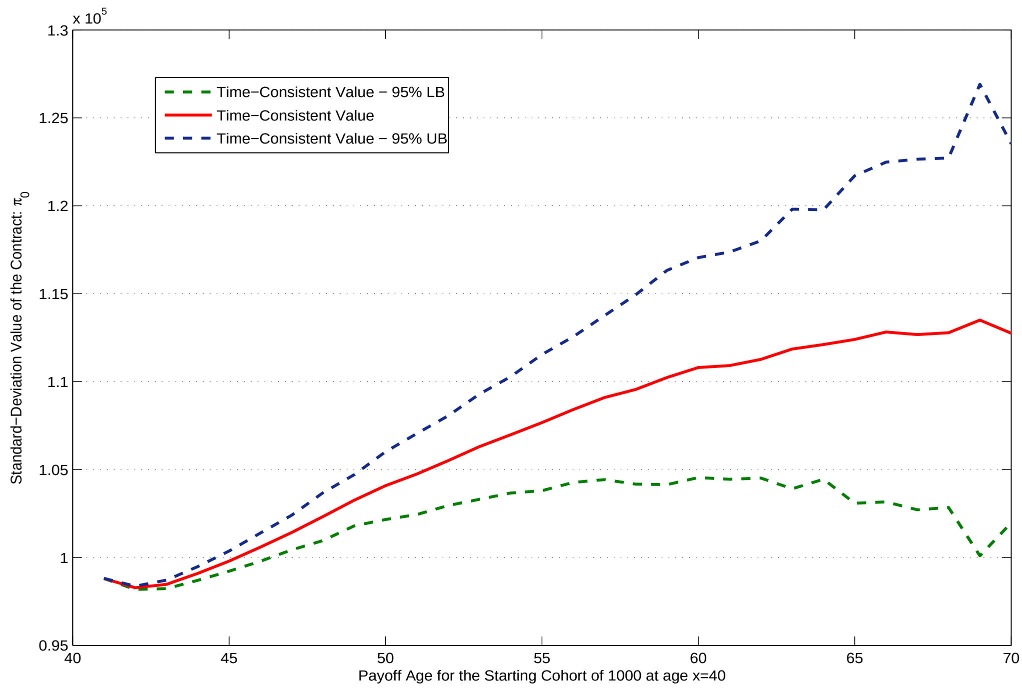


Figure 3: Time-Consistent and Market-Consistent Standard-Deviation actuarial price of the European participating pension contract over different maturities $T = 1, 2, ..., 30$ for a policyholder of age $x = 40$.

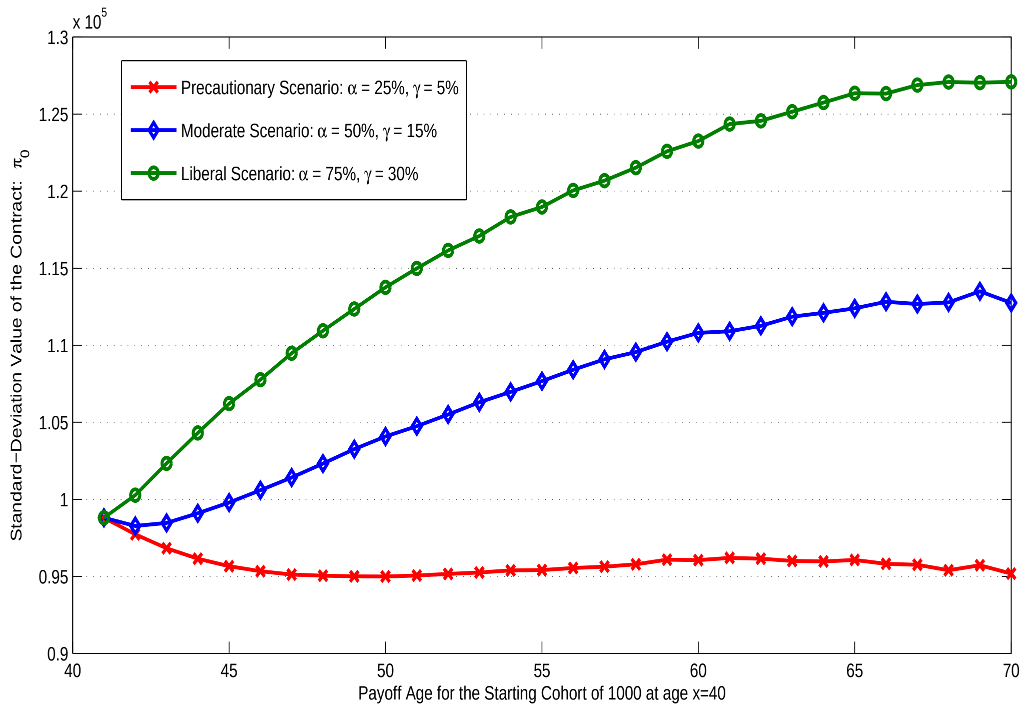


Figure 4: Comparison of the Time-Consistent and Market-Consistent Standard-Deviation price under different funding policy scenarios for different maturities.

this effect by changing the level of the funding ratio, $1+\gamma$ (or equivalently the target buffer ratio, γ), and distribution ratio, α.

We compare the time-consistent and market-consistent value of the contract with three combined level of α and γ that represent three different funding policy scenarios consist of:

- **Precautionary policy**, where we set $\alpha = 25\%$ and $\gamma = 30\%$,
- **Moderate policy**, where we set $\alpha = 50\%$ and $\gamma = 15\%$, and
- **Liberal policy**, where we set $\alpha = 75\%$ and $\gamma = 5\%$.

In the precautionary policy, for example, the fund management requires a relatively high level of funding ratio, $1+\gamma = 130\%$, and only if that holds will distribute relatively a low portion, $\alpha = 25\%$ of the extra buffer on top of the guaranteed rate $r_G = 2\%$ as the bonus. The opposite happens with the liberal policy. The rest of the parameters stay unchanged with respect to the previous graph.

The comparative values are provided in Figure 4 where the precautionary funding policy requires lower risk premium and price comparing to the liberal and moderate policies. Looking into the numbers, the time-consistent and market-consistent prices by the liberal policy, for a contract with maturity of $T = 15$ years is roughly 25% higher than precautionary policy while for a contract with $T = 30$ years this difference is around 34%.

4.2.2. Comparison of the Different Valuation Methods

We compare the market-consistent Standard-Deviation actuarial values for the participating pension contract with four different valuation methods:

- Time-consistent price,
- Price under EIOPA directives,
- One-period price and,
- The expected value of the discounted payoff.

Figure 5 demonstrate the graphical evolution of the above-mentioned values for different maturities. The parameters are the same as those provided in Subsection 4.2.1 for the ``Moderate'' funding policy.

For a 30-years contract, the discounted expected value is around 91.7×10^3, which is even lower than the expected value of a 1-year contract (by 6.4%). The one-period Standard-Deviation price spins around 97.5×10^3 where the actuarial risk loading is around 5.8×10^3 (6.3%). The EIOPA and time-consistent prices are 106.4^3 (9.2% higher than the one-period) and 112.7^3, respectively, and time-consistency risk premium is 13.5%.

All of the four different valuation at time zero in Figure 5 are affected by three main factors consist of: time to maturity, T, that is mostly reflected in the ``loading'' part of the price and increases the price when it is longer; the discount factor, $exp(-rT)$, that decrease the price when T increases; and the survival probability (longevity risk), ${}_Tp_x$, that basically has direct relationship by the price but decreases during maturity T, ending up with lower price. For a short-term contract, the effect of discounting and drop in the survival probability is higher than time to maturity and this result in a drop in the price in the first 3-4 years. In the middle-term contracts, the maturity highers the price and dominate the effect of the drop in survival probability and discounting. We can observe the fact

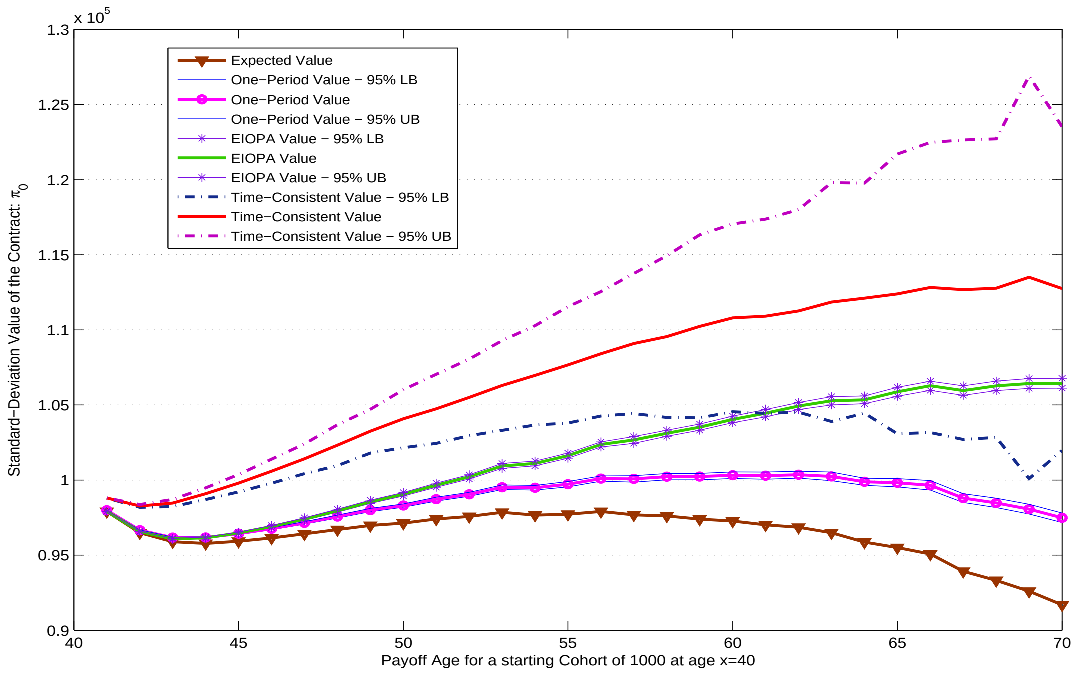


Figure 5: Comparison of the Market-Consistent Standard-Deviation actuarial price and 95% confidence interval, calculated by three different valuation method consist of: One-period, EIOPA directives and Time-Consistent method for different maturities.

that, this dominance in Time-consistent and EIOPA values is more considerable. Finally in long-term contracts, the discount factor dominates maturity and survival probability by decreasing the long-term expected value and one-period price. However, in case of the time-consistent and EIOPA price, the backward iteration and summation of annual risk-loading calculations (see equation (3.33) and (4.14)), as an implicit factor, resonates with maturity and survival probability to neutralize the effect of discount factor and as a results, the price stays slightly increasing.

We also observe that the LSMC method used for time-consistent valuation produces much wider confidence interval than the one-period and EIOPA price which is calculated by a finite difference interval method. The finite difference interval used in EIOPA and one-period valuation, calculates the price through a mathematical formula of the generated scenarios of underlying risk drivers, where the source of the standard error is difference between continuous-time distribution and discrete one. In LSMC method, we use the same discrete approximation for scenario generating that brings the same error as of the finite difference method. However, besides that, there is another error from residuals of the regression between the price and the underlying risk drivers. Only if the regression relationship can explain the whole variability of the price by the underlying risk drivers (roughly saying $R^2 \simeq 1$), we may have roughly same accuracy. This explains why, although by structure for a one-year contract with payoff age of $x + T = 41$ the price resulted by the three methods must be the same, LSMC price is slightly different. Note that, the accuracy of LSMC method depends on quality of the regression and the choice of basis functions and number of scenarios.

Also it worth to consider that for longer-term contracts, EIOPA price may fall in the 95% confidence interval of the time-consistent price. This may be interpreted via the fact that valuation under EIOPA directives also partly use the extra variation imposed from the middle-time information, while we believe the time-consistent method use those information is a more structured and complete ways.

Last but not least, the time-consistency risk premium increases when maturity goes longer. The numerical results show that for a 10-year contract, this extra premium is 5.8×10^3 (5.6% of the price), while for a 20-year and 30-years contract it increase respectively to 10.5×10^3 (9.5% of the price) and 15.3×10^3 (13.5% of the price).This is due to difference in numerical technique that shows a trivial difference in the result.

References

[1] Artzner, P., Delbaen, F., Eber, J., Heath, D., and Ku, H. (2007). Coherent multiperiod risk adjusted values and bellman's principle. *Annals of Operations Research*, 152(1):5--22.

[2] Barrieu, P. and El Karoui, N. (2009). Pricing, hedging, and designing derivatives with risk measures. Princeton University Press.

[3] Bion-Nadal, J. (2009). Time consistent dynamic risk processes. *Stochastic Process Appl.*, 119(2):633--654.

[4] Bühlmann, H. (1970). *Mathematical methods in risk theory*. Springer Verlag, Berlin.

[5] Carriere, J. (1996). Valuation of the early-exercise price for options using simulations and nonparametric regression. *Insurance: mathematics and Economics*, 19(1):19--30.

[6] Cheridito, P., Delbaen, F., and Kupper, M. (2006). Coherent and convex monetary risk measures for unbounded cadlag processes. *Finance and Stochastics*, 9(3):369--387.

[7] Cvitanic, J. and Karatzas, I. (1992). Convex duality in constrained portfolio optimization. *Annals of applied probability*, 2(4):767--818.

[8] Delbaen, F. and Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300(3):463--520.

[9] Delbaen, F. and Schachermayer, W. (1996). The variance-optimal martingale measure for continuous processes. *Bernoulli*, pages 81--105.

[10] Föllmer, H. and Schweizer, M. (1989). Hedging by sequential regression: An introduction to the mathematics of option trading. *ASTIN Bulletin*, 18(2):147--160.

[11] Frittelli, M. and Gianin, E. R. (2004). *Dynamic Convex Risk Measures, Risk measures for the 21st Century*. Wiley, New York.

[12] Goovaerts, M. and Laeven, R. (2008). Actuarial risk measures for financial derivative pricing. *Insurance: Mathematics and Economics*, 42(2):540--547.

[13] Grosen, A. and Jorgensen, P. L. (2000). Fair valuation of life insurance liabilities: The impact of interest rate guarantees, surrender options, and bonus policies. *Insurance: Mathematics and Economics*, 26:37--57.

[14] Hodges, S. and Neuberger, A. (1989). Optimal replication of contingent claims under transaction costs. *Review of Futures Markets*, 8(2):222--239.

[15] Jobert, A. and Rogers, L. (2008). Valuations and dynamic convex risk measures. *Mathematical Finance*, 18(1):1--22.

[16] Kaas, R., Goovaerts, M., Dhaene, J., and Denuit, M. (2008). *Modern Actuarial Risk Theory: Using R*. Springer Verlag.

[17] Kramkov, D. and Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904--950.

[18] Kupper, M., Cheridito, P., and Filipovic, D. (2008). Dynamic risk measures, valuations and optimal dividends for insurance. In *Mini-Workshop: Mathematics of Solvency*. Mathematisches Forschungsinstitut Oberwolfach.

[19] Lee, R. D. and Carter, L. R. (1992). Modeling and forecasting u.s. mortality. *Journal of the American Statistical Association*, 87(419):659--671.

[20] Longstaff, F. A. and Schwartz, E. S. (2001). Valuing american options by simulation: A simple least-squares approach. *The Review of Financial Studies*, 14(1):113--147.

[21] Maccheroni, F., Marinacci, M., and Rustichini, A. (2006). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447--1498.

[22] Malamud, S., Trubowitz, E., and Wüthrich, M. (2008). Market consistent pricing of insurance products. *ASTIN Bulletin*, 38(2):483--526.

[23] Møller, T. (2002). On valuation and risk management at the interface of insurance and finance. *British Actuarial Journal*, 8(4):787--827.

[24] Musiela, M. and Zariphopoulou, T. (2004b). A valuation algorithm for indifference prices in incomplete markets. *Finance Stochastic*, 8(3):399--414.

[25] Pelsser, A. and Salahnejhad, A. (2015). Time-consistent actuarial valuation.

[26] Pelsser, A. and Stadje, M. (2014). Time-consistent and market-consistent evaluations. *Mathematical Finance*, 24(1):25--65.

[27] Peng, S. (2004). Filtration consistent nonlinear expectations and evaluations of contingent claims. *Acta Mathematicae Applicatae Sinica (English Series)*, 20(2):191--214.

[28] Rogers, L. (2001). Duality in constrained optimal investment and consumption problems: a synthesis. In *Workshop on Financial Mathematics and Econometrics held in Montreal*. Springer.

[29] Rosazza Gianin, E. (2006). Risk measures via g-expectations. *Insurance Mathematics and Economics*, 39(1):19--34.

[30] Schweizer, M. (1995). On the minimal martingale measure and the föllmer-schweizer decomposition. *Stochastic analysis and applications*, 13(5):573--600.

[31] Wüthrich, M. V., Bühlman, H., and Furrer, H. (2010). *Market-Consistent Actuarial Valuation*. Springer, Berlin Hridelberg, 2nd edition. European Actuarial Academy.