# Time-Consistent Actuarial Valuations*

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**Abstract**

Time-consistent valuations (i.e. pricing operators) can be created by backward iteration of one-period valuations. In this paper we investigate the continuous-time limits of well-known actuarial premium principles when such backward iteration procedures are applied. This method is applied to an insurance risk process in the form of a diffusion process and a jump process in order to capture the heavy tailed nature of insurance liabilities. We show that in the case of the diffusion process, the one-period time-consistent Variance premium principle converges to the non-linear exponential indifference price. Furthermore, we show that the Standard-Deviation and the Cost-of-Capital principle converge to the same price limit. Adding the jump risk gives a more realistic picture of the price. Furthermore, we no longer observe that the different premium principles converge to the same limit since each principle reflects the effect of the jump differently. In the Cost-of-Capital principle, in particular the VaR operator fails to capture the jump risk for small jump probabilities, and the time-consistent price depends on the distribution of the premium jump.

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# 1 Introduction

Standard actuarial premium principles usually consider a static premium calculation problem: what is today's price of an insurance contract with payoff at time $T$. Textbooks such as those by Bühlmann (1970), Gerber (1979), and Kaas et al. (2008) provide examples of this. The study of risk measures and the closely related concept of monetary risk measures have also been studied in static settings by authors such as Artzner et al. (1999) and Cheridito et al. (2005). The study of utility indifference valuations has mainly confined itself to static settings as well. Different applications can be found in papers by Young and Zariphopoulou (2002), Henderson (2002), Hobson (2004), Musiela and Zariphopoulou (2004) and Monoyios (2006), and the book by Carmona (2009).

Financial pricing usually considers a "dynamic" pricing problem, and looks at how the price evolves over time until the final payoff date $T$. This dynamic perspective is driven by the focus on hedging and replication. The literature was started by the seminal paper of Black and Scholes (1973) and has been immensely generalized to broad classes of securities and stochastic processes; see Delbaen and Schachermayer (1994).

In recent years, researchers have begun to investigate risk measures in a dynamic setting, where the question of constructing time-consistent (or "dynamic") risk measures has been investigated. See Riedel (2004), Cheridito et al. (2006), Roorda et al. (2005), Rosazza Gianin (2006), and Artzner et al. (2007). As an example, Stadje (2010) showed how a large class of dynamic convex risk measures in continuous time can be derived from the limit of their discrete time versions. Moreover, Jobert and Rogers (2008) showed how time-consistent valuations can be constructed through the backward induction of static one-period risk measures (or "valuations"). And later, Pelsser and Stadje (2014) studied time and market consistency of the well-known actuarial principles in a dynamic setting by using a two-step valuation method.

Insurance risk can be modeled in a stochastic way by using a diffusion process. However, it is usual that insurance risks exhibit jump type movements in their evolution, and the data usually contain a number of extreme events and stylized facts usually exist such as fat-tailed and skewed distributions. This justifies the usage of a jump component to draw a realistic inference about the dynamic pricing framework. Merton (1976) introduced the jump-diffusion model to price options by assuming discontinuity in returns. The model was developed extensively for financial modeling, actuarial valuation and the pricing of different derivatives and contingent claims in incomplete markets. There are numerous works about the jump process in finance; see for example Cont and Tankov (2012). For an introduction to the application of diffusion and jump processes in insurance see, for example, Korn et al. (2010) and for more specific actuarial applications see Biffis (2005), Verrall and Wüthrich (2012), Chen and Cox (2009), and Jang (2007). Some researchers have generalized the concept of time-consistent dynamic risk measures by using jump-diffusion processes when underlying risks include jumps. See for example Bion-Nadal (2008). The idea was developed in actuarial valuation using Backward Stochastic Differential Equations (BSDE) and $g$-expectations as more powerful tools to deal with non-linear pricing operators such as different premium principles. There are also a number of studies about modeling jumps with BSDEs in valuation and portfolio choice. See for example the textbook by Delong (2013) and the paper by Laeven and Stadje (2014).

In this paper we investigate well-known actuarial premium principles such as the Variance principle and the Standard-Deviation principle, and we study their time-consistent extension. We first consider one-period valuations, then extend this to a multi-period setting using the backward iteration method of Jobert and Rogers (2008) for a given discrete time-step $(t, t+\Delta t)$, and finally consider the continuous-time limit for $\Delta t \to 0$. A more general setting to model the insurance risk could be "infinite activity Lévy process" where it allows for infinite number of jumps for any finite time interval. However, as it does not seem realistic for an insurance process to have infinite number of jumps when $(t, t+\Delta t)$ is infinitesimally small, we waive the infinite activity Lévy process and we focus on investigating the method with simple diffusion and jump-diffusion processes.

We apply backward iteration to a simple diffusion model to show that the one-period Variance premium principle converges to the non-linear exponential indifference valuation. Furthermore, we study the continuous time limit of the one-period Standard-Deviation principle and the Cost-of-Capital principle, and establish that in the diffusion setting, they converge to the same limit represented by an expectation under an equivalent martingale measure. We apply the same approach to the jump-diffusion setting and show that the time-consistent prices for different premium principles in the limit converge to different results than in the diffusion case. We mainly use Itô's formula to calculate different functions of the process $y(t)$, and the Feynman-Kaç theorem to obtain our results. See for example the book by Shreve (2010) about working with martingales and Itô's formula. As an exception, in the Cost-of-Capital principle under the jump setting, we have to make inference about the distribution of the insurance process under the VaR operator. To do so, we will assume the jump process as a special case of the Lévy process and find its characteristic function. To get more insight about the Lévy process and its applications, see for example Figueroa-López (2012) and the textbook by Barndorff-Nielsen et al. (2001).

We apply this method to a health process to price a stylized life insurance product and we use a Markov chain approximation to discretize the time and state space of the underlying insurance process. See for example Kushner and Dupuis (2001), Duan et al. (2003), and Tang and Li (2007) for the idea of using a Markov chain approximation to price contingent payoffs in theory and application.

The rest of this paper is organized as follows. In Section 2 we define the time-consistent valuation operators and explain about the backward iteration method used to construct it. In Section 3 we derive the time-consistent extension of the Variance premium principle with and without discounting. Section 3 also includes a benchmark version of this premium and the Mean Value principle as a more general pricing rule. In Section 4, we derive the time-consistent value of the Standard-Deviation and Cost-of-Capital premium principles by using the same method. In both sections, we assume that the underlying pure insurance risks follow a diffusion process and we represent the results by means of the related Partial Differential Equation (PDE). In Section 5, we assume that the underlying process includes a Poisson jump component and we derive the time-consistent value for the principles (that we used in sections 3 and 4) in the form of the Partial Integro-Differential Equations (PIDEs). In Section 6, we apply our idea by giving examples of pricing for a stylized insurance product using the Markov chain method. We summarize and conclude in Section 7.

## 2 Time-Consistent Valuation Operators

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the underlying probability space and $X(\omega)$ and $Y(\omega)$ be the stochastic insurance risk processes defined over the $\sigma$-algebra $\mathscr{F}$. Indexing for the time $0 \le t \le T$, we form the filtration $\mathscr{F}_t$ as the collection of the $\sigma$-algebras. In this paper, we limit ourselves to the square integrable functions and denote the space of such random variables as $\mathscr{L}^2(\Omega, \mathscr{F}_t, \mathbb{P})$.

Time consistency postulates that the order of riskiness of different portfolios measured by a dynamic risk measure in the future time is consistent with their riskiness at any time prior to that point in time and remains the same. It suggests that if at any time $t$ the position $A$ forms a higher risk than position $B$, the level of risk will be higher for all $s < t$. The next definition formulates the time consistency of a risk measure.

**Definition 2.1** *A dynamic risk measure $(\rho_t)$ is Time-Consistent if and only if, for all $0 \le t \le T$ and $\forall X, Y \in L^2(\mathscr{F}_t)$,*

$$\rho_T(X) \le \rho_T(Y) \quad P-a.s. \quad \Rightarrow \quad \rho_t(X) \le \rho_t(Y) \quad P-a.s. \tag{2.1}$$

*or equivalently by its "recursive" form for $\forall s = \Delta t, 2\Delta t, ..., T-t$, we have $\rho_t = \rho_t(-\rho_{t+s})$,*

where $\rho_t : \mathscr{L}^2(\mathscr{F}_T) \to \mathscr{L}^2(\mathscr{F}_t)$ is a conditional risk measure for all $T \ge t$. The definition for non-negative risks (e.g. insurance losses) then becomes,

$$\rho_t = \rho_t(\rho_{t+s}) \tag{2.2}$$

Similar notions of time consistency can be found in Föllmer and Penner (2006), Cheridito and Stadje (2009), and Acciaio and Penner (2011).

We construct the time-consistent valuation operators for the insurance risks by the recursive form (2.2) and we use the backward induction method introduced by Jobert and Rogers (2008). In general we assume that the insurance process evolves during the time period $[0, T]$ and that at maturity time $T$ it falls into a bounded state space where we can also define the state space of the contingent payoff. Based on this method, time consistency can be achieved for the price operator by decomposing the valuation operator into a family of one-period pricing operators that can only be valuated in shorter intermediate time periods.

To derive the time-consistent actuarial value at the present time $t = 0$, we divide the valuation period $[0, T]$ into a discrete set $\{0, \Delta t, 2\Delta t, ..., T - \Delta t, T\}$ so that we can perform a multi-period valuation by applying the one-period pricing operator to all sub-intervals denoted by $(t, t + \Delta t)$. We use well-known actuarial premium principles such as the Variance, Standard-Deviation and Cost-of-Capital principles as pricing operators. Our aim is to apply the backward iteration method to all subintervals $(t, t + \Delta t) \in [0, T]$ to obtain the value of the related premium principle at time zero. We start with a payoff state space that is equal to the terminal values at time $T$ and calculate the one-period price at time $T - \Delta t$ for the last sub-interval $(T - \Delta t, T)$. This value space is derived by conditioning on the information available at $T - \Delta t$ and will look like a new payoff state space from the time $t - 2\Delta t$ viewpoint. Next, we repeat the one-period valuation process for the interval $(T - 2\Delta t.T - \Delta t)$. Conditional on the information available at $T - 2\Delta t$, we then obtain a new value state which plays the role of the new payoff state space for the former time period. The set of these conditional values can be used repeatedly as a new payoff state space for the former time points. We continue this backward valuation procedure for all subintervals of the form $(t, t+\Delta t)$ to gradually reach the time period $(0, \Delta t)$, where we derive the price of the actuarial risk at time zero.

The method is relatively straightforward and provides a discrete time valuation for the time-consistent actuarial premium principles. To derive the theoretical formulation of the time-consistent actuarial premium principle for a typical time interval $(t, t+\Delta t)$, we obtain the continuous time limit of the premium operator at time $t$, on the premium value at time $t+\Delta t$ when $\Delta t \to 0$. This will lead to a PDE if the underlying insurance risk is a diffusion process and will lead to a PIDE if the underlying process has a jump component. The results can also be validated via a (bi/quadrinomial) discretization of the underlying process and by applying the same valuation method when $\Delta t \to 0$. In the applied situation, we achieve an approximation of the time-consistent premium by increasing the number of $(t, t+\Delta t)$ subintervals in $[0, T]$, which will decrease the size of $\Delta t$.

Let the mapping $\Pi_t : \mathscr{L}^2(\mathscr{F}_t) \to \mathscr{L}^2(\mathscr{F}_s)$ for $0 \le s \le t$ be the conditional one-period actuarial valuation operator (e.g. premium principle) with respect to $\mathscr{F}(t)$. We denote the price of the insurance risk (i.e. insurance premium) at time $t$ by $\pi(t, y(t))$. Then, $\pi(t, y(t))$ can be derived for any time interval $(t, t+\Delta t)$, by applying $\Pi_t$ to the payoff random variable at time $t+\Delta t$ denoted by $\pi(t+\Delta t, y(t, t+\Delta t))$ as below,

$$\pi(t, y) = \Pi_t\left[\pi(t+\Delta t, y(t+\Delta t))\right] \tag{2.3}$$

In a backward iteration procedure, $\pi(t+\Delta t, y(t+\Delta t))$ is supposed to be the conditional value with respect to $\mathscr{F}_{t+\Delta t}$ obtained one step further. We may also show "$y(t)$" as "$y$" in some formulations in this paper.

We also implicitly assume that the insurance contract has only a payoff at maturity time $T$ and there is no implicit partial middle-time payment mechanism such as early exercise option. Note that we assume annuities as a series of payoffs in different maturities.

# 3 Variance Pricing

We start by considering an unhedgeable insurance process $y(t)$, which is given by means of a diffusion equation:

$$dy(t) = a(t, y(t))\,dt + b(t, y(t))\,dW(t). \tag{3.1}$$

We assume for $t \ge 0$, that $\mathscr{F}_t$ is the related filtration for $W_t$ and that $y(t)$ is an Itô process with $a(t, y(t))$ and $b(t, y(t))$ as adapted processes where $y(t)$ is still square integrable process.

Note that discounting is usually ignored in the standard actuarial literature (see for example Kaas et al., 2008). To facilitate the discussion, we will first derive the continuous-time limit of the Variance principle without using discounting in Section 3.1. We will then consider a case with discounting in Section 3.2, by means of a constant rate of discount for simplicity.

## 3.1 Variance Principle

If we consider an insurance contract with a payoff at time $T$, defined as a function $f\big(y(T)\big)$, then the actuarial Variance principle $\Pi^{\mathrm{v}}_t[]$ is defined as (see e.g. Kaas et al., 2008)

$$\Pi^{\mathrm{v}}_t[f(y(T))] = \mathbb{E}[f(y(T))|\mathscr{F}_t] + \tfrac{1}{2}\alpha\,\mathbb{V}\mathrm{ar}_t[f(y(T))|\mathscr{F}_t], \tag{3.2}$$

where $\mathbb{E}_t[.|\mathscr{F}_t]$ and $\mathbb{V}\mathrm{ar}[.|\mathscr{F}_t]$ denote the expectation and variance operators conditional on the information available at time $t$ under the "real-world" probability measure $\mathbb{P}$. To keep the notation simple, we will use $\mathbb{E}_t[]$ and $\mathbb{V}\mathrm{ar}_t[]$. The one-period Variance price can be obtained explicitly by substituting (3.2) into (2.3):

$$\pi^{\mathrm{v}}\big(t, y(t)\big) = \mathbb{E}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right] + \tfrac{1}{2}\alpha\mathbb{V}\mathrm{ar}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right]. \tag{3.3}$$

We assume that $\pi^{\mathrm{v}}$ is smooth enough to be twice continuously differentiable in $t$ and $y = y(t)$; and that $\pi_t$, $\pi_y$ and $\pi_{yy}$ are continuous functions of $(t, y(t))$. To keep the calculation notation simple from now on by integration on $t \le s \le t+\Delta t$, all integrands including $\pi_t\big(s, y(s)\big)$, $\pi_y\big(s, y(s)\big)$ and $\pi_{yy}\big(s, y(s)\big)$ will be represented by $\pi_t$, $\pi_y$ and $\pi_{yy}$, respectively unless they are fully represented.

To obtain the Variance price at (3.3), we derive the stochastic process for $\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)$ and $\big(\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\big)^2$. To avoid too many parentheses in the notation, we will also show "$(\pi^{\mathrm{v}})^2$" as "$\pi^{\mathrm{v}2}$". By using an integral form of the Itô formula, we then obtain

$$\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big) = \pi^{\mathrm{v}}\big(t, y(t)\big) + \int_t^{t+\Delta t}\Big(\pi^{\mathrm{v}}_t + a\pi^{\mathrm{v}}_y + \frac{1}{2}b^2\pi^{\mathrm{v}}_{yy}\Big)ds + \int_t^{t+\Delta t} b\pi^{\mathrm{v}}_y dW(s) \tag{3.4}$$

and

$$\begin{aligned}\pi^{\mathrm{v}2}\big(t+\Delta t, y(t+\Delta t)\big) &= \pi^{\mathrm{v}2}\big(t, y(t)\big) + \int_t^{t+\Delta t}\Big((\pi^{\mathrm{v}2})_t + a(\pi^{\mathrm{v}2})_y + \frac{1}{2}b^2(\pi^{\mathrm{v}2})_{yy}\Big)ds + \int_t^{t+\Delta t} b(\pi^{\mathrm{v}2})_y dW(s)\\ &= \pi^{\mathrm{v}2}\big(t, y(t)\big) + \int_t^{t+\Delta t}\Big(2\pi^{\mathrm{v}}\pi^{\mathrm{v}}_t + 2a\pi^{\mathrm{v}}\pi^{\mathrm{v}}_y + b^2(\pi^{\mathrm{v}2}_y + \pi^{\mathrm{v}}\pi^{\mathrm{v}}_{yy})\Big)ds\\ &\quad + \int_t^{t+\Delta t} 2b\pi^{\mathrm{v}}\pi^{\mathrm{v}}_y dW(s)\end{aligned} \tag{3.5}$$

where we applied the Itô formula to $\pi^2$ in the first equality and used the chain rule in the second equality to take derivatives with respect to $t$ and $y$. We should mention that the adapted $a$ and $b$ processes, and the derivatives under integration are all stochastic functions of $(s, y(s))$. Based on the martingale property of the Brownian motion with respect to filtration $\mathscr{F}_t$ for $t \le s$, we obtain the following by taking the conditional expectation of both sides of the above equation:

$$\mathbb{E}\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\,\middle|\,\mathscr{F}_t\right] = \pi^{\mathrm{v}}\big(t, y(t)\big) + \mathbb{E}_t\int_t^{t+\Delta t}\Big(\pi^{\mathrm{v}}_t + a\pi^{\mathrm{v}}_y + \frac{1}{2}b^2\pi^{\mathrm{v}}_{yy}\Big)ds \tag{3.6}$$

and

$$\mathbb{E}\left[\pi^{\mathrm{v}2}\big(t+\Delta t, y(t+\Delta t)\big)\,\middle|\,\mathscr{F}_t\right] = \pi^{\mathrm{v}2}\big(t, y(t)\big) + \mathbb{E}_t\int_t^{t+\Delta t}\left[2\pi^{\mathrm{v}}(s, y(s))\Big(\pi^{v}_t + a\pi^{v}_y + \frac{1}{2}b^2\pi^{v}_{yy}\Big) + (b\pi^{\mathrm{v}}_y)^2\right]ds. \tag{3.7}$$

From (3.6) we also have

$$\begin{aligned}\Big[\mathbb{E}\big[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)|\mathscr{F}_t\big]\Big]^2 &= \pi^{\mathrm{v}2}\big(t, y(t)\big) + 2\pi^{\mathrm{v}}(t, y(t))\mathbb{E}_t\int_t^{t+\Delta t}\Big(\pi^{\mathrm{v}}_t + a\pi^{\mathrm{v}}_y + \frac{1}{2}b^2\pi^{\mathrm{v}}_{yy}\Big)ds\\ &\quad + \left(\mathbb{E}_t\int_t^{t+\Delta t}\Big(\pi^{\mathrm{v}}_t + a\pi^{\mathrm{v}}_y + \frac{1}{2}b^2\pi^{\mathrm{v}}_{yy}\Big)ds\right)^2.\end{aligned} \tag{3.8}$$

Substituting (3.6), (3.7) and (3.8) into (3.3) and simplifying the notation gives us

$$
\begin{aligned}
\pi(t,y(t)) = \pi(t,y(t)) + \mathbb{E}_t &\left[\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}}(s,y(s)) + a\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(s,y(s)) + \frac{1}{2}\alpha\left(b\pi_y^{\mathrm{v}}(s,y(s))\right)^2\right)ds\right] \\
&+\tfrac{1}{2}\alpha\left[2\mathbb{E}_t\int_t^{t+\Delta t}\left(\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(t,y(t))\right)\left(\pi_t^{\mathrm{v}}(s,y(s)) + a\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(s,y(s))\right)ds\right] \\
&-\tfrac{1}{2}\alpha\left(\mathbb{E}_t\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}}(s,y(s)) + a\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(s,y(s))\right)ds\right)^2
\end{aligned}
\tag{3.9}
$$

If we divide both sides of the equation by $\Delta t$ and take the limit when $\Delta t \to 0$, we get the following after simplifying the notation:

$$
\begin{aligned}
0 = \mathbb{E}_t &\left[\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}}(s,y(s)) + a\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(s,y(s)) + \frac{1}{2}\alpha\left(b\pi_y^{\mathrm{v}}(s,y(s))\right)^2\right)ds\right] \\
&+\alpha\mathbb{E}_t\left[\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\left(\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(t,y(t))\right)\left(\pi_t^{\mathrm{v}}(s,y(s)) + a\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(s,y(s))\right)ds\right] \\
&-\tfrac{1}{2}\alpha\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\mathbb{E}_t\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}}(s,y(s)) + a\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(s,y(s))\right)ds\right)^2 \\
&= \pi_t^{\mathrm{v}}(t,y(t)) + a\pi_y^{\mathrm{v}}(t,y(t)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(t,y(t)) + \frac{1}{2}\alpha\left(b\pi_y^{\mathrm{v}}(t,y(t))\right)^2 \\
&+\alpha\mathbb{E}_t\left[\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\pi^{\mathrm{v}}(s,y(s))\left(\pi_t^{\mathrm{v}}(s,y(s)) + a\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(s,y(s))\right)ds\right] \\
&-\alpha\mathbb{E}_t\left[\pi^{\mathrm{v}}(t,y(t))\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}}(s,y(s)) + a\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(s,y(s))\right)ds\right] \\
&-\tfrac{1}{2}\alpha\mathbb{E}_t\left(\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\right)ds\right)\left(\mathbb{E}_t\lim_{\Delta t\to 0}\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\right)ds\right)
\end{aligned}
\tag{3.10}
$$

where, in the last term, we suppressed $(s, y(s))$ for all the integrands to shorten the notation. In the first term, we can assume the whole integrand as a continuous differentiable function

$$f(s,y(s)) = \pi_t^{\mathrm{v}}(s,y(s)) + a(s,y(s))\pi_y^{\mathrm{v}}(s,y(s)) + \frac{1}{2}b^2(s,y(s))\pi_{yy}^{\mathrm{v}}(s,y(s)) + \frac{1}{2}\alpha\left(b(s,y(s))\pi_y^{\mathrm{v}}(s,y(s))\right)^2.$$

Therefore by the definition of the integration for such a function, the limit of integral in the first equality above will clearly be equal to $f(t, y(t))$. By the same rule of ordinary calculus, the second and third terms are equal in the limit to

$$\pi^{\mathrm{v}}(t,y(t))\left(\pi_t^{\mathrm{v}}(t,y(t)) + a\pi_y^{\mathrm{v}}(t,y(t)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(t,y(t))\right)$$

and they cancel out to zero. The fourth term is the production of two expectation limits where the first integral uses the ordinary calculus as above in the limit result in $\pi_t^{\mathrm{v}}(t,y(t)) + a\pi_y^{\mathrm{v}}(t,y(t)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}(t,y(t))$, and the second integral is equal to zero, making the whole term equal to zero. Therefore we obtain a partial differential equation for the Variance price,

$$\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}} + \frac{1}{2}\alpha\left(b\pi_y^{\mathrm{v}}\right)^2 = 0 \tag{3.11}$$

where we have again suppressed the dependence on $t$ and $y(t)$ to lighten the notation.

Note that due to the appearance of the quadratic term $(b\pi_y)^2$, equation (3.11) is a semi-linear PDE that describes the behavior of the Variance price $\pi^{\mathrm{v}}(t,y)$ as a function of $t$ and $y$. The PDE is subject to the boundary condition $\pi^{\mathrm{v}}\big(T,y(T)\big) = f\big(y(T)\big)$, which is the payoff for the insurance contract at time $T$. This PDE is equivalent to a Backward Stochastic Differential Equations (BSDE) with the quadratic driver $g(t,Z) = \frac{1}{2}\alpha(bZ)^2$. The existence of the solutions of BSDE has been investigated in numerous studies. See for example Delong (2013).

### 3.1.1 Explicit Solution of the PDE

In this particular case, we can construct the solution of (3.11) explicitly by employing a Hopf-Cole transformation of the solution that removes the non-linearity from the PDE. The result is only valid if $\alpha$ is a constant. Consider the auxiliary function $h^{\mathrm{v}}(t,y) := \exp\{\alpha\pi^{\mathrm{v}}(t,y)\}$. The original function $\pi^{\mathrm{v}}(t,y)$ can be obtained from the inverse relation $\pi^{\mathrm{v}}(t,y) = \frac{1}{\alpha}\ln h^{\mathrm{v}}(t,y)$. If we now apply the chain-rule of differentiation, we can express the partial derivatives of $\pi^{\mathrm{v}}()$ in terms of $h^{\mathrm{v}}()$ as

$$\pi^{\mathrm{v}}_t = \frac{1}{\alpha}\frac{h^{\mathrm{v}}_t}{h^{\mathrm{v}}}, \quad \pi^{\mathrm{v}}_y = \frac{1}{\alpha}\frac{h^{\mathrm{v}}_y}{h^{\mathrm{v}}}, \quad \pi^{\mathrm{v}}_{yy} = \frac{1}{\alpha}\frac{h^{\mathrm{v}}_{yy}h^{\mathrm{v}} - (h^{\mathrm{v}}_y)^2}{(h^{\mathrm{v}})^2}. \tag{3.12}$$

If we substitute these expressions into (3.11), the non-linear terms are canceled and we obtain a linear PDE for $h^{\mathrm{v}}(t,y)$:

$$h^{\mathrm{v}}_t + ah^{\mathrm{v}}_y + \tfrac{1}{2}b^2 h^{\mathrm{v}}_{yy} = 0. \tag{3.13}$$

Hence, by considering the transformed function $h^{\mathrm{v}}(t,y)$, we have managed to obtain a linear PDE for $h^{\mathrm{v}}()$. The boundary condition at $T$ is given by $h^{\mathrm{v}}(T,y(T)) = \exp\{\alpha\pi^{\mathrm{v}}(T,y(T))\} = \exp\{\alpha f(y(T))\}$. Using the Feynman-Kaç formula, we can express the solution of (3.13) as

$$h^{\mathrm{v}}(t,y) = \mathbb{E}_t\left[e^{\alpha f(y(T))}\middle| y(t) = y\right], \tag{3.14}$$

where the expectation is taken with respect to the stochastic process $y(t)$ defined in equation (3.1) conditional on the information that at time $t$ the process $y(t)$ is equal to $y$. From the representation (3.14), it immediately follows that we can express $\pi^{\mathrm{v}}(t,y)$ as

$$\pi^{\mathrm{v}}(t,y) = \frac{1}{\alpha}\ln\mathbb{E}_t\left[e^{\alpha f(y(T))}\middle| y(t) = y\right]. \tag{3.15}$$

The form of the Variance price in the expectation part is equal to the moment generating function of the time $T$ payoff function $f(y(T))$, where for any known distribution of $f$ it will be easy to find a unique closed form formula for the premium. Also note that this representation of $\pi^{\mathrm{v}}()$ is identical to the exponential indifference price, which has been studied extensively in recent years. See, for example, Henderson (2002), Young and Zariphopoulou (2002), and Musiela and Zariphopoulou (2004). For an overview of recent advances in indifference pricing, we refer to the book by Carmona (2009).

To summarize this section, we have established that the continuous-time limit of the iterated actuarial Variance principle is the exponential indifference price when $\alpha$ is constant.

## 3.2 Variance Pricing With Discounting

Up to now we have ignored discounting in our derivation. (Or equivalently, we assumed that the interest rate is equal to zero.) In a time-consistent setting, it is important to take discounting into consideration, as money today cannot be compared to money tomorrow.

If we consider the definition of the Variance principle given in (3.2), it seems that we are adding apples and oranges. The first term $\mathbb{E}_t[f(y(T))]$ is a quantity in monetary units (say €) at time $T$. However, the second term $\mathbb{V}\mathrm{ar}_t[f(y(T))]$ is basically the expectation of $f(y(T))^2$, and is therefore a quantity in units of $(€)^2$. We can rectify this situation by understanding that the parameter $\alpha$ is not a dimensionless quantity, but is a quantity expressed in units of 1/€. This should not come as a surprise. The parameter $\alpha$ is similar to the absolute risk aversion parameter introduced by the seminal paper of Pratt (1964) in which he derives the Variance principle as an approximation "in the small" of the price that an economic agent facing a decision under uncertainty should ask.

To stress in our notation the units in which the absolute risk aversion $\alpha$ is expressed, we will rewrite the absolute risk aversion as the relative risk aversion $\gamma$ (also introduced by Pratt, 1964), which is a dimensionless quantity, divided by a benchmark wealth-level $X(T)$, which is expressed in € at time $T$. If we now assume a constant rate of interest $r$, we can set our benchmark wealth as $X(T) = X_0 e^{rT}$. We can then rewrite our Variance principle as

$$\Pi^{\mathrm{v}}_t[f(y(T))] = \mathbb{E}_t[f(y(T))] + \tfrac{1}{2}\frac{\gamma}{X_0 e^{rT}}\mathbb{V}\mathrm{ar}_t[f(y(T))]. \tag{3.16}$$

Note that $\Pi^{\mathrm{v}}_t[]$ leads to a "forward" price expressed in units of € at time $T$.

Given the enhanced definition (3.16) of the Variance principle including discounting, we can now proceed as in Section 3.1. For a one-period valuation, we apply the following expression for the price:

$$\pi^{\mathrm{v}}\big(t, y(t)\big) \;=\; e^{-r\Delta t}\Big(\mathbb{E}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right] \;+\; \tfrac{1}{2}\frac{\gamma}{X_0 e^{r(t+\Delta t)}}\mathbb{V}\mathrm{ar}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right]\Big). \tag{3.17}$$

Note that we have included an additional discounting term $e^{-r\Delta t}$ to discount the values from time $t+\Delta t$ back to time $t$. Multiplying both sides of (3.17) by $e^{r\Delta t}$, we have rearranged the equation (3.9) as below:

$$\begin{aligned}(e^{r\Delta t}-1)\pi(t,y(t)) = \mathbb{E}\int_t^{t+\Delta t}&\left(\pi^{\mathrm{v}}_t + a\pi^{\mathrm{v}}_y + \frac{1}{2}b^2\pi^{\mathrm{v}}_{yy} + \frac{1}{2}\alpha\big(b\pi^{\mathrm{v}}_y\big)^2\right)ds\\
&+\tfrac{1}{2}\frac{\gamma}{X_0e^{rt}}\left[2\mathbb{E}\int_t^{t+\Delta t}\left(\pi^{\mathrm{v}}(s,y(s))-\pi^{\mathrm{v}}(t,y(t))\right)\left(\pi^{\mathrm{v}}_t + a\pi^{\mathrm{v}}_y + \frac{1}{2}b^2\pi^{\mathrm{v}}_{yy}\right)ds\right]\\
&-\tfrac{1}{2}\frac{\gamma}{X_0e^{rt}}\left(\mathbb{E}_t\int_t^{t+\Delta t}\left(\pi^{\mathrm{v}}_t(s,y(s)) + a\pi^{\mathrm{v}}_y + \frac{1}{2}b^2\pi^{\mathrm{v}}_{yy}\right)ds\right)^2\end{aligned} \tag{3.18}$$

Using a similar derivation as before, and dividing both sides by $\Delta t$ and taking the limit when $\Delta t \to 0$, we arrive at the following partial differential equation for $\pi^{\mathrm{v}}(t,y)$:

$$\pi^{\mathrm{v}}_t + a\pi^{\mathrm{v}}_y + \tfrac{1}{2}b^2\pi^{\mathrm{v}}_{yy} + \tfrac{1}{2}\frac{\gamma}{X_0e^{rt}}(b\pi^{\mathrm{v}}_y)^2 - r\pi^{\mathrm{v}} = 0. \tag{3.19}$$

This non-linear PDE can again be linearized by considering $h^{\mathrm{v}}(t,y) = \exp(\frac{\gamma}{X_0e^{rt}}\pi^{\mathrm{v}}(t,y))$ transformation, which leads to the following expression for the solution of (3.19):

$$\pi^{\mathrm{v}}(t,y) = \frac{X_0e^{rt}}{\gamma}\ln\mathbb{E}\left[e^{\frac{\gamma}{X_0e^{rT}}f(y(T))}\,\middle|\, y(t)=y\right]. \tag{3.20}$$

This result shows that the discounting is incorporated into the non-linear pricing formula, by expressing all units relative to the "benchmark wealth" $X(t) = X_0 e^{rt}$.[1] See the chapter written by Musiela and Zariphopoulou (2009) in the book by Carmona (2009).

### 3.2.1 Current price as benchmark

In the previous subsection we took the benchmark wealth to be a risk-free investment $X_0 e^{rt}$. Another interesting example can be found when we consider the current price $\pi(t,y)$ as the benchmark wealth.

This leads to a new pricing operator, which we will denote by $\pi^{\mathrm{p}}()$. The one-step valuation is then given as

$$\pi^{\mathrm{p}}\big(t,y(t)\big) = e^{-r\Delta t}\Big(\mathbb{E}_t[\pi^{\mathrm{p}}\big(t+\Delta t, y(t+\Delta t)\big)] + \tfrac{1}{2}\gamma\frac{\mathbb{V}\mathrm{ar}_t[\pi^{\mathrm{p}}\big(t+\Delta t, y(t+\Delta t)\big)]}{\mathbb{E}_t[\pi^{\mathrm{p}}\big(t+\Delta t, y(t+\Delta t)\big)]}\Big). \tag{3.21}$$

Hence, we assume that we want to measure the variance of $\pi^{\mathrm{p}}()$ relative to the expected value of $\pi^{\mathrm{p}}()$. Obviously, this will only be well-defined if $\pi^{\mathrm{p}}(t,y)$ is strictly positive for all $(t,y)$.

When we employ the Itô formula for $\pi^P$ and take the limit for $\Delta t \to 0$, we obtain the following PDE:

$$\pi^{\mathrm{p}}_t + a\pi^{\mathrm{p}}_y + \tfrac{1}{2}b^2\pi^{\mathrm{p}}_{yy} + \tfrac{1}{2}\frac{\gamma}{\pi^{\mathrm{p}}}(b\pi^{\mathrm{p}}_y)^2 - r\pi^{\mathrm{p}} = 0. \tag{3.22}$$

Again, we can study the solution of (3.22) by employing a transformation of the solution that removes the non-linearity from the PDE. Consider the auxiliary function $h^{\mathrm{p}}(t,y) := (\pi^{\mathrm{p}}(t,y))^{1/q}$. The original function can be obtained from the inverse relationship $\pi^{\mathrm{p}}(t,y) = (h^{\mathrm{p}}(t,y))^q$. If we now apply the chain rule, we can express the partial derivatives of $\pi^{\mathrm{p}}$ in terms of $h^{\mathrm{p}}$ as

$$\pi^{\mathrm{p}}_t = q(h^{\mathrm{p}})^{q-1}h^{\mathrm{p}}_t, \quad \pi^{\mathrm{p}}_y = q(h^{\mathrm{p}})^{q-1}h^{\mathrm{p}}_y, \quad \pi^{\mathrm{p}}_{yy} = q(h^{\mathrm{p}})^{q-1}\left(\frac{q-1}{h^{\mathrm{p}}}(h^{\mathrm{p}}_y)^2 + h^{\mathrm{p}}_{yy}\right). \tag{3.23}$$

If we substitute these expressions into (3.22) and simplify, we obtain

$$h^{\mathrm{p}}_t + ah^{\mathrm{p}}_y + \tfrac{1}{2}b^2\left(\frac{(1+\gamma)q-1}{h^{\mathrm{p}}}(h^{\mathrm{p}}_y)^2 + h^{\mathrm{p}}_{yy}\right) - \frac{r}{q}h^{\mathrm{p}} = 0. \tag{3.24}$$

If we choose $q = 1/(1+\gamma)$, then the non-linear terms cancel out and we obtain a linear PDE for $h^{\mathrm{p}}(t,y)$:

$$h^{\mathrm{p}}_t + ah^{\mathrm{p}}_y + \tfrac{1}{2}b^2h^{\mathrm{p}}_{yy} - r(1+\gamma)h^{\mathrm{p}} = 0. \tag{3.25}$$

The boundary condition at $T$ is given by $h^{\mathrm{p}}(T,y(T)) = \pi^{\mathrm{v}}(T,y(T))^{1+\gamma} = f(y(T))^{1+\gamma}$. If we use the Feynman-Kaç formula, we can express the solution of (3.25) as

$$h^{\mathrm{p}}(t,y) = \mathbb{E}_t\left[e^{-r(1+\gamma)(T-t)}f(y(T))^{1+\gamma}\middle| y(t) = y\right], \tag{3.26}$$

where the expectation is taken with respect to the stochastic process $y(t)$ defined in equation (3.1) conditional on the information that at time $t$ the process $y(t)$ is equal to $y$. From the representation (3.26), it immediately follows that we can express $\pi^{\mathrm{p}}(t,y)$ as

$$\pi^{\mathrm{p}}(t,y) = e^{-r(T-t)}\left(\mathbb{E}_t\left[f(y(T))^{1+\gamma}\middle| y(t) = y\right]\right)^{\frac{1}{1+\gamma}}. \tag{3.27}$$

[1] For general results concerning "benchmark pricing" in a linear setting, we refer to Platen (2006) and the book by Platen and Heath (2006).

Note that this representation of the price $\pi^{\mathrm{p}}()$ also arises in the study of indifference pricing under power-utility functions, and the related notion of pricing under "$q$-optimal" measures. See, for example, Hobson (2004) and Henderson and Hobson (2009).

### 3.3 Mean Value Principle

The examples we gave in the previous subsections are all special cases of the Mean Value principle, which is defined as

$$\Pi_t^{\mathrm{m}}[f(y(T))] = v^{-1}\left(\mathbb{E}_t[v(f(y(T)))]\right) \tag{3.28}$$

for any convex and increasing function $v()$ (see Kaas et al., 2008, Chap. 5).

Once more, we need to pay attention to units. If we want to apply a general function $v()$ to a value (expressed in units of €), we need to make sure that the argument of $v()$ is dimensionless. The easiest way to achieve this is to express the argument for $v()$ in "forward terms". For a single time step of $(t, t+\Delta t)$, we therefore obtain the following expression for the price:

$$\pi^{\mathrm{m}}\big(t, y(t)\big) = v^{-1}\Big(\mathbb{E}_t\left[v\big(e^{-r\Delta t}\pi^{\mathrm{m}}\big(t+\Delta t, y(t+\Delta t)\big)\big)\right]\Big). \tag{3.29}$$

We can rewrite this definition as

$$v\left(\frac{\pi^{\mathrm{m}}\big(t, y(t)\big)}{e^{rt}}\right) = \mathbb{E}_t\left[v\left(\frac{\pi^{\mathrm{m}}\big(t+\Delta t, y(t+\Delta t)\big)}{e^{r(t+\Delta t)}}\right)\right], \tag{3.30}$$

from which it is immediately clear that the "distorted" value $v(\pi^{\mathrm{m}}(t,y)/e^{rt})$ is linear and that it therefore satisfies the Feynman-Kaç formula. Therefore, its solution corresponds exactly to the solutions we found in the previous subsections.

As $v()$ is a Borel-measurable function, and if we assume $\mathbb{E}_t|v(\pi^{\mathrm{m}}(t,y)| < \infty$, it becomes clear that the stochastic process $v\big(\pi^{\mathrm{m}}\big(t, y(t)\big)/e^{rt}\big)$ is a local martingale as the conditional expectation $\mathbb{E}_t[.]$ is a martingale (see Shreve, 2010, Lemma 6.4.2). We can use this consideration to find the corresponding PDE for the price $\pi^{\mathrm{m}}(t,y)$. We can simplify this by defining the new process as $\pi^{\mathrm{mf}}(t,y) := \pi^{\mathrm{m}}(t,y)/e^{rt}$, which is the price expressed in forward terms. We use the Itô formula derivation for both stochastic processes $\pi^{\mathrm{mf}}(t,y)$, and also $v\left(\pi^{\mathrm{mf}}(t,y)/e^{rt}\right)$ with respect to $\pi^{\mathrm{mf}}(t,y)$. By applying the Itô formula for $\pi^{\mathrm{mf}}(t,y)$ with respect to (3.1), we get

$$\pi^{\mathrm{mf}}(t,y) = \int_0^t (\pi_t^{\mathrm{mf}} + a\pi_y^{\mathrm{mf}} + \tfrac{1}{2}b^2\pi_{yy}^{\mathrm{mf}})ds + \int_0^t b\pi_y^{\mathrm{mf}} dW(s), \tag{3.31}$$

where we assumed $y(0) = \pi^{\mathrm{mf}}(0, y(0)) = 0$ at time $t = 0$. By applying the differential form of the Itô formula to function $v(\pi^{\mathrm{mf}}(t,y))$ we get,

$$dv\left(\pi^{\mathrm{mf}}(t,y)\right) = \left[\big(\pi_t^{\mathrm{mf}} + a\pi_y^{\mathrm{mf}} + \tfrac{1}{2}b\pi_{yy}^{\mathrm{mf}}\big)v_y(\pi^{\mathrm{mf}}) + \tfrac{1}{2}(b\pi_y^{\mathrm{mf}})^2 v_{yy}(\pi^{\mathrm{mf}})\right]ds + b\pi_y^{\mathrm{mf}} v_y(\pi^{\mathrm{mf}})dW(s). \tag{3.32}$$

Note that by considering $\pi^{\mathrm{mf}}(t,y)$, $v()$ is no longer assumed to be a function of $t$.

As $v\left(\pi^{\mathrm{mf}}(t,y)\right)$ is a martingale process, the drift term can be set equal to zero:

$$\big(\pi_t^{\mathrm{mf}} + a\pi_y^{\mathrm{mf}} + \tfrac{1}{2}b\pi_{yy}^{\mathrm{mf}}\big)v_y(\pi^{\mathrm{mf}}) + \tfrac{1}{2}(b\pi_y^{\mathrm{mf}})^2 v_{yy}(\pi^{\mathrm{mf}}) = 0.$$

If we divide both sides by $v_y(\pi^{\mathrm{mf}})$, we obtain the PDE for $\pi^{\mathrm{mf}}$,

$$\pi_t^{\mathrm{mf}} + a\pi_y^{\mathrm{mf}} + \tfrac{1}{2}b\pi_{yy}^{\mathrm{mf}} + \tfrac{1}{2}\frac{v_{yy}(\pi^{\mathrm{mf}})}{v_y(\pi^{\mathrm{mf}})}(b\pi_y^{\mathrm{mf}})^2 = 0, \tag{3.33}$$

where this special derivation is true for any time step and we can relax the assumption of taking the limit when $\Delta t \to 0$. If we substitute the $\pi^{\mathrm{mf}} = e^{-rt}\pi^{\mathrm{m}}$ in (3.33) and simplify the notation, the corresponding PDE for the discounted Mean Value price will be

$$\pi_t^{\mathrm{m}} + a\pi_y^{\mathrm{m}} + \tfrac{1}{2}b\pi_{yy}^{\mathrm{m}} + \tfrac{1}{2}\frac{v_{yy}(\pi^{\mathrm{m}})}{v_y(\pi^{\mathrm{m}})}(b\pi_y^{\mathrm{m}})^2 - r\pi^{\mathrm{m}} = 0. \tag{3.34}$$

In both equations (3.33) and (3.34), we observe that the coefficient $v_{yy}/v_y$ in front of the non-linear term can be identified as the "local risk aversion", induced by the function $v()$ at the current value $\pi^{\mathrm{mf}}()$. Note that since the function $v()$ is increasing and convex by assumption, $v_{yy}/v_y$ is positive. Both forms of the PDE for the Mean Value principle are similar to the PDE of the Variance principle and have a quadratic driver for the equivalent BSDE in a time-consistent framework.

# 4 Standard-Deviation Pricing

## 4.1 Standard-Deviation Principle

Another well-known actuarial pricing principle is the Standard-Deviation principle, defined as

$$\Pi_t^{\mathrm{s}}[f(y(T))] = \mathbb{E}_t[f(y(T))] + \beta\sqrt{\mathbb{V}\mathrm{ar}_t[f(y(T))]} \tag{4.1}$$

(see Kaas et al., 2008). Please note that in this case we also need to be careful about the dimensionality of the parameter $\beta$. Even though the expectation and the standard deviation are expressed in units of €, they both have different "time scales". If we use smaller time scales (as we will be doing when considering the limit for $\Delta t \to 0$) then, due to the diffusion term $dW$ of the process $y$, we have the property that the expectation of any function $f(y)$ scales linearly with $\Delta t$, but the standard deviation scales with $\sqrt{\Delta t}$. This means that the standard deviation term will literally overpower the expectation term for small $\Delta t$. Therefore, the only way to obtain a well-defined limit for $\Delta t \to 0$ is if we take $\beta\sqrt{\Delta t}$ as the parameter for the Standard-Deviation principle for a time step $(t, t+\Delta t)$.

Another way of understanding this result is to consider the following example. If we want to compare a standard deviation measured over an annual time step with a standard deviation measured over a monthly time step, we have to scale the annual outcome with $\sqrt{1/12}$ to get a fair comparison.

Given the above discussion on dimensionality and the time scales, we will then get the following expression for the one-step price:

$$\pi^{\mathrm{s}}\big(t, y(t)\big) = e^{-r\Delta t}\Big(\mathbb{E}_t[\pi^{\mathrm{s}}\big(t+\Delta t, y(t+\Delta t)\big)] + \beta\sqrt{\Delta t}\sqrt{\mathbb{V}\mathrm{ar}_t[\pi^{\mathrm{s}}\big(t+\Delta t, y(t+\Delta t)\big)]}\Big). \tag{4.2}$$

Using a similar derivation by means of the Itô formula in Section 3.1, we can represent the alternative equation of (3.18) as

$$
\begin{aligned}
(e^{r\Delta t}-1)\pi^{\mathrm{s}}(t,y(t)) &= \mathbb{E}\left[\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{s}}(s,y(s))+a\pi_y^{\mathrm{s}}(s,y(s))+\frac{1}{2}b^2\pi_{yy}^{\mathrm{s}}(s,y(s))\right)ds\right] \\
&+\beta\sqrt{\Delta t}\sqrt{2\mathbb{E}\left[\int_t^{t+\Delta t}\left(\pi^{\mathrm{s}}(s,y(s))-\pi^{\mathrm{s}}(t,y(t))\right)\left(\pi_t^{\mathrm{s}}+a\pi_y^{\mathrm{s}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{s}}\right)ds\right]+\mathbb{E}\left[\int_t^{t+\Delta t}\left(b\pi_y^{\mathrm{s}}\right)^2 ds\right]-\left(\mathbb{E}_t\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}}+a\pi_y^{\mathrm{v}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\right)ds\right)^2}
\end{aligned}
\tag{4.3}
$$

where we have suppressed the dependence of all the derivative terms of $\pi$ on $(s,y(s))$ to lighten the notation. Dividing both sides by $\Delta t$ and taking the limit when $\Delta t\to 0$, we obtain,

$$
\begin{aligned}
r\pi^{\mathrm{s}}(t,y(t)) &= \mathbb{E}\left[\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{s}}+a\pi_y^{\mathrm{s}}+\frac{1}{2}b^2\pi_{yy}^{\mathrm{s}}\right)ds\right] \\
&+\beta\sqrt{2\mathbb{E}\left[\lim_{\Delta t\to 0}\tfrac{1}{\Delta t}\int_t^{t+\Delta t}\left(\pi^{\mathrm{s}}(s,y(s))-\pi^{\mathrm{s}}(t,y(t))\right)\left(\pi_t^{\mathrm{s}}+a\pi_y^{\mathrm{s}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{s}}\right)ds\right]+\mathbb{E}\left[\lim_{\Delta t\to 0}\tfrac{1}{\Delta t}\int_t^{t+\Delta t}\left(b\pi_y^{\mathrm{s}}\right)^2 ds\right]}
\end{aligned}
\tag{4.4}
$$

where we omit the third term in the square root function since we proved before that its limit will be zero when $\Delta t\to 0$. The first term by definition of the limit in the right hand side of the equation is the integrand itself in terms of $(t,y(t))$. The first integral under the square root is zero in the limit and the second integral is equal to $(b\pi^{\mathrm{s}}(t,y(t)))^2$ by definition of the limit. Therefore, we arrive at the following partial differential equation for $\pi^{\mathrm{s}}(t,y(t))$:

$$
\pi_t^{\mathrm{s}}+a\pi_y^{\mathrm{s}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{s}}+\beta\sqrt{(b\pi_y^{\mathrm{s}})^2}-r\pi^{\mathrm{s}}=0,
\tag{4.5}
$$

which can be rewritten as

$$
\pi_t^{\mathrm{s}}+a\pi_y^{\mathrm{s}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{s}}+\beta b|\pi_y^{\mathrm{s}}|-r\pi^{\mathrm{s}}=0.
\tag{4.6}
$$

This is again a semi-linear PDE that can be represented by a BSDE with a Lipschitz driver, $g(t,Z)=\beta|bZ|$. However, the semi-linearity is much more benign in this case. Whenever the sign of the partial derivative $\pi_y^{\mathrm{s}}$ does not change anywhere in the domain of $y$ (i.e. the function $\pi^{\mathrm{s}}$ either monotonically increases or monotonically decreases in $y$), then (4.6) is reduced to the linear PDE:

$$
\pi_t^{\mathrm{s}}+(a\pm\beta b)\pi_y^{\mathrm{s}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{s}}-r\pi^{\mathrm{s}}=0,
\tag{4.7}
$$

where the sign of $\pm\beta b$ depends on the (uniquely defined) sign of $\pi_y^{\mathrm{s}}$.

Using the Feynman-Kaç formula, we can represent the solution of (4.7) as follows:

$$
\pi^{\mathrm{s}}(t,y)=\mathbb{E}_t^{\mathbb{S}}\left[e^{-r(T-t)}f(y(T))\middle|y(t)=y\right],
\tag{4.8}
$$

where $\mathbb{E}_t^{\mathbb{S}}[]$ denotes the expectation at time $t$ with respect to the "risk-adjusted" process $y^{\mathbb{S}}$ defined as

$$
dy^{\mathbb{S}}=\left(a(t,y)\pm\beta b(t,y)\right)dt+b(t,y)\,dW^{\mathbb{S}}.
\tag{4.9}
$$

The risk-adjusted process is consistent with the concept of actuarial prudence, where the insurer calculates the premium using an adjusted drift to make a more conservative assessment of expectation. Mathematically, the drift rate is adjusted upwards $(a+\beta b)$ if the payoff $f(y)$ monotonically increases in $y$, and is adjusted downwards $(a-\beta b)$ if $f(y)$ monotonically decreases in $y$. So, the risk adjustment is always in the "upwind" direction of the risk, making the price $\pi^{\mathrm{s}}$ more expensive than the real-world expectation $\mathbb{E}[f(y)]$.

## 4.2 Cost-of-Capital Principle

Another actuarial pricing principle is the Cost-of-Capital principle. This was introduced by a Swiss insurance supervisor as a part of the method used to calculate solvency capitals for insurance companies (Keller and Luder, 2004).[2] The Cost-of-Capital method has been widely adopted by the insurance industry in Europe, and has also been prescribed as the standard method by the European Insurance and Pensions Supervisor for the Quantitative Impact Studies (see EIOPA, 2010).

The Cost-of-Capital principle is based on the following economic reasoning. We first consider the "expected loss" $\mathbb{E}[f(y(T)]$ of the insurance claim $f(y(T))$ as a basis for pricing. In addition, the insurance company needs to hold a capital buffer against the "unexpected loss". This buffer is calculated as a Value-at-Risk (VaR) over a time horizon (typically 1 year) and a probability threshold $q$ (usually 0.995 for insurance). The unexpected loss is then calculated as $\mathrm{VaR}_q\big[f(y(T))-\mathbb{E}[f(y(T))]\big]$. The capital buffer is borrowed from the shareholders of the insurance company; however, there is a small probability $(1-q)$ that the capital buffer is needed to cover an unexpected loss. Hence, the shareholders require a compensation for this risk in the form of a "cost of capital". This cost of capital needs to be included in the pricing of the insurance contract. If we denote the cost of capital by $\delta$, then the Cost-of-Capital principle is given by

$$\Pi_t^{\mathrm{c}}[f(y(T))] = \mathbb{E}_t[f(y(T))] + \delta \mathrm{VaR}_{q,t}\Big[f(y(T)) - \mathbb{E}_t[f(y(T))]\Big]. \tag{4.10}$$

Note that we also need to be careful about the dimensionality of the different terms in this case. First, we are comparing VaR quantities at different time scales, and these have to be scaled back to a per annum basis. To do this we divide the VaR term by $\sqrt{\Delta t}$. We must then realize that the cost of capital $\delta$ behaves like an interest rate: it is the compensation the insurance company needs to pay its shareholders for borrowing the buffer capital over a certain period. The cost of capital is expressed as a percentage per annum; hence over a time-step $\Delta t$ the insurance company will have to pay a compensation of $\delta\Delta t$ per € of buffer capital. As a result, we obtain a "net scaling" of $\delta\Delta t/\sqrt{\Delta t} = \delta\sqrt{\Delta t}$. Note that this is the same scaling as for the Standard-Deviation principle.

For a single time-step, we therefore get the following expression for the Cost-of-Capital price:

$$\begin{aligned}\pi^{\mathrm{c}}\big(t,y(t)\big) = e^{-r\Delta t}\Big(&\mathbb{E}_t[\pi^{\mathrm{c}}\big(t+\Delta t, y(t+\Delta t)\big)] \\ &+ \delta\sqrt{\Delta t}\mathrm{VaR}_{q,t}\Big[\pi^{\mathrm{c}}\big(t+\Delta t, y(t+\Delta t)\big) - \mathbb{E}_t[\pi^{\mathrm{c}}\big(t+\Delta t, y(t+\Delta t)\big)]\Big]\Big).\end{aligned} \tag{4.11}$$

We recall $\pi^{\mathrm{c}}(t+\Delta t, y(t+\Delta t))$ and $\mathbb{E}_t\big[\pi^{\mathrm{c}}(t+\Delta t, y(t+\Delta t))\big]$ from equations (3.4) and (3.6)

[2]For a critical discussion on the risk measure implied by the Swiss Solvency Test, we refer to Filipovic and Vogelpoth (2008).

by suppressing $(s, y(s))$ for all integrand terms and putting them into (4.11)

$$
\begin{aligned}
e^{r\Delta t}\pi^{c}\big(t, y(t)\big) =& \pi^{c}\big(t, y(t)\big) + \mathbb{E}\left[\int_t^{t+\Delta t}\Big(\pi^{c}_t(s, y(s)) + a\pi^{c}_y(s, y(s)) + \frac{1}{2}b^2\pi^{c}_{yy}(s, y(s))\Big)ds\right] \\
&+ \delta\sqrt{\Delta t}\mathrm{VaR}_{1-q,t}\Bigg(\int_t^{t+\Delta t}\Big(\pi^{c}_t(s, y(s)) + a\pi^{c}_y(s, y(s)) + \frac{1}{2}b^2\pi^{c}_{yy}(s, y(s))\Big)ds \\
&- \mathbb{E}_t\left[\int_t^{t+\Delta t}\Big(\pi^{c}_t(s, y(s)) + a\pi^{c}_y(s, y(s)) + \frac{1}{2}b^2\pi^{c}_{yy}(s, y(s))\Big)ds\right] \\
&+ \int_t^{t+\Delta t} b\pi^{c}_y(s, y(s))dW(s)\Bigg),
\end{aligned}
\tag{4.12}
$$

where we recall that the $a$ and $b$ drift and diffusion rates under the integration are also functions of $s$ and $y(s)$ for $s > t$.

In the above valuation equation, the continuous time limit of the drift term (integration with respect to $ds$) is equal to its conditional expectation when we divide it by $\Delta t$. By definition of the limit, as we assume $f(s, y(s)) = \pi^{c}_t(s, y(s)) + a\pi^{c}_y(s, y(s)) + \frac{1}{2}b^2\pi^{c}_{yy}(s, y(s))$ is a continuous differentiable function, we have

$$
\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\Big(\pi^{c}_t(s, y(s))+a\pi^{c}_y(s, y(s))+\frac{1}{2}b^2\pi^{c}_{yy}(s, y(s))\Big)ds = \pi^{c}_t(t, y(t))+a\pi^{c}_y(t, y(t))+\frac{1}{2}b^2\pi^{c}_{yy}(t, y(t)).
\tag{4.13}
$$

By linearity of the expectation operator, the limit of the conditional expectation of the above integral when $\Delta t \to 0$, with respect to $\mathscr{F}_t$, is also equal to $f(t, y(t)) = \pi^{c}_t(t, y(t)) + a\pi^{c}_y(t, y(t)) + \frac{1}{2}b^2\pi^{c}_{yy}(t, y(t))$. Thus, for the drift term of $\pi(t+\Delta t, y(t+\Delta t))$ we have the following equality:

$$
\begin{aligned}
&\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\Big(\pi^{c}_t(s, y(s)) + a\pi^{c}_y(s, y(s)) + \frac{1}{2}b^2\pi^{c}_{yy}(s, y(s))\Big)ds \\
&\quad = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[\int_t^{t+\Delta t}\Big(\pi^{c}_t(s, y(s)) + a\pi^{c}_y(s, y(s)) + \frac{1}{2}b^2\pi^{c}_{yy}(s, y(s))\Big)ds\,\middle|\,\mathscr{F}_t\right].
\end{aligned}
\tag{4.14}
$$

Since we will later divide the VaR function in (4.12) by $\Delta t$ and take its limit to find the continuous time limit of the premium, we can omit the term $\int_t^{t+\Delta t} f(s, y(s))ds - \mathbb{E}_t[\int_t^{t+\Delta t} f(s, y(s))ds]$ in the VaR by using (4.14). The only remaining term under the VaR function will then be an Itô integral.

The valuation of the Itô integral under the $\mathrm{VaR}_{1-q,t}$ function is a critical part of this premium. We denote this integral as,

$$
Z(t+\Delta t) = Z(t) + \int_t^{t+\Delta t} b(s, y(s))\pi^{c}_y(s, y(s))dW(s).
\tag{4.15}
$$

In general, the integrand $b(s, y(s))\pi^{c}(s, y(s))$ in (4.15) for $s > t$ is an adapted stochastic process. In this situation, it is difficult to draw inferences about the distribution of the above Itô integral and to give a more direct calculation for $\mathrm{VaR}_{q,t}$. Although we do not know the analytical distribution of $Z(t+\Delta t)$, we can obtain its first two moments with respect to the filtration $\mathscr{F}_t$. As the Itô integral is a martingale, its conditional expectation with respect to the filtration $\mathscr{F}_t$ is zero,

$$
\mathbb{E}\left[\int_t^{t+\Delta t} b(s, y(s))\pi^{c}_y(s, y(s))dW(s)\,\middle|\,\mathscr{F}_t\right] = 0,
\tag{4.16}
$$

where its variance can be obtained based on the Itô isometry for stochastic integrands as follows:

$$\begin{aligned}\mathbb{V}\mathrm{ar}\Big[\int_t^{t+\Delta t} b(s,y(s))\pi_y^c(s,y(s))dW(s)|\mathscr{F}_t\Big] &= \mathbb{E}_t\Big[\Big(\int_t^{t+\Delta t} b(s,y(s))\pi_y^c(s,y(s))dW(s)\Big)^2\Big]\\ &= \int_t^{t+\Delta t}\mathbb{E}_t\Big[\Big(b(s,y(s))\pi_y^c(s,y(s))\Big)^2\Big]ds.\end{aligned} \tag{4.17}$$

Since we want to compute the continuous time limit of the price in an Euler-Maruyama approximation setting when $\Delta t \to 0$, we assume $Z(t+\Delta t) - Z(t)$ as a partition $(t, t+\Delta t)$ of the process $Z$ with drift zero in $[0, T]$. Kloeden and Platen (1999) have discussed the Euler-Maruyama discretization of the stochastic processes. Using the weak convergence of this approximation, we have

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t} b(s,y(s))\pi_y^c(s,y(s))dW(s) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[b(t,y(t))\pi_y^c(t,y(t))\Delta W(t)\Big], \tag{4.18}$$

where $\Delta W(t) = W(t+\Delta t) - W(t)$ is an independent and identically distributed normal random variable with expected value zero and variance $\Delta t$ for all $0 < t \leq T$. Note that at time $t$, $b(t,y(t))\pi_y^c(t,y(t))$ is non-random and when $\Delta t$ is small, the distribution of $Z(t+\Delta t)$ is approximately normal and we can conclude that when $\Delta t \to 0$,

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t} b(s,y(s))\pi_y^c(s,y(s))dW(s) \sim N\left(0, \frac{\Big(b(t,y(t))\pi_y^c(t,y(t))\Big)^2}{\Delta t}\right). \tag{4.19}$$

This also shows that in (4.17), $\lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{V}\mathrm{ar}\Big[\int_t^{t+\Delta t} b(s,y(s))\pi_y^c(s,y(s))dW(s)|\mathscr{F}_t\Big] = (b\pi_y^c)^2$.

By using this approximation and the "translation and scaling invariance" property of the VaR function with respect to a non-negative constant, we can calculate the VaR term in (4.12) when $\Delta t \to 0$:

$$\lim_{\Delta t\to 0}\mathrm{VaR}_{1-q,t}\Big[\frac{1}{\Delta t}\int_t^{t+\Delta t} b(s,y(s))\pi_y^c(s,y(s))dW(s)\Big] = \frac{1}{\sqrt{\Delta t}}b(t,y(t))\Big|\pi_y^c(t,y(t))\Big|\,\Phi^{-1}(1-q). \tag{4.20}$$

Dividing both sides of (4.12) by $\Delta t$, and taking the limit when $\Delta t \to 0$, results in

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{(e^{r\Delta t}-1)\pi^c\big(t,y(t)\big)}{\Delta t} &= \mathbb{E}\Big[\lim_{\Delta t\to 0}\tfrac{1}{\Delta t}\textstyle\int_t^{t+\Delta t}\Big(\pi_t^c(s,y(s)) + a\pi_y^c(s,y(s)) + \tfrac{1}{2}b^2\pi_{yy}^c(s,y(s))\Big)ds\Big]\\ &\quad+\delta\sqrt{\Delta t}\mathrm{VaR}_{1-q,t}\Big[\lim_{\Delta t\to 0}\tfrac{1}{\Delta t}\textstyle\int_t^{t+\Delta t} b\pi_y^c(s,y(s))dW(s)\Big].\end{aligned} \tag{4.21}$$

Similar to the case of the Variance and Standard-Deviation principles, by general definition of the limit and substituting for VaR from (4.20), we derive the related PDE for the Cost-of-Capital premium principle as

$$r\pi^c(t,y(t)) = \pi_t^c(t,y(t)) + a(t,y(t))\pi_y^c(t,y(t)) + \tfrac{1}{2}b^2(t,y(t))\pi_{yy}^c(t,y(t)) + \delta b(t,y(t))\Big|\pi_y^c(t,y(t))\Big|\,\Phi^{-1}(1-q), \tag{4.22}$$

where a simplified notation for $\Phi^{-1}(1-q) = k$ gives

$$\pi_t^c + a\pi_y^c + \tfrac{1}{2}b^2\pi_{yy}^c + \delta kb|\pi_y^c| - r\pi^c = 0. \tag{4.23}$$

This PDE is the same as the one we obtained in (4.6) for the Standard-Deviation price, except for the factor $\delta k$, which replaces $\beta$ in front of $b|\pi^{c}_{y}|$. This should not come as a surprise, since the $(1-q)$-quantile of $y(t+\Delta t)$ for a small time-step $\Delta t$ converges to $k$ times the standard deviation $b\sqrt{\Delta t}$, and hence the Cost-of-Capital pricing operator $\pi^{c}()$ should converge to the Standard-Deviation pricing operator $\pi^{s}()$ with $\beta = \delta k$.

If the payoff $f(y(T))$ is monotonous in $y(T)$, we can represent the Cost-of-Capital price $\pi^{c}(t, y)$ in the same way as the Standard-Deviation price (4.8) with respect to the risk-adjusted process $y$:

$$dy = \big(a(t, y) \pm \delta k b(t, y)\big)\, dt + b(t, y)\, dW. \tag{4.24}$$

# 5 Pricing under Jump Process

In this section, we extend the concept of time-consistent actuarial pricing by adding a jump component to the valuation process. In fact, we generalize the backward iteration of the one-period valuation of the insurance premium principles when the unhedgeable insurance process can also jump by an stochastic arrival time.

Let $(\Omega, \mathscr{F}_t, \mathbb{P})$ $t \geq 0$ be the filtered probability space, $N(t)$ be the Poisson arrival time process of the jumps with intensity $\lambda$, and $C(t)$ be the bounded jump size random variable in $Ł^{\infty}(\mathscr{F}_t)$ with $\mathbb{E}[C(t)] = \beta$. If we exhibit the left continuous version of $y(t)$ as $y(t_-)$, when the $N(t)$-th jump is observed at time $t$, the jump size can be shown as $C(t) = y(t) - y(t_-) = J(t) - J(t_-)$. We use the model of Merton (1976) where the insurance process $y(t)$ follows the jump process of the form

$$dy(t) = a(t, y(t))dt + b(t, y(t))dW(t) + dJ(t), \tag{5.1}$$

with

$$J(t) = \sum_{0<s\leq t} C(s)dN(s)$$

where $J(t)$ is an adapted right continuous pure jump process with $J(0) = 0$ and $dN(s) \in \{0, 1\}$. We assume we have finitely many jumps in any finite time interval of the form $(t, t + \Delta t]$. Moreover, $W(t)$, $N(t)$ and $C(t)$ are $\mathscr{F}_t$-measurable processes with independent increment. Note that $N(t)$ and $C(t)$ are assumed to be independent while together they form a compound Poisson process which is also $\mathscr{F}_t$-measurable with independent increment.

## 5.1 Variance Pricing with Jump

In this section we directly apply the case of Variance pricing with discounting and we employ the one-period valuation of this premium principle to obtain a time-consistent price. We recall (3.17),

$$\pi^{v}\big(t, y(t)\big) = e^{-r\Delta t}\Big(\mathbb{E}_t[\pi^{v}\big(t+\Delta t, y(t+\Delta t)\big)] + \tfrac{1}{2}\frac{\gamma}{X_0 e^{r(t+\Delta t)}}\mathbb{V}\mathrm{ar}_t[\pi^{v}\big(t+\Delta t, y(t+\Delta t)\big)]\Big)$$

as the main pricing rule. We assume that $y(t)$ is a jump process defined in (5.1) and that $\pi^{v}(t, y(t))$ is a sufficiently smooth function and twice continuously differentiable with respect to both $y$ and $t$. Therefore, we use the Itô-Doeblin formula for the jump process to represent the values of $\pi^{\mathrm{v}}(t+\Delta t+y(t+\Delta t))$ and $\pi^{v2}(t+\Delta t+y(t+\Delta t))$ as

$$\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big) = \pi^{\mathrm{v}}\big(t, y(t)\big) + \int_t^{t+\Delta t}\Big(\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\Big)ds + \int_t^{t+\Delta t} b\pi_y^{\mathrm{v}} dW(s) + \sum_{t<s\le t+\Delta t}\big[\pi^{\mathrm{v}}(s, y(s)) - \pi^{\mathrm{v}}(s_-, y(s_-))\big] \quad (5.2)$$

and

$$\pi^{\mathrm{v}2}\big(t+\Delta t, y(t+\Delta t)\big) = \pi^{\mathrm{v}2}\big(t, y(t)\big) + \int_t^{t+\Delta t}\Big((\pi^{\mathrm{v}2})_t + a(\pi^{\mathrm{v}2})_y + \frac{1}{2}b^2(\pi^{\mathrm{v}2})_{yy}\Big)ds + \int_t^{t+\Delta t} b(\pi^{\mathrm{v}2})_y dW(s) + \sum_{t<s\le t+\Delta t}\big[\pi^{v2}(s, y(s)) - \pi^{v2}(s_-, y(s_-))\big], \quad (5.3)$$

where we applied the Itô formula to the $\pi^2$ function in the second equality. For the sake of clarity, we should mention again that the derivative terms under the integrations in both equations are functions of $(s, y(s))$ for $t < s \le t+\Delta t$. We consider $\pi^{\mathrm{v}}(s_-, y(s_-))$ as the left continuous version of the $\pi^{\mathrm{v}}$ and $\pi^{\mathrm{v}}(s, y(s)) - \pi^{\mathrm{v}}(s_-, y(s_-)$ as the possible jump size of $\pi^{\mathrm{v}}$ at time $s$. If there is no jump at $s$, this is equal to zero. The continuous part in both of the above stochastic processes corresponds to the case where $y(t)$ has a simple diffusion with no jump, as seen in (3.4) and (3.5). Similar to our approach in sections 3.1 and 3.2, we have to reform the values of $\mathbb{E}_t[.]$ and $\mathbb{V}\mathrm{ar}_t[.]$ for a case where the basic insurance process $y(t)$ follows a jump process. Taking conditional expectations gives

$$\begin{aligned}\mathbb{E}_t\Big[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\Big] &= \pi^{\mathrm{v}}\big(t, y(t)\big) + \mathbb{E}_t\left[\int_t^{t+\Delta t}\Big(\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\Big)ds\right] \\ &\quad + \mathbb{E}_t\left[\sum_{t<s\le t+\Delta t}\big[\pi^{\mathrm{v}}(s, y(s)) - \pi^{\mathrm{v}}(s_-, y(s_-))\big]\right] \\ &= \pi^{\mathrm{v}}\big(t, y(t)\big) + \mathbb{E}_t\left[\int_t^{t+\Delta t}\Big(\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\Big)ds\right] \\ &\quad + \lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\big[\pi^{\mathrm{v}}(s, y(s)) - \pi^{\mathrm{v}}(s_-, y(s_-))\big]\end{aligned} \quad (5.4)$$

and

$$\begin{aligned}\mathbb{E}_t\Big[\pi^{\mathrm{v}2}\big(t+\Delta t, y(t+\Delta t)\big)\Big] &= \pi^{\mathrm{v}2}\big(t, y(t)\big) + \mathbb{E}_t\left[\int_t^{t+\Delta t}\Big((\pi^{\mathrm{v}2})_t + a(\pi^{\mathrm{v}2})_y + \frac{1}{2}b^2(\pi^{\mathrm{v}2})_{yy}\Big)ds\right] \\ &\quad + \mathbb{E}_t\left[\sum_{t<s\le t+\Delta t}\big[\pi^{\mathrm{v}2}(s, y(s)) - \pi^{\mathrm{v}2}(s_-, y(s_-))\big]\right] \\ &= \pi^{\mathrm{v}2}\big(t, y(t)\big) + \mathbb{E}_t\left[\int_t^{t+\Delta t}\Big[2\pi^{\mathrm{v}}(s, y(s))\Big(\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\Big) + (b\pi_y^{\mathrm{v}})^2\Big]ds\right] \\ &\quad + \lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\Big[\pi^{\mathrm{v}2}(s, y(s)) - \pi^{\mathrm{v}2}(s_-, y(s_-))\Big],\end{aligned} \quad (5.5)$$

where $\mathbb{E}_{(t,t+\Delta t)}$ exhibits the expectation of the possible jump during $(t, t+\Delta t]$. In the second expectation, we used the chain rule to take derivatives of $\pi^{\mathrm{v}2}$ with respect to $t$ and $y$. The expectation of the summation term is rewritten as a product of jump arrival intensity $\lambda\Delta t$

for the time interval $(t, t+\Delta t)$, and $\mathbb{E}_{(t,t+\Delta t)}$. The justification comes from the fact that $N(t+\Delta t) - N(t)$ also follows the Poisson distribution with parameter $\lambda \Delta t$ and then,

$$\begin{aligned}
\mathbb{E}_t\left[\sum_{t<s\leq t+\Delta t}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right]\right] &= \mathbb{E}_t\left[\sum_{i=N(t)}^{N(t+\Delta t)}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right]\right] \\
&= \textstyle\sum_{k=0}^{\infty}\mathbb{E}_t\left[\sum_{i=1}^{k}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right] \mid N(t+\Delta t)-N(t)=k\right] \\
&= \sum_{k=0}^{\infty}\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right]k\frac{(\lambda\Delta t)^k}{k!}e^{-\lambda\Delta t} \\
&= \lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right]e^{-\lambda\Delta t}\textstyle\sum_{k=1}^{\infty}\frac{(\lambda\Delta t)^(k-1)}{(k-1)!} \\
&= \lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right],
\end{aligned} \tag{5.6}$$

As there is finite number of possible jumps with frequency $N(t)$, the summation over $(t, t+\Delta t)$ in the left-hand side can be written as a summation over index "$i$" in the right hand side. Also from (5.4) we have

$$\begin{aligned}
\left[\mathbb{E}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right]\right]^2 &= \pi^{\mathrm{v}2}\big(t,y(t)\big) + \mathbb{E}_t^2\left[\int_t^{t+\Delta t}\left(\pi_t^v + a\pi_y^v + \frac{1}{2}b^2\pi_{yy}^v\right)ds\right] \\
&+ \lambda^2\Delta t^2\mathbb{E}_{(t,t+\Delta t)}^2\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right] \\
&+ 2\pi^{\mathrm{v}}(t,y(t))\mathbb{E}_t\left[\int_t^{t+\Delta t}\left(\pi_t^v + a\pi_y^v + \frac{1}{2}b^2\pi_{yy}^v\right)ds\right] \\
&+ 2\pi^{\mathrm{v}}(t,y(t))\lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right] \\
&+2\lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right]\mathbb{E}_t\left[\textstyle\int_t^{t+\Delta t}\left(\pi_t^v + a\pi_y^v + \frac{1}{2}b^2\pi_{yy}^v\right)ds\right].
\end{aligned} \tag{5.7}$$

Note that when $\Delta t \to 0$ (equivalently, $s \to t$), the number of possible jumps declines such that $N(t+\Delta t) - N(t) \in \{0,1\}$ and the only candidate for a probable jump in the value of $\pi^{\mathrm{v}}$ is time $t$. So, we assume that the limit of the $\mathbb{E}_{(t,t+\Delta t)}$ when $\Delta t \to 0$ can be expressed as

$$\lim_{\Delta t\to 0}\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right] = \mathbb{E}\left[\pi^{\mathrm{v}}(t,y(t)+C(t)) - \pi^{\mathrm{v}}(t,y(t))\right], \tag{5.8}$$

where there is no need for the subscription $N(t+\Delta t) - N(t)$ in the expectation term above and $C(t)$ is the jump size random variable at time $t$ (i.e. $\pi^{\mathrm{v}}(t, y(t)+C(t))$ is random in the sense of jump size).

The variance term in the premium principle using (5.5) and (5.7) is then:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right] &= \mathbb{E}_t\left[\pi^{\mathrm{v}2}\big(t+\Delta t, y(t+\Delta t)\big)\right] - \left[\mathbb{E}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right]\right]^2 \\
&= \mathbb{E}_t\left[2\textstyle\int_t^{t+\Delta t}\left(\pi^{\mathrm{v}}(s) - \pi^{\mathrm{v}}(t)\right)\left(\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\right)ds + \int_t^{t+\Delta t}(b\pi_y^{\mathrm{v}})^2ds\right] \\
&+ \lambda\Delta t\left[\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}2}(s) - \pi^{\mathrm{v}2}(s_-)\right] - 2\pi^{\mathrm{v}}(t)\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}}(s) - \pi^{\mathrm{v}}(s_-)\right]\right] \\
&- \mathbb{E}_t^2\left[\int_t^{t+\Delta t}\left(\pi_t^v + a\pi_y^v + \frac{1}{2}b^2\pi_{yy}^v\right)ds\right] \\
&- \lambda^2\Delta t^2\mathbb{E}_{(t,t+\Delta t)}^2\left[\pi^{\mathrm{v}}(s) - \pi^{\mathrm{v}}(s_-)\right] \\
&- 2\lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{v}}(s) - \pi^{\mathrm{v}}(s_-)\right]\mathbb{E}_t\left[\textstyle\int_t^{t+\Delta t}\left(\pi_t^{\mathrm{v}} + a\pi_y^{\mathrm{v}} + \frac{1}{2}b^2\pi_{yy}^{\mathrm{v}}\right)ds\right].
\end{aligned} \tag{5.9}$$

Note that we suppressed the variable $y(s)$ from function $\pi^{\mathrm{v}}$ above to shorten the notations. Also, the term $\mathbb{E}_t^2$ can be written as a product of two $\mathbb{E}_t$, where when we divide by $\Delta t$ and take the limit, one $\mathbb{E}_t$ will tend to zero and the other will be equal to the integrand inside at $(t, y(t))$; which means that all in all its limit will be zero. Therefore, if we divide everything by $\Delta t$, the variance term has the limit below when $\Delta t \to 0$,

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathrm{Var}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right] &= (b\pi_y^{\mathrm{v}}(t,y(t))^2\\ &+\lambda\Big[\mathbb{E}\left[\pi^{v2}(y(t)+C(t))-\pi^{v2}(y(t))\right]-2\pi^{\mathrm{v}}(y(t))\mathbb{E}\left[\pi^{\mathrm{v}}(y(t)+C(t))-\pi^{\mathrm{v}}(y(t))\right]\Big]\\ &= (b\pi_y^{\mathrm{v}}(t,y(t))^2+\lambda\mathbb{E}\left[\Big(\pi^{\mathrm{v}}(t,y(t)+C(t))-\pi^{\mathrm{v}}(t,y(t))\Big)^2\right], \end{aligned} \tag{5.10}$$

where after dividing by $\Delta t$, the other terms in (5.9) under the limit operation are zero by definition. Note that $\lambda\mathbb{E}\left[\big(\pi^{\mathrm{v}}(t,y(t)+C(t))-\pi^{\mathrm{v}}(t,y(t))\big)^2\right]$ represents the variance of the compound Poisson jump process divided by $\Delta t$. Similarly, the limit can be obtained for the expectation term in (5.4)

$$\lim_{\Delta t\to 0}\tfrac{1}{\Delta t}\mathbb{E}_t\left[\pi^{\mathrm{v}}\big(t+\Delta t, y(t+\Delta t)\big)\right]=\pi^{\mathrm{v}}(t,y(t))+\pi_t^{\mathrm{v}}+a\pi_y^{\mathrm{v}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{v}}+\mathbb{E}\left[\pi^{\mathrm{v}}(t,y(t)+C(t))-\pi^{\mathrm{v}}(t,y(t))\right]. \tag{5.11}$$

Finally if we substitute for (5.4) and (5.9) in (3.17), divide both sides by $\Delta t$, take the limit when $\Delta t\to 0$ and simplify the notation, we get the new form of differential equation for Variance pricing including a jump component:

$$\begin{aligned}&\pi_t^{\mathrm{v}}+a\pi_y^{\mathrm{v}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{v}}+\tfrac{1}{2}\frac{\gamma}{X_0e^{rt}}(b\pi_y^{\mathrm{v}})^2-r\pi^{\mathrm{v}}\\ &\quad+\lambda\left[\mathbb{E}\Big[\pi^{\mathrm{v}}(y(t)+C(t))-\pi^{\mathrm{v}}(y(t))\Big]+\tfrac{1}{2}\frac{\gamma}{X_0e^{rt}}\mathbb{E}\left[\big(\pi^{\mathrm{v}}(y(t)+C(t))-\pi^{\mathrm{v}}(y(t))\big)^2\right]\right]=0,\end{aligned} \tag{5.12}$$

where we substitute $\alpha=\frac{\gamma}{X_0e^{rt}}$. Considering $y(t)$ as a special Lévy process with the jump size random variable $C(t)$ and the Lévy measure $v(dc)$, we can exhibit (5.12) by a more standard formulation,

$$\begin{aligned}&\pi_t^{\mathrm{v}}+a\pi_y^{\mathrm{v}}+\tfrac{1}{2}b^2\pi_{yy}^{\mathrm{v}}+\tfrac{1}{2}\frac{\gamma}{X_0e^{rt}}(b\pi_y^{\mathrm{v}})^2-r\pi^{\mathrm{v}}\\ &\quad+\lambda\int\left(\pi^{\mathrm{v}}(y(t)+c)-\pi^{\mathrm{v}}(y(t))+\tfrac{1}{2}\frac{\gamma}{X_0e^{rt}}\big(\pi^{\mathrm{v}}(y(t)+c)-\pi^{\mathrm{v}}(y(t))\big)^2\right)v(dc)=0.\end{aligned} \tag{5.13}$$

The above equation is a Partial Integro-Differential Equation (PIDE), as the expectation terms can be rephrased in the form of integrals of the premium jump on the jump size in the related sample space. (5.13) is a semi-linear PIDE where it includes quadratic terms of both continuous and jump components. The quadratic term again represents that the equivalent BSDE for this PIDE will have a quadratic driver $g(t,Z)=\frac{1}{2}\frac{\gamma}{X_0e^{rt}}(bZ)^2$. It also includes the probability of one jump for any point at time $t>0$ by means of the parameter $\lambda$. Conditional on a "one-jump" event, the integral (expectation) terms then formulate the effect of the jump size on the value of $\pi^{\mathrm{v}}(t,y)$. It is also clear that the PDE in (3.19) is a special case of the (5.13) PIDE where there is no jump in the insurance process.

## 5.2 Mean Value Price with Jump

In the previous case we assumed a simple jump-diffusion process (5.1) to drive the underlying risk process $y(t)$ and we obtained the proper PIDE to describe the time-consistent Variance premium principle with a jump. Again, to find the PIDE for the Mean Value principle in the jump case, we need to reform the equation (3.30) as the pricing rule. To do so, we still need the martingale property for $v\left(\frac{\pi^{\mathrm{m}}(t,y(t))}{e^{rt}}\right)$, where $\frac{\pi^{\mathrm{m}}(t,y(t))}{e^{rt}} = \pi^{\mathrm{mf}}(t,y(t))$. The implicit compound Poisson process to describe the jumps in (5.1) is not enough to achieve the martingale property for $\pi^{\mathrm{mf}}(t,y(t))$. Instead we use the compensated version of the Poisson process in (5.1) as below,

$$dy(t) = \left[a(t,y(t)) + \lambda\right] dt + b(t,y(t))\,dw(t) + d\tilde{J}(t) \tag{5.14}$$

or alternatively,

$$dy(t) = \left[a(t,y(t)) + \lambda C(t,y(t_-))\right] dt + b(t,y(t))\,dw(t) + C(t,y(t_-))\,d\tilde{N}(t), \tag{5.15}$$

where $\tilde{J}$ is built with the compensated Poisson process $\tilde{N}(t) = N(t) - \lambda t$ with intensity $\lambda$ and random jump size $C(t,y(t_-))$ at any time $t$. As we need to evaluate $v(\pi^{\mathrm{mf}}(t,y(t)))$, we can apply the Itô formula in two steps for $\pi^{\mathrm{mf}}(t,y(t))$ with respect to $t$ and $y(t)$ and then for $v(\pi^{\mathrm{mf}})$ with respect to $\pi^{\mathrm{mf}}$. The resulted stochastic processes for $\pi^{\mathrm{mf}}$ is

$$\begin{aligned}\pi^{\mathrm{mf}}(t,y) = \Big(\pi^{\mathrm{mf}}_t + \lambda\mathbb{E}[\pi^{\mathrm{mf}}(t,y(t)+C(t)) - \pi^{\mathrm{mf}}(t,y(t))] + a\pi^{\mathrm{mf}}_y + \tfrac{1}{2}b^2\pi^{\mathrm{mf}}_{yy}\Big)\,ds + b\pi^{\mathrm{mf}}_y\,dW(t)\\ + \big(\pi^{\mathrm{mf}}(t,y(t)+C(t)) - \pi^{\mathrm{mf}}(t,y(t))\big)\,d\tilde{N}(t)\end{aligned} \tag{5.16}$$

and then for $v\left(\pi^{\mathrm{mf}}\right)$ we have,

$$\begin{aligned}dv\left(\pi^{\mathrm{mf}}(t,y)\right) = \Big\{\Big[\pi^{\mathrm{mf}}_t + \lambda\mathbb{E}[\pi^{\mathrm{mf}}(t,y(t)+C(t)) - \pi^{\mathrm{mf}}(t,y(t))] + a\pi^{\mathrm{mf}}_y + \tfrac{1}{2}b\pi^{\mathrm{mf}}_{yy}\Big]\,v_y\left(\pi^{\mathrm{mf}}\right)\\ + \tfrac{1}{2}(b\pi^{\mathrm{mf}}_y)^2 v_{yy}\left(\pi^{\mathrm{mf}}\right) + \lambda\mathbb{E}\left[v\left(\pi^{\mathrm{mf}}(t,y(t)+C(t))\right) - v\left(\pi^{\mathrm{mf}}(t,y(t))\right)\right]\Big\}dt + b\pi^{\mathrm{mf}}_y v_y\left(\pi^{\mathrm{mf}}\right)dW(t)\\ + \left[v\left(\pi^{\mathrm{mf}}(t,y(t)+C(t))\right) - v\left(\pi^{\mathrm{mf}}(t,y(t))\right)\right]d\tilde{N}(t).\end{aligned} \tag{5.17}$$

According to (3.30) and the martingale property of $\mathbb{E}_t[v\left(\pi^{\mathrm{mf}}(t+\Delta t, y(t+\Delta t))\right)]$, the compensated Poisson jump process of $v\left(\pi^{\mathrm{mf}}(t,y(t))\right)$ in (5.17) should also be martingale. So, we set the drift term above equal to zero:

$$\begin{aligned}\Big[\pi^{\mathrm{mf}}_t + a\pi^{\mathrm{mf}}_y + \tfrac{1}{2}b\pi^{\mathrm{mf}}_{yy} + \lambda\mathbb{E}[\pi^{\mathrm{mf}}(t,y(t)+C(t)) - \pi^{\mathrm{mf}}(t,y(t))]\Big]\,v_y\left(\pi^{\mathrm{mf}}\right) + \tfrac{1}{2}(b\pi^{\mathrm{mf}}_y)^2 v_{yy}\left(\pi^{\mathrm{mf}}\right)\\ + \lambda\mathbb{E}\left[v\left(\pi^{\mathrm{mf}}(t,y(t)+C(t))\right) - v\left(\pi^{\mathrm{mf}}(t,y(t))\right)\right] = 0.\end{aligned} \tag{5.18}$$

We can simplify this by dividing the whole equation by $v_y$ to obtain the pide for the forward term $\pi^{\mathrm{mf}}$:

$$\begin{aligned}&\pi^{\mathrm{mf}}_t + a\pi^{\mathrm{mf}}_y + \tfrac{1}{2}b\pi^{\mathrm{mf}}_{yy} + \tfrac{1}{2}\frac{v_{yy}\left(\pi^{\mathrm{mf}}\right)}{v_y\left(\pi^{\mathrm{mf}}\right)}(b\pi^{\mathrm{mf}}_y)^2\\ &+ \lambda\mathbb{E}\left[\pi^{\mathrm{mf}}(t,y(t)+C(t)) - \pi^{\mathrm{mf}}(t,y(t)) + \frac{v\left(\pi^{\mathrm{mf}}(t,y(t)+C(t))\right) - v\left(\pi^{\mathrm{mf}}(t,y(t))\right)}{v_y\left(\pi^{\mathrm{mf}}(t,y(t))\right)}\right] = 0.\end{aligned} \tag{5.19}$$

Again we substitute for $\pi^{\mathrm{mf}} = e^{-rt}\pi^{\mathrm{m}}$ in (5.19). After we simplify the notation, the corresponding PIDE for the discounted Mean Value principle with jump is then

$$\begin{aligned}
&\pi^{\mathrm{m}}_t + a\pi^{\mathrm{m}}_y + \tfrac{1}{2}b\pi^{\mathrm{m}}_{yy} + \tfrac{1}{2}\frac{v_{yy}(\pi^{\mathrm{m}})}{v_y(\pi^{\mathrm{m}})}(b\pi^{\mathrm{m}}_y)^2 - r\pi^{\mathrm{m}} \\
&+\lambda\int\left(\pi^{\mathrm{m}}(t,y(t)+c) - \pi^{\mathrm{m}}(t,y(t)) + \frac{v\left(\pi^{\mathrm{m}}(t,y(t)+c)\right) - v\left(\pi^{\mathrm{m}}(t,y(t))\right)}{v_y\left(\pi^{\mathrm{m}}(t,y(t))\right)}\right)v(dc) = 0. \qquad (5.20)
\end{aligned}$$

We recognize that the continuous part of the PIDE is the same as the related PDE for the Mean Variance principle in the diffusion case including a positive "local risk aversion" for increasing and convex function $V()$. Conditional on the event of the jump with instantaneous rate of $\lambda$, the PIDE captures the effect of the premium jump by means of the term $\pi^{\mathrm{m}}(t,y(t)+C(t)) - \pi^{\mathrm{m}}(t,y(t))$ as well as the relative difference of the convex function $v(\pi)$ as a result of the jump with respect to the differentiation of $v()$ without a jump. If we assume $v()$ as a nonlinear function, then the PIDE represents the jump effect on the price in both a linear and nonlinear sense.

## 5.3 Standard-Deviation Pricing with Jump

To obtain the time-consistent Standard-Deviation price we have to revalue the principle formula in (4.2) under the jump process:

$$\pi^{\mathrm{s}}\big(t,y(t)\big) = e^{-r\Delta t}\Big(\mathbb{E}_t[\pi^{\mathrm{s}}\big(t+\Delta t, y(t+\Delta t)\big)] + \beta\sqrt{\Delta t}\sqrt{\mathbb{V}\mathrm{ar}_t[\pi^{\mathrm{s}}\big(t+\Delta t, y(t+\Delta t)\big)]}\Big).$$

We embed the same values of $\mathbb{E}_t$ and $\mathbb{V}\mathrm{ar}_t$ we obtained in (5.4) and (5.9) for the jump process, into this equation. We then obtain,

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}e^{r\Delta t}\pi^{\mathrm{s}}\big(t,y(t)\big) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}_t[\pi^{\mathrm{s}}\big(t+\Delta t, y(t+\Delta t)\big)] + \beta\sqrt{\lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{V}\mathrm{ar}_t\left[\pi^{\mathrm{s}}\big(t+\Delta t, y(t+\Delta t)\big)\right]}.$$

If we substitute for (5.11) and (5.10) in the above equation, we obtain the appropriate PIDE for the Standard-Deviation principle as below:

$$\begin{aligned}
&\pi^{\mathrm{s}}_t + a\pi^{\mathrm{s}}_y + \tfrac{1}{2}b^2\pi^{\mathrm{s}}_{yy} - r\pi^{\mathrm{s}}(y(t)) + \lambda\int\left(\pi^{\mathrm{s}}\left(y(t)+c\right) - \pi^{\mathrm{s}}(y(t))\right)v(dc) \\
&\qquad + \beta\sqrt{\lambda\int\left(\left(\pi^{\mathrm{s}}(y(t)+c) - \pi^{\mathrm{s}}(y(t))\right)^2\right)v(dc) + (b\pi^{\mathrm{s}}_y)^2} = 0. \qquad (5.21)
\end{aligned}$$

The Standard-Deviation PIDE presents the jump effect on the premium by using the first and second moments of the premium jump $\pi^{\mathrm{s}}\pi^{\mathrm{s}}(y+c) - \pi^{\mathrm{s}}(y)$. The loading part of the equation with coefficient $\beta$ consists of the conditional quadratic premium jump and quadratic term $(b\pi^{\mathrm{s}}_y)^2$, where the square root function makes it impossible to rewrite a linear version of this PIDE. If there is no jump, $\lambda = 0$, the PIDE will be summarized to the PDE in (4.6) or (4.7).

## 5.4 Cost-of-Capital Principle with Jump

The Cost-of-Capital premium principle can also be valued by assuming a jump process for the underlying insurance process. The one-step pricing formula is the same as (4.11). For the case involving the jump process, we use the Itô-Doeblin representations that we provided in (5.2) and (5.4) to substitute instead of $\pi^{\mathrm{c}}(t+\Delta t, y(t+\Delta t))$ and its conditional expectation to substitute in (4.11), respectively. The pricing rule will be

$$\begin{aligned} e^{r\Delta t}\pi^{\mathrm{c}}\big(t,y(t)\big) &= \pi^{\mathrm{c}}\big(t,y(t)\big) + \mathbb{E}_t\int_t^{t+\Delta t}\Big(\pi_t^{\mathrm{c}}(s,y(s)) + a\pi_y^{\mathrm{c}}(s,y(s)) + \frac{1}{2}b^2\pi_{yy}^{\mathrm{c}}(s,y(s))\Big)ds \\ &+ \lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{c}}(s,y(s)) - \pi^{\mathrm{c}}(s_-,y(s_-))\right] + \delta\sqrt{\Delta t}\mathrm{VaR}_{1-q,t}\Bigg[\int_t^{t+\Delta t} b\pi_y^{\mathrm{c}}(s,y(s))dW(s) \\ &+ \sum_{t<s\le t+\Delta t}\left[\pi^{\mathrm{c}}(s,y(s)) - \pi^{\mathrm{c}}(s_-,y(s_-))\right] - \lambda\Delta t\mathbb{E}_{(t,t+\Delta t)}\left[\pi^{\mathrm{c}}(s,y(s)) - \pi^{\mathrm{c}}(s_-,y(s_-))\right]\Bigg], \end{aligned} \tag{5.22}$$

where according to (4.14), the term $\int_t^{t+\Delta t} f(s,y(s))ds - \mathbb{E}_t[\int_t^{t+\Delta t} f(s,y(s))ds]$ under the VaR function has been omitted from the valuation, for the same reason we explained in subsection 4.2. Note that we multiplied $\mathrm{VaR}_{q,t}$ by $\sqrt{\Delta t}$ to scale down the annual $\mathrm{VaR}_q$ to the $\Delta t$-related version, $\mathrm{VaR}_{q_{\Delta t}}$. This is consistent with the usual Variance-Covariance method of calculating VaR.

To compute this premium, we need some insights into the distribution of the process under the VaR term. The whole terms under the VaR function are a special Lévy jump-diffusion process containing: a Brownian motion with drift zero and diffusion $b(s,y(s))\pi_y^{\mathrm{c}}(s,y(s))$ and a compound Poisson process for a jump component with intensity $\lambda\Delta t$, compensated by its expected value between $(t,t+\Delta t)$. If we assume stationary and independent increments, it is possible to identify the characteristic function of the above Lévy process and find its marginal distribution.

To compute the continuous time limit of (5.22), we divide both sides by $\Delta t$ and we take the limit when $\Delta t\to 0$. Taking into account the limit of $\frac{1}{\Delta t}\mathbb{E}_t\left[\pi\big(t+\Delta t, y(t+\Delta t)\big)\right]$ from (5.11) and the limit of the expectation of the compound Poisson jump from (5.8), we derive

$$\begin{aligned} &\lim_{\Delta t\to 0}\frac{(e^{r\Delta t}-1)\pi^{\mathrm{c}}\big(t,y(t)\big)}{\Delta t} = \pi_t^{\mathrm{c}}\big(t,y(t)\big) + a\pi_y^{\mathrm{c}}\big(t,y(t)\big) + \tfrac{1}{2}b^2\pi_{yy}^{\mathrm{c}}\big(t,y(t)\big) + \lambda\mathbb{E}\left[\pi^{\mathrm{c}}(t,y(t)+C(t)) - \pi^{\mathrm{c}}(t,y(t))\right] \\ &+ \lim_{\Delta t\to 0}(\delta\sqrt{\Delta t})\mathrm{VaR}_{1-q,t}\Big[\lim_{\Delta t\to 0}\tfrac{1}{\Delta t}\int_t^{t+\Delta t} b\pi_y^{\mathrm{c}}(s,y(s))dW(s) + \lim_{\Delta t\to 0}\tfrac{1}{\Delta t}\textstyle\sum_{t<s\le t+\Delta t}\left[\pi^{\mathrm{c}}(s,y(s)) - \pi^{\mathrm{c}}(s_-,y(s_-))\right]\Big], \end{aligned} \tag{5.23}$$

where we used the translation and scale invariance properties of VaR to omit the conditional expectation of the premium jump from the VaR function as

$$\lim_{\Delta t\to 0}\delta\sqrt{\Delta t}\frac{1}{\Delta t}\left[\lambda\Delta t\mathbb{E}\left[\pi^{\mathrm{c}}(t,y(t)+C(t)) - \pi^{\mathrm{c}}(t,y(t))\right]\right] = 0. \tag{5.24}$$

In equation (4.19) in section 4.2 we inferred that the limit of the Itô integral in the VaR operator is normally distributed. The summation term $\sum_{t<s\le t+\Delta t}\left[\pi^{\mathrm{v}}(s,y(s)) - \pi^{\mathrm{v}}(s_-,y(s_-))\right]$, however, is a compound Poisson process with intensity $\lambda\Delta t$. Therefore, the terms under the VaR operator in (5.23) constitute a convolution. We assume that the Itô integral and compound Poisson jumps are independent, as so are the frequency and size of the premium jump, and we calculate the characteristic function $\psi(\theta)$ of the convolution. We denote the convolution of the normal and compound Poisson random variables by $M$, the premium jump

by $D(s) = \pi^{\mathrm{v}}(s, y(s)) - \pi^{\mathrm{v}}(s_-, y(s_-))$ and the compound Poisson process as $X = \sum_{t<s\leq t+\Delta t} D(s)$. Considering the fact that $\psi_{\sqrt{\Delta t}M}(\theta) = \psi_M(\sqrt{\Delta t}\theta)$ and $\psi_{\frac{1}{\Delta t}X}(\theta) = \psi_X(\frac{\theta}{\Delta t})$ and the independence assumption, the characteristic function of the convolution under VaR is

$$\begin{aligned}
\psi_{\sqrt{\Delta t}M}(\theta) &= \exp\left[-\frac{(b\pi_y)^2(\sqrt{\Delta t}\theta)^2}{2\Delta t} + \lambda\Delta t\left(\psi_{\frac{1}{\Delta t}X}(\sqrt{\Delta t}\theta) - 1\right)\right] \\
&= \exp\left[-\frac{(b\pi_y)^2\theta^2}{2} + \lambda\Delta t\left(\psi_X\left(\frac{\theta}{\sqrt{\Delta t}}\right) - 1\right)\right].
\end{aligned} \tag{5.25}$$

The distribution of the convolution depends on the distribution of the premium jump and thus on the form of $\psi_X$. If we assume normally distributed premium jumps, $D \sim N(\mu, \sigma^2)$, the characteristic function of the whole convolution turns to

$$\begin{aligned}
\psi_{\sqrt{\Delta t}M}(\theta) &= \exp\left[-\frac{(b\pi_y)^2\theta^2}{2} + \lambda\Delta t\left(\frac{i\mu\theta}{\sqrt{\Delta t}} - \frac{\sigma^2\theta^2}{2\Delta t} - 1\right)\right] \\
&= \exp\left[-\frac{(b\pi_y)^2\theta^2}{2} + \lambda\sqrt{\Delta t}(i\mu\theta) - \frac{\lambda\sigma^2\theta^2}{2} - \lambda\Delta t\right].
\end{aligned} \tag{5.26}$$

If we take the limit of $\psi_{\sqrt{\Delta t}M}(\theta)$ when $\Delta t \to 0$, we obtain

$$\lim_{\Delta t\to 0} \psi_{\sqrt{\Delta t}M}(\theta) = \exp\left[-\frac{b^2\pi_y^2\theta^2}{2} + \frac{\lambda\sigma^2\theta^2}{2}\right] \tag{5.27}$$

which shows that the asymptotic distribution of the compound Poisson process with coefficient $\sqrt{\Delta t}/\Delta t$ is normal with mean zero and variance $\lambda\sigma^2$, where the zero mean was justified earlier in (5.24). Hence, the convolution is normal with mean zero and variance $b^2\pi_y^2 + \lambda\sigma^2$, and by using the scale invariance property, the limit of the VaR term in (5.23) will be equal to $\sqrt{b^2(t, y(t))\pi^{\mathrm{c}2}{}_y(t, y(t)) + \lambda\sigma^2}\,\Phi^{-1}(1-q)$. Finally (5.23) gives the resulted PIDE as

$$\begin{aligned}
r\pi^{\mathrm{c}}(t, y(t)) =& \pi_t^{\mathrm{c}}(t, y(t)) + a(t, y(t))\pi_y^{\mathrm{c}}(t, y(t)) + \tfrac{1}{2}b^2(t, y(t))\pi_{yy}^{\mathrm{c}}(t, y(t)) \\
&+ \delta\Phi^{-1}(1-q)\sqrt{b^2(t, y(t))\pi^{\mathrm{c}2}{}_y(t, y(t)) + \lambda\mathbb{V}\mathrm{ar}\left[\pi^{\mathrm{c}}(t, y(t) + C(t)) - \pi^{\mathrm{c}}(t, y(t))\right]} \\
&+ \lambda\mathbb{E}\left[\pi^{\mathrm{c}}(t, y(t) + C(t)) - \pi^{\mathrm{c}}(t, y(t))\right],
\end{aligned} \tag{5.28}$$

where taking $\Phi^{-1}(1-q) = k$ and changing to integral notation, the PIDE is:

$$\begin{aligned}
&\pi_t^{\mathrm{c}} + a\pi_y^{\mathrm{c}} + \tfrac{1}{2}b^2\pi_{yy}^{\mathrm{c}} - r\pi^{\mathrm{c}} + \lambda\int \left(\left(\pi^{\mathrm{s}}(y(t)+c) - \pi^{\mathrm{s}}(y(t))\right)\right) v(dc) \\
&\quad + \delta k\sqrt{b^2\pi^{\mathrm{c}2}{}_y + \lambda\int \left(\left(\pi^{\mathrm{s}}(y(t)+c) - \pi^{\mathrm{s}}(y(t))\right)^2 - \mathbb{E}^2\left[\pi^{\mathrm{c}}(y(t)+c) - \pi^{\mathrm{c}}(y(t))\right]\right) v(dc)} = 0.
\end{aligned} \tag{5.29}$$

Looking back at the derivation of the PIDE, it is clear that the loading term of the premium (VaR term) is independent of the expected premium jump. The PIDE also shows that if the premium jump is normally distributed, the Cost-of-Capital price is able to capture a quadratic jump effect on the price (i.e. the variance of the premium jump size) that makes it very similar to the Standard-Deviation price, which presents the second moment of the premium jump. The rest of the terms for the Cost-of-Capital and Standard-Deviation prices are the same. The quadratic driver of the PIDE is forced to be linearized by the square root function in both the Standard-Deviation and Cost-of-Capital principles. In the non-jump case, the PIDE converges the PDE in (4.23).

The underlying distribution of the premium jump size is effective on the Cost-of-Capital price of the insurance process with jump. If we change the distribution of the premium jump, the continuous time limit of the Cost-of-Capital premium will result in a different PIDE. For example, if the premium jump has an exponential distribution with parameter $\alpha$, then it will turn (5.25) into

$$\begin{aligned}\psi_{\sqrt{\Delta t}M}(\theta) &= \exp\left[-\frac{(b\pi_y)^2\theta^2}{2} + \lambda\Delta t\left((1 - i\frac{\theta}{\sqrt{\Delta t}}\alpha^{-1})^{-1} - 1\right)\right] \\ &= \exp\left[-\frac{(b\pi_y)^2\theta^2}{2} + \lambda\Delta t\left(\frac{i\theta}{\lambda\sqrt{\Delta t} - i\theta}\right)\right]\end{aligned} \tag{5.30}$$

and by taking the limit when $\Delta t \to 0$, the exponential part tends to zero and we have

$$\lim_{\Delta t\to 0}\psi_{\sqrt{\Delta t}M}(\theta) = \exp\left[-\frac{b^2\pi_y^2\theta^2}{2}\right]. \tag{5.31}$$

This is the characteristic function of the normal distribution with mean zero and variance $b^2\pi_y^{\mathrm{c}\,2}$ and by means of equation (5.23) gives the PIDE as

$$\pi_t^{\mathrm{c}} + a\pi_y^{\mathrm{c}} + \tfrac{1}{2}b^2\pi_{yy}^{\mathrm{c}} + \delta k|b\pi_y^{\mathrm{c}}| - r\pi^{\mathrm{c}} + \lambda\int\left(\left(\pi^{\mathrm{s}}(y(t)+c) - \pi^{\mathrm{s}}(y(t))\right)\right)\nu(dc) = 0. \tag{5.32}$$

We observe a different PIDE in the sense that the quadratic jump term has disappeared from the VaR perspective and only the jump effect is captured via the expectation term of the Cost-of-Capital premium principle. The non-jump case still converges to the Cost-of-Capital PDE in (4.23).

# 6 Numerical Method

In this section we apply the idea of time-consistent valuation to price a simplified insurance contract to give a real-world example of this method and its differences to the normal one-step valuation. We apply the multi-step pricing operator to the time-consistent version and divide any time period $T - t$ into $n$ steps with a length of $\Delta t$. We use the same backward iteration method to calculate the value of the premium for an insurance risk. As we modeled earlier, the unhedgeable risk process can be described either by a simple diffusion process in (3.1) or a jump-diffusion process in (5.1). In time step $(t, t+\Delta t)$, we have the increment as below,

$$\begin{aligned}&\text{Simple Diffusion:} \quad \Delta y(t) = \mu(t, y(t))\Delta t + \sigma(t, y(t))\Delta W(t) \\ &\text{jump-diffusion:} \quad \Delta y(t) = \mu(t, y(t))\Delta t + \sigma(t, y(t))\Delta W(t) + \Delta J.\end{aligned}$$

We are interested in the price of any contract at time $t \geq 0$ that offers a contingent payoff at $T$ or any time depending on $T$. To price the contract, we will use the premium principles that we used in the time-consistent contexts in the previous sections. To implement the idea of time-consistent valuation, we will use the Markov chain method to approximate the underlying process and payoff function, where the pricing rules will be one of the previously mentioned premium principles. The Markov chain provides a straightforward method to apply the valuation task in each sub-period for the payoff and calculate the price in a dynamic way. This method is frequently used to price path dependent derivatives such as American options, barrier options, etc. See for example Duan et al. (2003) and Monoyios (2004).

## 6.1 Setting for a Simple Life Insurance Payoff

Suppose we have a stylized life-insurance contract for the period of $[0, T]$. We are monitoring the health of an individual as a diffusion process, say $y(t)$. The person is alive as long as $y(t) > 0$ and dies when $y(t)$ hits zero. Therefore, the insurance contract has a payoff 1 at time $T$ (i.e. the survival benefit), if $y(t) > 0$ for all $0 < t < T$. Another stylized contract pays the benefit 1 at $T$ if $y(t)$ hits the level zero before $T$, where the individual dies. Let us define the first hitting time at level $x > 0$ for the process $y(t)$ as below,

$$\tau_x = \min\{t \geq 0; \quad y(t) = x\}.$$

If we assume $y(t) = W(t)$ is a Brownian motion, it is not hard to prove that $\mathbb{P}(\tau_x < \infty) = 1$ but $\mathbb{E}(\tau_x) = \infty$. The health process can offer a more realistic picture if we assume a negative drift $\mu < 0$ as any individual's health gradually deteriorates and the individual comes closer to death. Naturally, the health quality of an individual can fluctuate daily due to different factors like nutrition, exercise, diseases etc, which means $\sigma > 0$.

Based on the above properties of the Brownian motion $W(t)$, such as "symmetry", for a constant $\mu$ and $\sigma$, the distribution function of the first hitting time of the level zero by the process $y$ with the initial value of $y(t)$ and the maturity time $T$ is,

$$\mathbb{P}\left(\tau_0 < T - t \middle| y(t)\right) = \Phi\left(\frac{-y(t) - \mu(T-t)}{\sigma\sqrt{T-t}}\right) + \exp\left(\frac{-2\mu y(t)}{\sigma^2}\right)\Phi\left(\frac{-y(t) + \mu(T-t)}{\sigma\sqrt{T-t}}\right). \quad (6.1)$$

We will use this probability and the corresponding survival function for the hitting time $\tau_0$ to calculate the analytical solution of the PDEs obtained for each premium principle.

The physical setting for the value and payoff of the above stylized product is basically a simple control problem for the underlying stochastic process with constant boundary levels over time. It is ideal and more realistic, regarding the natural situation of any individual, that $\mu(t, y(t))$ and $\sigma(t, y(t))$ be stochastic processes depending on time and the health condition of the individual in the previous time step. However, to keep our demonstration simple, we assume a constant $\mu$ and $\sigma$ in this paper.

To tackle the approximation problem with regard to efficiency of the calculation, we need to determine the boundary region of the health process either by "time" or "state space". As the valuation period is up to time $T$, this formulation for the payoff function becomes an inference about the hitting (stopping) time probability of a diffusion process in a fixed finite time period $[0, T]$. For the state space, we look for the possibility and the time of hitting the level zero as a lower bound for the non-negative process $y(t)$. Although the payoff is determined by the health process hitting the zero level, we need to consider the upper boundary of the states to avoid the approximation error when we implement the idea later by means of the Markov chain.

In theory, the probability that any finite level will be hit is non-zero for a Brownian motion with or without drift. However, to make a more efficient calculation, we select the finite boundary in the upside level (lets say $y_{\max}$). As $y(t)$ has the standard deviation $\sigma\sqrt{T}$ on $[0,T]$, we assume

$$y_{\max} = y(0) + k\sigma\sqrt{T}, \tag{6.2}$$

where $k \geq 3$ makes the probability of hitting $y_{\max}$ very small. Although this probability will be negligible for a reasonably large $k$ and negative drift, we will reduce the sample space in the calculation phase by conditioning the probability on the over-$y_{\max}$ hits. We therefore define the boundary region for the approximation in both time and state dimensions as $[0, y_{\max}] \times [0, T]$.

## 6.2 Markov Chain Implementation

The Markov chain method has been used extensively as a numerical tool for control problems, particularly in the dynamic valuation of contingent payoffs such as American options. See for example Kushner and Dupuis (2001) and Yin and Zhang (2012). The backward iteration of the one step valuation can be applied by means of the Markov chain method to the underlying (original) health process, discretized by both time horizon and state space. We define the approximating Markov chain on the related state space by using a finite difference interval $\Delta y$ such that the first moments of the chain are matched to those of the original process $y(t)$, as $\Delta y \to 0$. Note that $\Delta y$ can also be interpreted as a discrete time parameter of the Markov chain and can be defined as a function of time step $\Delta t$.

Basically, we use a Markov chain with a lattice structure of approximation for $y(t)$ in a discrete-time and finite state space. The model gives enough flexibility to evaluate a wide range of derivatives and risk products in a dynamic way with time varying or stochastic risk drivers like interest rate or volatility and different pricing formulas. Duan et al. (2003) have provided a generally applied frame for the method used to price American option, by applying the Black-Scholes model and GARCH option pricing model.

### 6.2.1 Pricing by Simple Diffusion Health Process

We start with a term life insurance for time horizon $T$ that pays benefit 1 on the event of death if $\tau_0 < T$ and we use the Variance premium principle as a pricing rule. The time space consists of numerous small time steps $\Delta t$, and the payoff can be recursively defined as below for all $s \in \{t, t+\Delta t, t+2\Delta t, ..., T-\Delta t\}$:

$$B(s, y(s)) = \mathbb{E}\left[e^{-r\Delta t}B(s+\Delta t, y(s+\Delta t))\,\middle|\,\mathscr{F}_s\right] + \tfrac{1}{2}\alpha \mathbb{V}\mathrm{ar}[e^{-r\Delta t}B(s+\Delta t, y(s+\Delta t))|\mathscr{F}_s] \tag{6.3}$$

with the terminal condition

$$B(T, y(T)) = \begin{cases} 1, & \exists t \in (0,T] \;\; y(t) \leq 0; \\ 0, & \forall t \in (0,T] \;\; y(t) > 0. \end{cases} \tag{6.4}$$

We implicitly assume that if for any $t \leq T$, $y(t)$ hits zero, the process will be killed and will remain zero till time $T$ when the payoff will be made. We repeat this valuation operation in the backward iteration method to price the product at time zero, starting from $B(T, y(T))$. As we mentioned before, we use constant interest rate, drift rate and volatility.To find the transition matrix of the Markov chain, we consider the method in Duan et al. (2003), which calculates the transition probabilities over all states in the range $(0, y_{\max})$. However, to make the calculations faster, we use a special matrix that involves a recombining trinomial tree.

Suppose the health process $y(t)$ is a simple diffusion. To obtain the non-negative probability constraints in the transition matrix, we use the "adaptive recombining trinomial tree" technique, in which the middle tree node follows the local drift and the up/down nodes follow the volatility for each time step. We match the local mean and variance of the underlying process and the Markov state space. See for example, Tang and Li (2007) for more details about the method. The state difference interval will be constructed as

$$\Delta y(t) = \begin{cases} \Delta y_d(t) & = & -\sigma\sqrt{k\Delta t}, \\ \Delta y_m(t) & = & 0, \\ \Delta y_u(t) & = & \sigma\sqrt{k\Delta t} \end{cases} \tag{6.5}$$

where an common value of $k = 3$ also can match the local kurtosis and reduce the distribution error to speed up the convergence of the chain. Similar method in Figlewski and Gao (1999) and Baule and Wilkens (2004), produced the trinomial transition probabilities as follows

$$\begin{array}{lcl} p_d & = & 1/6 - \frac{\mu\sqrt{3\Delta t}}{6\sigma}, \\ p_m & = & 2/3, \\ p_u & = & 1/6 + \frac{\mu\sqrt{3\Delta t}}{6\sigma}, \end{array} . \tag{6.6}$$

where $p_u \geq 0, p_m \geq 0, p_d \geq 0$ and $p_u + p_m + p_d = 1$ and the state difference interval is constructed in a way that the local kurtosis will be matched and the distribution error will decline. The result for the scope of our stylized example is consistent with the nature of the health process, where for a negative drift $\mu$ we expect a larger downward probability $p_d$ (and smaller upward probability $p_u$), to push the process closer to zero.

### 6.2.2 Pricing by jump-diffusion Health Process

We enter a simple jump component into the trinomial tree to investigate its effect on the price of the product. Generally, most of the methods for random-sized Poisson jump components are studied with the aim of finding the tree probabilities so that the discrete time Markov process including a jump matches the first local moments of the continuous time jump-diffusion process. For more about the applications of the method to price the options, see for example Amin (1993) and Yuen and Yang (2009).

Considering the same criteria, Hilliard and Schwartz (2005) investigated how to use a jump-diffusion model to price derivatives. They used a bivariate tree approach to separate the diffusion and jump parts and used the same methods to match the local moments. They assumed that the size of the jump in discrete time also has a grid containing jump nodes constructed by the integer product of the jump size's finite difference interval. After that, the jump-diffusion discrete time approximation will be the summation of the diffusion and jump parts.

We use a simplified version of the above techniques to separate the jump and diffusion parts in the implemented Markov chain setting. To keep the problem simple, we assume a constant jump size $J$ such that

$$\left\lceil \frac{J}{\Delta y(s)} \right\rceil = K, \tag{6.7}$$

where $K \geq 2$. As the number of valuation steps increases, the state difference $\Delta y(s)$ decreases and $K$ increases so that $J$ remains constant.

We also implement the transition probabilities for a valuation time step $\Delta t$, in the form of a skewed quadrinomial, by mixing the arrival time rate of jump $\lambda$ and trinomial tree transition probabilities as below,

$$\pi(i,j,\Delta t) = \begin{cases} \lambda\Delta t, & j = K; \\ (1-\lambda\Delta t)\begin{cases} p_d = & 1/6 - \frac{\mu\sqrt{3\Delta t}}{6\sigma}, & j = i-1; \\ p_m = & 2/3, & j = i; \\ p_u = & 1/6 + \frac{\mu\sqrt{3\Delta t}}{6\sigma}, & j = i+1. \end{cases} \end{cases} \tag{6.8}$$

Based on this formulation, we assume that any jump event, will be large enough to nullify the effect of the diffusion part for the evolution of the underlying health process. If there is no jump, we can reduce the sample space for the diffusion part and distort the trinomial transition probabilities so that we can define the entire process in one probability space. This can be considered a very simple and special case of the regime switching between the jump and diffusion parts, so that there is only a possible jump in the first regime and diffusion instead of a jump in the second regime.

## 6.3 Simulation

We apply the above method to calculate the time-consistent price of the contract with both diffusion and jump-diffusion processes. To compare the time-consistent price obtained from the diffusion and jump process, we also need to match the local moments of the diffusion process with regard to those of jump process. Therefore, we recall the locally matched processes for constant drift, volatility and jump size as below,

$$\begin{array}{ll} \text{Simple Diffusion:} & dy(t) = (\mu + \lambda J)dt + (\sqrt{\sigma^2 + \lambda J})dW(t) \\ \text{jump-diffusion:} & dy(t) = \mu(t, y(t))dt + \sigma dW(t) + JdN(t). \end{array} \tag{6.9}$$

In the above formulation we implicitly assume that no more than one jump should be possible for a small time step. Using the locally matched diffusion process above and (6.7), we update the transition probabilities in (6.6) as

$$\begin{cases} p_d &= 1/6 - \frac{(\mu+\lambda K\Delta y)\sqrt{3\Delta t}}{6\sqrt{\sigma^2+\lambda K\Delta y}}, \\ p_m &= 2/3, \\ p_u &= 1/6 + \frac{(\mu+\lambda K\Delta y)\sqrt{3\Delta t}}{6\sqrt{\sigma^2+\lambda K\Delta y}}. \end{cases} \tag{6.10}$$

The alternative transition probabilities for the jump case stays the same as (6.8).

### 6.3.1 Variance Price

We calculate the time-consistent Variance premium principle for a $T$-year term life insurance. We do this for both the death and survival benefits based on the stylized health process. Note that in this numerical work, we do not solve the related Variance PDE, but we directly calculate the Variance premium for the shorter time steps starting with the terminal time $T$ state space and apply the backward iteration method to reach the time $t < T$ price.

It is important to examine the convergence of the Markov chain trinomial tree approximation to the analytical time-consistent price. The time-consistent solution for the case of the Variance price was derived in (3.20) as $\pi^{\mathrm{V}}(t,y) = \frac{X_0 e^{rt}}{\gamma} \ln \mathbb{E}\left[\exp\left(\frac{\gamma}{X_0 e^{rT}} f(y(T))\right)\Big| y(t) = y\right]$. According to the Markov chain discretization, the payoff for the death benefit is 1 when $\tau_0 < T - t$ and 0 in all other cases. The apposite is valid for the survival benefit where the payoff is 1 if $\tau_0 \geq T - t$. If we assume $\mathbb{P}(\tau_0 < T) = p$ as the probability of a Bernoulli event, which can be calculated by the equation (6.1), the analytical price will be obtained as

$$\pi^{\mathrm{V}}(t,y) = \frac{X_0 e^{rt}}{\gamma} \ln \mathbb{E}\left[e^{\frac{\gamma}{X_0 e^{rT}} \mathbb{I}_{\{\tau_0 < T-t\}}}\Big| y(t) = y\right] = \frac{X_0 e^{rt}}{\gamma} \ln\left(1 - p + p e^{\frac{\gamma}{X_0 e^{rT}}}\right), \tag{6.11}$$

where for $\alpha = \gamma / X_0 e^{rt}$, the simpler notation is $\pi^{\mathrm{V}}(t,y) = \frac{1}{\alpha} \ln\left[1 - p + p \times \exp(\alpha e^{r(T-t)})\right]$.

We calculate the time-consistent price approximation for both types of coverage, based on the following set of sample parameters: the drift $\mu = -0.2$, the diffusion coefficient $\sigma = 0.4$, the initial value at time $t$, $y(t) = 1$, the time duration $T - t = 1$, annual discount rate $r = 0.05$, the relative risk aversion per benchmark wealth level $\frac{\gamma}{X_0} = 0.1$, and the jump arrival time rate $\lambda = 0.03$, the expected jump size $K = 0.7y(t) = 0.7$, and the upper bound of the $y(T)$ state space will be driven as $y_{\max} = y(t) + 3\sigma\sqrt{(T-t)} = 2.2$. The probability of the first hitting time of the level zero (lower bound of the state space of $y(T)$), or equivalently the individual's death probability, can be calculated by (6.1) as $P(\tau_{y(\tau)=0} \leq T - t | y(t) = 1) = p = 0.03375$. Similarly the alternative conditional hitting time probability when taking into account the sample space reduction by the upper bound $y_{\max} = 2.2$ for $k = 3$ will be

$$P\left(\tau_{y(\tau)=0} < T - t \middle| y(t) = 1, y(\tau) \leq y_{\max}\right) = \frac{p}{P(y(\tau) \leq y_{\max})} = \frac{0.03375}{0.99967} = 0.033758,$$

where clearly the survival probability is $q = 1 - p = 0.96624$. Using the equation (6.11), the analytical time-consistent Variance price for the life insurance coverage will be $\pi^{\mathrm{V}}_{\mathrm{Death}}(t,y) = 0.03363$ and $\pi^{\mathrm{V}}_{\mathrm{Survival}}(t,y) = 0.92055$.

We provide a numerical approximation of the time-consistent Variance price operator for both the death and survival benefits using simplified diffusion and jump cases in discrete time. We use the backward iteration method, whereby the time steps $\Delta t$ become smaller by increasing the number of iterations, and we examine whether our approximation converges to the analytical continuous time limit of the price.

Figure 1 represents the convergence of the Markov chain trinomial tree approximation to the analytical time-consistent Variance premium for the diffusion case in which the number of time steps $(n)$ increases and the parameters are the same as above. Although we have no analytical solution for the obtained PIDE in the jump case in 5.12, the Variance price converges to the certain levels of 0.0499 and 0.9057 for the death and survival coverage, respectively. The difference in the price is reasonable as we have a one-sided downward jump in the health process. We still observe some perturbation in the Markov chain approximation, but the level of the relative difference between the values (i.e. the typical error) decreases when the number of steps increase. Figlewski and Gao (1999) explain that the reason for the typical errors is the lack of coincidence between the theoretical boundary levels and the highest state in the Markov chain. In our case, there is a lack of coincidence for the position of the time $t$ Markov chain premium in the lattice model with the analytical price, which always cause over/under value. Applying this method to the Standard-Deviation principle will give the same convergence result for both the diffusion and jump cases.

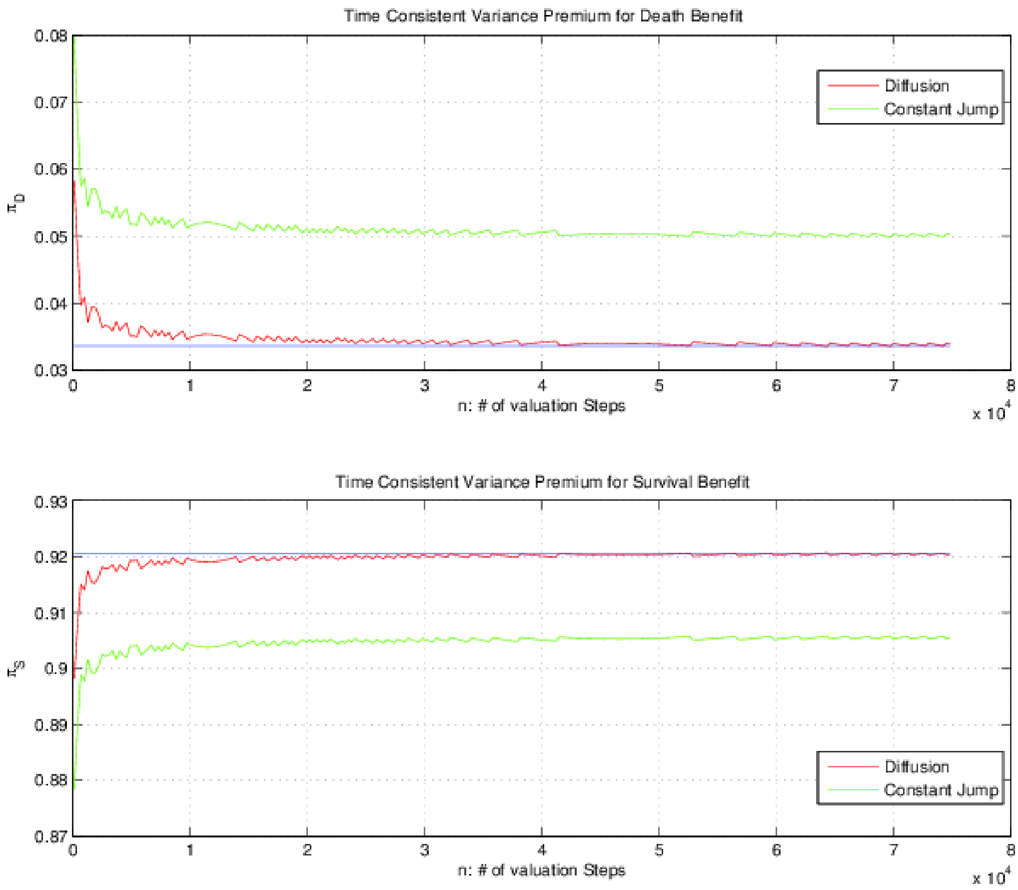


Figure 1: Markov Chain Simulation of the Time-Consistent Variance Premium for the Stylized Life Insurance Contract

### 6.3.2 Cost-of-Capital Price

We also compute the Markov chain approximation of the time-consistent Cost-of-Capital price for the above life insurance contract. The analytical solution of the Cost-of-Capital PDE for the diffusion case is given by the equation (4.8) under the risk-adjusted underlying process (4.24) as below,

$$\pi^{\mathrm{v}}(t,y) = \mathbb{E}\left[e^{-r(T-t)} f(y(T)) \middle| y(t) = y\right] = \mathbb{E}_t\left[e^{-r(T-t)} \mathbb{I}_{\{\tau_0 < T-t\}}\right] = e^{-r(T-t)} \times p, \tag{6.12}$$

where $p = \mathbb{P}(\tau_0 < T)$. There is no analytical solution for the jump-diffusion case. For the parameter values, we use the cost of capital $\delta = 0.1$ instead of the relative risk aversion. In order to give a better picture of the approximation evolution, we choose a relatively high jump intensity $\lambda = 0.1$ and probability level of the VaR, $1-q = 0.999$. The rest of the parameters are the same as those that we used in Variance pricing.

We use (6.9) as the underlying process. Since the payoff for the death benefit decreases monotonically in $y$, we use $(a+\lambda J-\delta kb)$ as the downward adjustment for the drift rate. The adjustment calculates the upwind price of the insurance risk as the drift rate decreases more by $-\delta kb$, pushing the process more towards the zero level, which means a higher probability of death from the insurer's perspective. Using the equation (6.1), the probability of the first hitting time of the level zero (death probability) is computed as $p = 0.04342$, where the conditional probability given the upper bound $y_{max} = 2.2$, is $P\left(\tau_0 < 1 \middle| y(t) = 1, y(\tau) \leq 2.2\right) = 0.043435$. On the other hand, since the survival benefit increases monotonically in $y$, we have to use $(a+\lambda J+\delta kb)$ as the upward adjustment for the drift rate, which gives a lower probability of hitting zero. This is interpreted as a higher price of the survival coverage for the insurer. Therefore, we obtain, $P\left(\tau_0 \geq 1 \middle| y(t) = 1, y(\tau) \leq 2.2\right) = 0.0.98976$. By using the formulation in (6.12), we obtain the analytical time-consistent value of the Cost-of-Capital premium for the life insurance coverage as $\pi^{\mathrm{c}}_{\mathrm{Death}}(t, y(t)) = 0.04132$ and $\pi^{\mathrm{c}}_{\mathrm{Survival}}(t, y(t)) = 0.9416$.

Figure 2 illustrates the Markov chain approximation of the time-consistent value of the Cost-of-Capital premium for different number of valuation steps in the backward iteration method. The upper graph illustrates the premium of the death coverage under the diffusion and jump-diffusion process, while the lower graph shows the same premium for the survival coverage. We start the valuation with just $n = 4$ steps and add four more steps to $n$ each time. In the above parameter set, the horizontal line is the analytical value of the time-consistent premium. In the case of death coverage modeled by a simple diffusion process, which increases the number of valuation steps, we observe a fast convergence of the Markov chain method to the analytical value.

However, for the jump-diffusion process, there is a downfall in the Markov chain approximation of the Cost-of-Capital premium on $n = 100$. The reason for this dramatic reduction of the premium can be explained by the fact that, when the probability of the jump event at any time interval $(t, t+\Delta t)$ is less than the VaR probability threshold in that period, $\lambda\Delta t < q$, the $\mathrm{VaR}_{1-q}$ function is not able to capture the effect of the jump. Therefore, in the point where $\lambda\Delta t = q$ and after that, the premium jump cannot be reflected in VaR, and the Cost-of-Capital premium drops. This is a substantial weakness in the Cost-of-Capital premium principle when dealing with rare jump events and it fails to capture part of the premium jump in the final value. In our example, for $\lambda = 0.1$ and $q = 0.001$, this happens when $n \geq 100$, $\lambda\Delta t \leq 0.001$. After the drop point, the Markov chain approximation converges to a special level of the premium that is significantly higher than the premium resulted by the simple diffusion process.

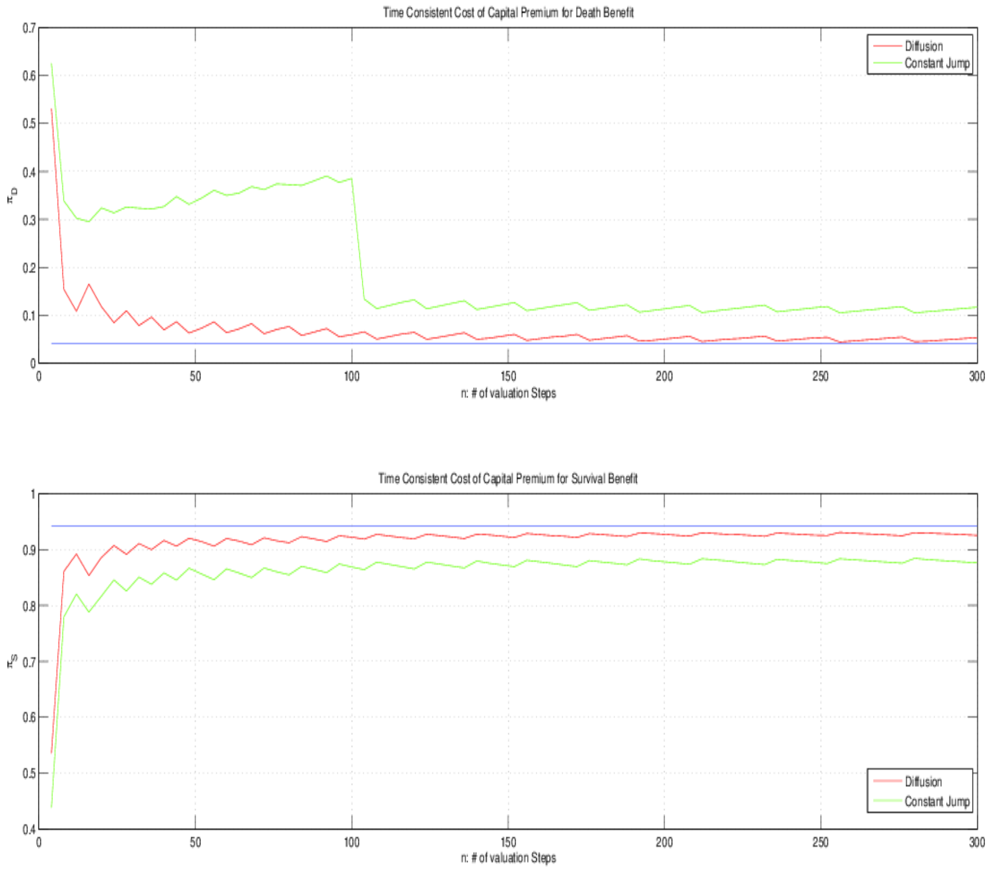


Figure 2: Markov Chain Simulation of the Time-Consistent Cost-of-Capital Premium for the Stylized Life Insurance Contract

For the survival coverage, the Markov chain premium approximation obtained by the diffusion processes converges to the analytical value of the time-consistent Cost-of-Capital premium (horizontal line). In the jump case, we observe a normal convergence with a decreasing perturbation rate without any sudden increase or decrease in the premium, while the number of valuation steps increases. The reason for this is that we use a one-sided jump in our example that moves downwards and is located on the left hand side of the survival risk distribution. As a result, it is not able to stimulate the VaR function by means of the jump probability level $\lambda\Delta t$. Nevertheless, part of the jump effect is always captured by the expectation operator of the Cost-of-Capital principle and when comparing this to the diffusion case, this justifies the lower survival premium in the jump case in the second part of the Figure 2.

# 7 Summary and Conclusions

In this paper we investigated a number of well-known actuarial premium principles, such as the Variance and Standard-Deviation principle, and studied their extension into a time-consistent direction. We constructed these extensions using one-period valuations, then we extended this to a multi-period setting by means of the backward iteration method of Jobert and Rogers (2008) for a given discrete time-step $\Delta t$, and finally we considered the continuous-time limit for $\Delta t \to 0$. We showed that the extended Variance premium principle converges to the non-linear exponential indifference valuation. Furthermore, we showed that the extended Standard-Deviation principle converges to an expectation under an equivalent martingale measure. Finally, we showed that the Cost-of-Capital principle, which is widely used by the insurance industry, converges to the same limit as that of the Standard-Deviation principle. In the above cases, we assumed that the underlying risk process is a simple diffusion process in which the continuous time limit of the time-consistent valuation results in a semi-linear Partial Differential Equation (PDE) that can be solved analytically with the Feynman-Kaç formula. To conduct a more realistic valuation, we added a Poisson jump component to the underlying risk process and obtained the time-consistent extension of the above premium principles in the form of different Partial Integro-Differential Equations (PIDEs) that can be solved numerically. There was no convergence in the price of the different premium principles in the jump case, but the effect of the jump component is reflected in the related PIDEs by different forms of premium jumps. In the Cost-of-Capital principle, the $\mathrm{VaR}_{1-q}$ operator failed to reflect the effect of the jump on the extended price where the probability of the jump in a single time step drops to less than the probability level of the quantile, $\lambda\Delta t < q$. This uncovers an important weakness that the Cost-of-Capital principle has in pricing the insurance risks containing the jump components in the time-consistent extension. The end of the paper is dedicated to using the Markov chain approximation to apply the backward iteration method and calculate the time-consistent value of a simple life insurance payoff. Here we observed the convergence of the numerical calculation to the analytical time-consistent solutions.

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