Time-consistent and Market-consistent Actuarial Valuations

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14-15 March 2016
High Performance Computing in Finance
Aberdeen Asset Management – London, UK

Research Outputs

3 Papers:

- Time-consistent actuarial valuations, *Insurance Mathematics and Economics, Volume 66, January 2016, Pages 97-112*.
- Market-consistent valuation by two-step operator and its application on life insurance pricing, *Working paper*.
- Time-consistent and Market-consistent valuation of the participating policy with hybrid profit-sharing, *Working paper*.

Motivation

- **Actuarial Pricing for Modern Life/pension Contract:**
    - Hybrid liabilities from financial and actuarial risks
    - Liabilities are not fully traded in the market.
    - Financial part is usually hedgeable while actuarial part is not.
    - Dynamic pricing needed due to path dependency & embedded options
- **Regulatory requirement for Market-consistent Valuation**
- **"Re-Valuation" Insurance liabilities**
- **Payoff in "Long-term"**
    - What happens in the middle time?!
    - How do we reflect this in pricing?
    - Require time-consistency for Actuarial price operators?

Ambitions

- Build Time-consistent Actuarial Pricing
- Combine Actuarial & Financial Pricing for Market-consistent Valuation
- "Applied Techniques" to implement Time-consistent and Market-consistent Pricing

Time-Consistency

- If $X_1 \geq X_2$ at time T, then $\Pi[t, X_1] \geq \Pi[t, X_2]$ for all $t < T$
- **Note:** The conditional expectation operator $\mathbb{E}^{\mathbb{Q}}[H \mid \mathcal{F}_t]$ is TC.
- Extension of "**tower property**" to actuarial operators

$$\Pi\left[H(T) \mid \mathcal{G}_t^A\right] = \Pi\left[\Pi\left[H(T) \mid \mathcal{G}_s^A\right] \mid \mathcal{G}_t^A\right] \quad \text{for} \quad t \leq s < T$$

- Well-known Actuarial operators **are NOT** time-Consistent

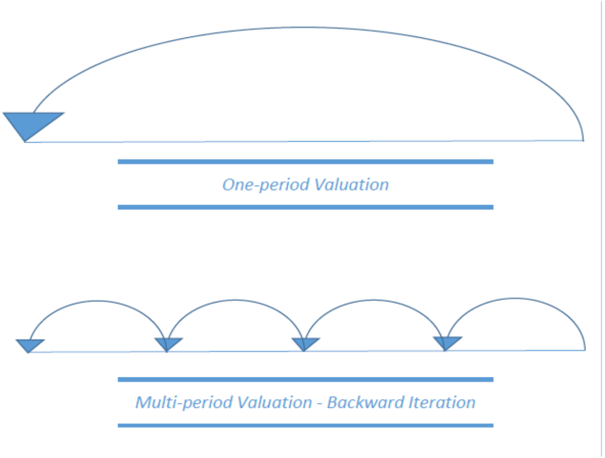
$$\text{Variance Price:} \quad \Pi[H] = \mathbb{E}[H] + \frac{1}{2}\alpha \mathbb{V}\mathrm{ar}[H], \qquad \alpha \geq 0 \tag{1a}$$

$$\text{Std-Deviation Price:} \quad \Pi[H] = \mathbb{E}[H] + \beta\sqrt{\mathbb{V}\mathrm{ar}[H]}, \qquad \beta \geq 0 \tag{1b}$$

$$\text{Cost-of-Capital Price:} \quad \Pi[H] = \mathbb{E}[H] + \delta \mathrm{VaR}_q\left[H - \mathbb{E}[H]\right], \quad \delta \geq 0 \tag{1c}$$

Construct the Time-consistent Valuation

[Jobert and Rogers, 2008]: Time-Consistent valuation can be constructed by the "**backward iteration**" of the static one-period valuation.



Continuous-time limit of the Time-consistent actuarial operators

Diffusion Insurance Process: $\mathrm{d}y_t = a(t, y_t)\mathrm{d}t + b(t, y_t)\,\mathrm{d}W_t$

- Variance premium principle $\rightarrow$ Exponential indifference price

$$\mathbb{E}[f(y_T)|y_t] + \tfrac{1}{2}\alpha\mathbb{V}\mathrm{ar}[f(y_T)|y_t] \equiv \frac{1}{\alpha}\ln\mathbb{E}_t\left[e^{\alpha f(y_T)}\middle|y_t\right]. \quad (2)$$

- Standard-Deviation principle

$$\mathbb{E}[f(y_T)|y_t] + \beta\sqrt{\mathbb{V}\mathrm{ar}[f(y_T)|y_t]} \equiv \mathbb{E}_t^{\$}\left[f(y_T)|y_t\right] \quad (3)$$

with "risk-adjusted" $\mathrm{d}y^{\$} = \big(a(t,y) \pm \beta b(t,y)\big)\,\mathrm{d}t + b(t,y)\,\mathrm{d}W^{\$}$.

- Cost-of-Capital principle

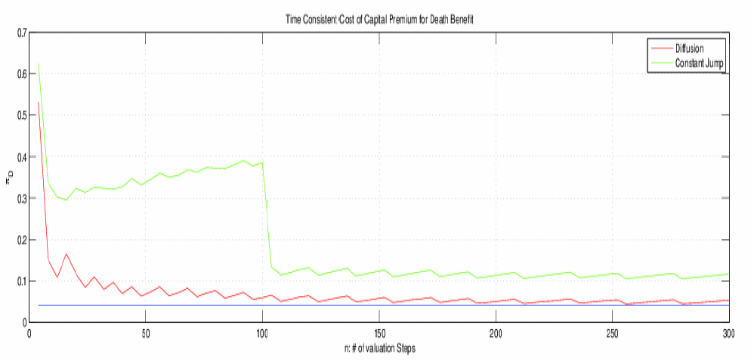
$$\mathbb{E}[f(y_T)|y_t] + \delta\mathrm{VaR}_q\left[f(y_T) - \mathbb{E}[f(y_T)|y_t]|y_t\right] \equiv \mathbb{E}^{\mathbb{C}}\left[f(y_T)|y_t\right] \quad (4)$$

with "risk-adjusted" $\mathrm{d}y^{\mathbb{C}} = \big(a(t,y) \pm \delta k b(t,y)\big)\,\mathrm{d}t + b(t,y)\,\mathrm{d}W^{\mathbb{C}}$.

Continuous-time limit of the Time-consistent actuarial operators

Jump-Diffusion Insurance Process:

- We found PIDEs. There exist a convergent time-consistent price.
- Each operator reflects the effect of the jump differently.
- VaR fails to capture part of the jump effect!!!



Market-Consistency

- x_t: traded hedgeable financial process
- y_t: unhedgeable insurance process
- $G(x_T, y_T)$: General hybrid claim
- $H^S(x_T)$: financial derivative

$$\Pi_{\mathcal{G}}(G + H^S) = \Pi_{\mathcal{G}}[G] + \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}\left[H^S\right] \tag{5}$$

- Generalised notion of "translation invariance" for financial risk
- Market-consistent valuation can not be "improved" by hedging
- Roughly saying: **If there is anything hedgeable (even in payoff G), it must be hedged via Market-consistent valuation!**

Two-step Valuation

- [Pelsser and Stadje, 2014]: Market-consistent valuation can be constructed by "**Two-step Market Evaluation**".

$$\Pi_{\mathcal{G}_t^A}\left[G(x_T, y_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[G\left(x_T, y_T\right) \;\middle|\; (y_t, x_T)\right] \middle| \; (y_t, x_t)\right]. \quad (6)$$

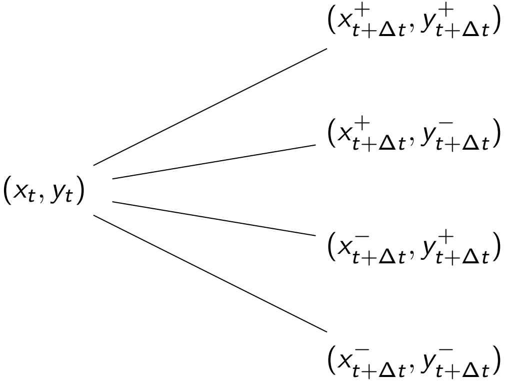
- **First/Inner step**: Fix the financial risk and apply the actuarial operator,

$$\Pi^{\mathbb{P}}\left[G\left(x_T, y_T\right) \;\middle|\; \sigma\left(\mathcal{G}_t^A, \mathcal{F}_T^S\right)\right] := G^S\left(x_T, y_t\right)$$

- The output is perfectly hedgeable.
- **Second/Outer step**: Conditional expectation under $\mathbb{Q}$
- Reflects the no-arbitrage argument for the hedgeable part of the general position.
- Gives initial capital needed to hedge the payoff/position.

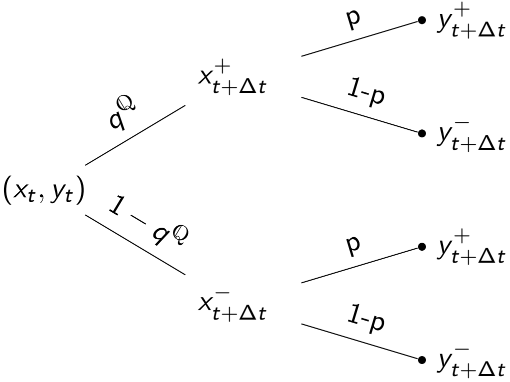
Quadrinomial Discretization

At a typical time-step $(t, t + \Delta t)$, every state (x_t, y_t) of the payoff at time t will develop to four different states of the world at time $t + \Delta t$,



Two-step Binomial Discretization

We pretend that first, x_t evolves ending to two different states at $t + \Delta t$. Only then, given each state of $x_{t+\Delta t}$, the process y_t moves:



Clean separation of financial and actuarial pricing in each "half-step".

Tech. cond: financial info arrives more frequently than insurance info

Market-Consistent & Time-Consistent

Combine Backward Iteration & Two-step Actuarial valuation. In discrete set $\{0, \Delta t, 2\Delta t, ..., T - \Delta t, T\}$ dividing $[0, T]$

- **Start** from T over $(T - \Delta t, T)$ and value $G(T, x(T), y(T))$ at $T - \Delta t$.

$$\pi_{\mathcal{G}^A}(T - \Delta t, x_{T-\Delta t}, y_{T-\Delta t}) = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left[G(T, x_T, y_T) \,\middle|\, \mathcal{G}^A_{T-\Delta t}, \mathcal{F}^S_T\right] \,\middle|\, \mathcal{G}^A_{T-\Delta t}, \mathcal{F}^S_{T-\Delta t}\right] \tag{7}$$

- $\pi(T - \Delta t, x_{T-\Delta t}, y_{T-\Delta t})$ is the **New payoff** in $(T - 2\Delta t, T - \Delta t)$.
- **Move Back** to $T - 2\Delta t$

$$\pi_{\mathcal{G}^A}(T - 2\Delta t, x, y) = \mathbb{E}^{\mathbb{Q}}\left[\Pi^{\mathbb{P}}\left(\pi_{\mathcal{G}^A}(T - \Delta t, x, y) \,\middle|\, \mathcal{G}^A_{T-2\Delta t}, \mathcal{F}^S_{T-\Delta t}\right) \,\middle|\, \mathcal{G}^A_{T-2\Delta t}, \mathcal{F}^S_{T-2\Delta t}\right].$$

- Repeat the procedure till $(0, \Delta t)$.

Contributions for the 2nd Paper

- Found continuous-time limit of the Time-consistent two-step valuation for some Actuarial operators.
- Found Some analytical solutions when the financial and actuarial risks are independent.
- Implemented regression-based computation method the for backward iteration of the two-step valuation.
- Calculated "**time-consistency risk premium**" = Time-consistent price - One-period Price

Application: Pricing the Participating Contract

- Payoff

$$G(P_T, r_T, \kappa_T) = \left(e^{-\int_0^T r_t dt}\right) \times P_T^{(h)} \times N_x(T). \tag{8}$$

- $N_x(T)$: Number of Survivors at Maturity T,
- P_T: Policy Reserve at Maturity
- $r_P(t)$: Policy Interest rate

$$P_t = P_{t-1} \times r_P(t).$$

- Three Underlying risk Drivers
  - A_t: Investment asset (Financial) - Black-Scholes Model
  - r_t: Interest rate (Financial) - Hull-White Model
  - κ_t: Longevity trend (Actuarial) - Lee-carter Model

Profit-Sharing Mechanism

- [Grosen and Jorgensen, 2000]: Path-dependent crediting mechanism

$$r_P(t) = \max\left\{ r_G \, , \, \alpha \left(\frac{A_{t-1}}{P_{t-1}} - (1+\gamma) \right) \right\} \quad t = 1, 2, ..., T \qquad (9)$$

- r_G: Guaranteed Interest rate
- γ: Target Buffer Ratio, Realistic Value: 10-20%
- α: Distribution Ratio, Realistic Value: 20-50%
- Similar to an **Option element** with strike value r_G.
- **Pure financial** mechanism: No longevity risk plays a role!!!

Hybrid Profit-Sharing Mechanism

- Hybrid crediting mechanism

$$r_P^{(h)}(t) = \max\left\{ r_G \,,\, \alpha \left(\frac{BE_0\left(N_x(T)\right)}{BE_t\left(N_x(T)\right)} \times \frac{A_{t-1}}{P_{t-1}} - (1+\gamma) \right) \right\} \qquad (10)$$

- **A**ll investment asset, interest rate and longevity play role!!!
- Price calculated by Numerical Methods; No Analytical Solution.

Market-Consistent Std-Dev Pricing

- **Multi-period** two-step Std-Dev price over $(t, t+1)$ (by **Backward iteration**),

$$\pi_t(A_{t+1}, r_{t+1}, \kappa_{t+1}) =$$
$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[\left(e^{-\int_t^{t+1} r_s \mathrm{d}s}\right)\pi_{t+1}(A_{t+1}, r_{t+1}, \kappa_{t+1}) \;\middle|\; \kappa_t, A_{t+1}, r_{t+1}\right]\right.$$
$$\left.+\beta\sqrt{\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\left(e^{-\int_t^{t+1} r_s ds}\right)\pi_{t+1}(A_{t+1}, r_{t+1}, \kappa_{t+1}) \mid \kappa_t, A_{t+1}, r_{t+1}\right]} \;\middle|\; \kappa_t, A_t, r_t\right]$$

with terminal condition

$$\pi_T(A_T, r_T, \kappa_T) = \left(e^{-\int_{T-1}^{T} r_s \mathrm{d}s}\right) \times P_T^{(h)} \times N_x(T) \tag{11}$$

- **One-period** two-step Std-Dev Price ("Traditional Actuarial Price")
- **Expected** Value

Numerical Techniques

- Conditional operators at each time step given **the state** of the underlying processes at **previous step**.
- **Methods**:
  - Nested Monte Carlo (Inefficient computation)
  - Markov Chain Discretization / Trinomial Tree (Inefficient in higher dimension)
  - Least-Square Monte-Carlo (Regress Now) (More Efficient Computation)

Figure 4. American option simulation tree

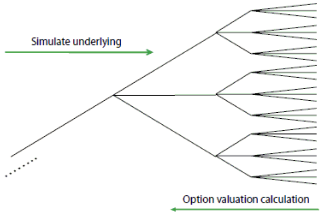
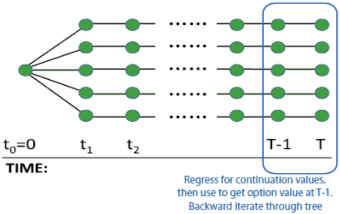


Figure 5. One inner scenario regression



LSMC for Two-step Std-Deviation Valuation

- [Longstaff & Schwartz, 2001] and [Glasserman & Yu, 2002] used regression-based methods to valuate American options.
- **First/Inner Step Regression**

$$\widehat{\mathbb{E}^{\mathbb{P}}}\left[\left(e^{-(r_T T - r_{T-1}(T-1))}\right)\ P_T^{(h)} \times N_x(T)\ \middle|\ A_T, r_T, \kappa_{T-1}\right] = \sum_{k=0}^{K-1} \widehat{a}_k^{(1,T)} e_k(A_T, r_T, \kappa_{T-1})$$

$$\widehat{\mathbb{E}^{\mathbb{P}}}\left[\left(\left(e^{-(r_T T - r_{T-1}(T-1))}\right)\ P_T^{(h)} \times N_x(T)\right)^2\ \middle|\ A_T, r_T, \kappa_{T-1}\right] = \sum_{k=0}^{K-1} \widehat{a}_k^{(2,T)} e_k(A_T, r_T, \kappa_{T-1})$$

- Calculate the conditional premium $\pi^s(S_T\,,\,\kappa_{T-\Delta t})$,

$$\pi^s\left(A_T,\, r_T,\, \kappa_{T-1}\right) = \widehat{\mathbb{E}^{\mathbb{P}}}\left[f_T^{(h)}\right] + \beta\sqrt{\widehat{\mathbb{E}^{\mathbb{P}}}\left[(f_T^{(h)})^2\right] - \left(\widehat{\mathbb{E}^{\mathbb{P}}}\left[f_T^{(h)}\right]\right)^2}.$$

- **Second/Outer Step Regression**

$$\widehat{\mathbb{E}^{\mathbb{Q}}}\left[\pi^s\left(A_T,\, r_T,\, \kappa_{T-1}\right)\right) \mid A_{T-1}, r_{t-1}, \kappa_{T-1}] = \sum_{k=0}^{K-1} b_k^T e_{\pi^s}(A_{T-1}, r_{T-1},\, \kappa_{T-1}). \tag{13}$$

TC Two-step Std-Dev Price vs Expected Value

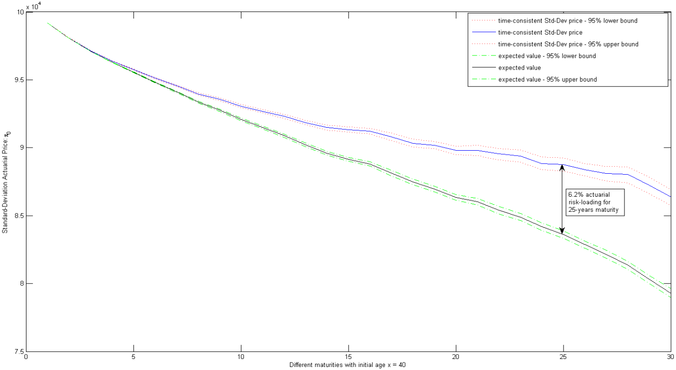


Figure: Time-consistent and market-consistent Standard-Deviation actuarial price vs. discounted expected value of the participating contract with 95% confidence interval and different maturities $T = 1, 2, ..., 30$. Parameter set: $A_0 = P_0 = 100$, $\sigma_A = 15\%$, $\sigma_r = 1\%$, $\rho_{A,r} = 0.25$, $r_G = 2\%$, $\alpha = 0.3$, $\gamma = 0.25$, $n = 1,000$, $N = 100$.

Proportion of the Risk-loadings in MC Price

Expected Value < 1-period Std-Dev Price < TC Std-Dev Price

Table: Values of the participating contract with different maturities and initial cohort of $N_{40}(0) = 1,000$ and the ratio of one-period risk loading and time-consistency risk premium on top of the expected value of the contract. Parameter set: $A_0 = P_0 = 100$, $\sigma_A = 15\%$, $\sigma_r = 1\%$, $\rho_{A,r} = 0.25$, $r_G = 2\%$, $\alpha = 0.3$, $\gamma = 0.25$, $n = 1,000$, $N = 100$.

	T					
Two-step Price	5	10	15	20	25	30
Expected-value	95,558.6	92,088.3	89,138.6	86,328.2	83,600.2	79,284.6
One-perod Std-Dev	95,574.6	92,123.6	89,198.5	86,422.7	83,746.6	79,513.5
Time-Consistent Std-Dev	95,752.7	93,072.1	91,342.2	89,802.1	88,761.6	86,365.5
One-period Risk-loading	0.02%	0.04%	0.07%	0.11%	0.19%	0.31%
Time-consistency Premium	0.19%	1.03%	2.40%	3.91%	5.99%	8.62%
Total Risk-loading	0.20%	1.07%	2.47%	4.02%	6.17%	8.93%
Ratio of TC Premium	91.8%	96.4%	97.2%	97.2%	97.0%	96.5%

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