Time-consistent and Market-consistent Actuarial Valuations

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Research Outputs

3 Papers:

Motivation

- **Actuarial Pricing for Modern Life/pension Contract:**
  - Hybrid liabilities from financial and actuarial risks
  - Liabilities are not fully traded in the market.
  - Financial part is usually hedgeable while actuarial part is not.
  - Dynamic pricing needed due to path dependency & embedded options

- **Regulatory requirement for Market-consistent Valuation**

- **“Re-Valuation” Insurance liabilities**

- **Payoff in “Long-term”**
  - What happens in the middle time?!
  - How do we reflect this in pricing?
  - Require time-consistency for Actuarial price operators?
Build Time-consistent Actuarial Pricing

Combine Actuarial & Financial Pricing for Market-consistent Valuation

“Applied Techniques” to implement Time-consistent and Market-consistent Pricing
Time-Consistency

- If $X_1 \geq X_2$ at time $T$, then $\Pi[t, X_1] \geq \Pi[t, X_2]$ for all $t < T$
- **Note:** The conditional expectation operator $\mathbb{E}^Q[H \mid \mathcal{F}_t]$ is TC.
- Extension of “tower property” to actuarial operators

$$\Pi[H(T) \mid G_t^A] = \Pi\left[\Pi[H(T) \mid G_s^A] \mid G_t^A\right] \quad \text{for} \quad t \leq s < T$$

- Well-known Actuarial operators are **NOT** time-Consistent

  Variance Price:  
  $$\Pi[H] = \mathbb{E}[H] + \frac{1}{2} \alpha \text{Var}[H], \quad \alpha \geq 0$$  
  \hspace{1cm} (1a)

  Std-Deviation Price:  
  $$\Pi[H] = \mathbb{E}[H] + \beta \sqrt{\text{Var}[H]}, \quad \beta \geq 0$$  
  \hspace{1cm} (1b)

  Cost-of-Capital Price:  
  $$\Pi[H] = \mathbb{E}[H] + \delta \text{VaR}_q[H - \mathbb{E}[H]], \quad \delta \geq 0$$  
  \hspace{1cm} (1c)
Construct the Time-consistent Valuation

[Jobert and Rogers, 2008]: Time-Consistent valuation can be constructed by the "backward iteration" of the static one-period valuation.
Continuous-time limit of the Time-consistent actuarial operators

Diffusion Insurance Process: $\frac{dy_t}{dt} = a(t, y_t)\,dt + b(t, y_t)\,dW_t$

- Variance premium principle $\rightarrow$ Exponential indifference price

$$E[f(y_T)|y_t] + \frac{1}{2}\alpha \text{Var}[f(y_T)|y_t] \equiv \frac{1}{\alpha} \ln E_t \left[ e^{\alpha f(y_T)} \right] |y_t]. \quad (2)$$

- Standard-Deviation principle

$$E[f(y_T)|y_t] + \beta \sqrt{\text{Var}[f(y_T)|y_t]} \equiv E^S_t[f(y_T)|y_t] \quad (3)$$

with “risk-adjusted” $\frac{dy^S}{dt} = (a(t, y) \pm \beta b(t, y))\,dt + b(t, y)\,dW^S$.

- Cost-of-Capital principle

$$E[f(y_T)|y_t] + \delta \text{VaR}_q [f(y_T) - E[f(y_T)|y_t]] |y_t] \equiv E^C_t[f(y_T)|y_t] \quad (4)$$

with “risk-adjusted” $\frac{dy^C}{dt} = (a(t, y) \pm \delta k b(t, y))\,dt + b(t, y)\,dW^C$. 
Continuous-time limit of the Time-consistent actuarial operators

Jump-Diffusion Insurance Process:
- We found PIDEs. There exist a convergent time-consistent price.
- Each operator reflects the effect of the jump differently.
- VaR fails to capture part of the jump effect!!!
Market-Consistency

- $x_t$: traded hedgeable financial process
- $y_t$: unhedgeable insurance process
- $G(x_T, y_T)$: General hybrid claim
- $H^S(x_T)$: financial derivative

\[ \Pi_G(G + H^S) = \Pi_G[G] + E^{Q_G}[H^S] \]  \hspace{1cm} (5)

- Generalised notion of “translation invariance” for financial risk
- Market-consistent valuation can not be “improved” by hedging
- Roughly saying: If there is anything hedgeable (even in payoff $G$), it must be hedged via Market-consistent valuation!
Two-step Valuation

- [Pelsser and Stadje, 2014]: Market-consistent valuation can be constructed by "Two-step Market Evaluation".

\[
\Pi_{G_t^A} [G(x_T, y_T)] = \mathbb{E}^Q \left[ \Pi^P \left[ G(x_T, y_T) \bigg| (y_t, x_T) \right] \bigg| (y_t, x_t) \right]. \quad (6)
\]

- **First/Inner step**: Fix the financial risk and apply the actuarial operator,

\[
\Pi^P \left[ G(x_T, y_T) \bigg| \sigma \left( G_t^A, \mathcal{F}_T^S \right) \right] := G^S (x_T, y_t)
\]

- The output is perfectly hedgeable.

- **Second/Outer step**: Conditional expectation under \( \mathbb{Q} \)

- Reflects the no-arbitrage argument for the hedgeable part of the general position.

- Gives initial capital needed to hedge the payoff/position.
Quadrinomial Discretization

At a typical time-step \((t, t + \Delta t)\), every state \((x_t, y_t)\) of the payoff at time \(t\) will develop to four different states of the world at time \(t + \Delta t\),

\[
(x_t, y_t) \rightarrow \begin{align*}
(x_t + \Delta t, y_t + \Delta t) \\
(x_t + \Delta t, y_t - \Delta t) \\
(x_t - \Delta t, y_t + \Delta t) \\
(x_t - \Delta t, y_t - \Delta t)
\end{align*}
\]
Two-step Binomial Discretization

We pretend that first, $x_t$ evolves ending to two different states at $t + \Delta t$. Only then, given each state of $x_{t+\Delta t}$, the process $y_t$ moves:

$$
\begin{align*}
(x_t, y_t) &
\rightarrow
\begin{cases}
(x^+_{t+\Delta t}, y^+_{t+\Delta t}) & \text{with probability } p \\
(x^-_{t+\Delta t}, y^-_{t+\Delta t}) & \text{with probability } 1-p
\end{cases}
\end{align*}
$$

Clean separation of financial and actuarial pricing in each “half-step”.

Tech. cond: financial info arrives more frequently than insurance info
Market-Consistent & Time-Consistent

Combine Backward Iteration & Two-step Actuarial valuation. In discrete set \( \{0, \Delta t, 2\Delta t, \ldots, T - \Delta t, T\} \) dividing \([0, T]\)

- **Start** from \( T \) over \((T - \Delta t, T)\) and value \( G(T, x(T), y(T)) \) at \( T - \Delta t \).

\[
\pi_{GA}(T - \Delta t, x_{T-\Delta t}, y_{T-\Delta t}) = \mathbb{E}^Q \left[ \prod \mathbb{P} \left[ G(T, x_T, y_T) \mid G_{T-\Delta t}^A, \mathcal{F}_T^S \right] \mid G_{T-\Delta t}^A, \mathcal{F}_T^S \right] \tag{7}
\]

- \( \pi(T - \Delta t, x_{T-\Delta t}, y_{T-\Delta t}) \) is the **New payoff** in \((T - 2\Delta t, T - \Delta t)\).

- **Move Back** to \( T - 2\Delta t \)

\[
\pi_{GA}(T - 2\Delta t, x, y) = \mathbb{E}^Q \left[ \prod \mathbb{P} \left( \pi_{GA}(T - \Delta t, x, y) \mid G_{T-2\Delta t}^A, \mathcal{F}_T^S \right) \mid G_{T-2\Delta t}^A, \mathcal{F}_T^S \right].
\]

- Repeat the procedure till \((0, \Delta t)\).
Contributions for the 2nd Paper

- Found continuous-time limit of the Time-consistent two-step valuation for some Actuarial operators.
- Found Some analytical solutions when the financial and actuarial risks are independent.
- Implemented regression-based computation method the for backward iteration of the two-step valuation.
- Calculated
  \[
  \text{“time-consistency risk premium”} = \text{Time-consistent price} - \text{One-period Price}
  \]
Application: Pricing the Participating Contract

- Payoff

\[ G(P_T, r_T, \kappa_T) = \left( e^{-\int_0^T r_t \, dt} \right) \times P_T^{(h)} \times N_x(T). \]  

(8)

- \( N_x(T) \): Number of Survivors at Maturity \( T \),
- \( P_T \): Policy Reserve at Maturity
- \( r_P(t) \): Policy Interest rate

\[ P_t = P_{t-1} \times r_P(t). \]

- Three Underlying risk Drivers
  - \( A_t \): Investment asset (Financial) - Black-Scholes Model
  - \( r_t \): Interest rate (Financial) - Hull-White Model
  - \( \kappa_t \): Longevity trend (Actuarial) - Lee-carter Model
Profit-Sharing Mechanism

- [Grosen and Jorgensen, 2000]: Path-dependent crediting mechanism

\[ r_P(t) = \max \left\{ r_G, \alpha \left( \frac{A_{t-1}}{P_{t-1}} - (1 + \gamma) \right) \right\} \quad t = 1, 2, \ldots, T \quad (9) \]

- \( r_G \): Guaranteed Interest rate
- \( \gamma \): Target Buffer Ratio, Realistic Value: 10-20%
- \( \alpha \): Distribution Ratio, Realistic Value: 20-50%
- Similar to an Option element with strike value \( r_G \).
- Pure financial mechanism: No longevity risk plays a role!!!
Hybrid Profit-Sharing Mechanism

- Hybrid crediting mechanism

\[
 r_P^{(h)}(t) = \max \left\{ r_G, \alpha \left( \frac{BE_0(N_x(T))}{BE_t(N_x(T))} \times \frac{A_{t-1}}{P_{t-1}} - (1 + \gamma) \right) \right\}
\]

- **All** investment asset, interest rate and longevity play role!!!
- Price calculated by Numerical Methods; No Analytical Solution.
Market-Consistent Std-Dev Pricing

- **Multi-period** two-step Std-Dev price over \((t, t + 1)\) (by Backward iteration),

\[
\pi_t(A_{t+1}, r_{t+1}, \kappa_{t+1}) =
\text{EQ} \left[ \text{EP} \left[ \left( e^{-\int_t^{t+1} r_s ds} \right) \pi_{t+1}(A_{t+1}, r_{t+1}, \kappa_{t+1}) \mid \kappa_t, A_{t+1}, r_{t+1} \right] \right]
+ \beta \sqrt{\text{Var} \left[ \left( e^{-\int_t^{t+1} r_s ds} \right) \pi_{t+1}(A_{t+1}, r_{t+1}, \kappa_{t+1}) \mid \kappa_t, A_{t+1}, r_{t+1} \right] \mid \kappa_t, A_t, r_t}
\]

with terminal condition

\[
\pi_T(A_T, r_T, \kappa_T) = \left( e^{-\int_{T-1}^{T} r_s ds} \right) \times P_T^{(h)} \times N_x(T)
\]

- **One-period** two-step Std-Dev Price ("Traditional Actuarial Price")
- **Expected Value**
Numerical Techniques

- Conditional operators at each time step given the state of the underlying processes at previous step.

**Methods:**
- Nested Monte Carlo (Inefficient computation)
- Markov Chain Discretization / Trinomial Tree (Inefficient in higher dimension)
- Least-Square Monte-Carlo (Regress Now) (More Efficient Computation)
LSMC for Two-step Std-Deviation Valuation


**First/Inner Step Regression**

\[
\hat{E}^P \left[ \left( e^{-\left( r_T - r_{T-1} (T-1) \right)} \right) P^{(h)}_T \times N_x(T) \mid A_T, r_T, \kappa_{T-1} \right] = \sum_{k=0}^{K-1} \hat{a}^{(1,T)}_k e_k(A_T, r_T, \kappa_{T-1})
\]
\[
\hat{E}^P \left[ \left( e^{-\left( r_T - r_{T-1} (T-1) \right)} \right) P^{(h)}_T \times N_x(T)^2 \mid A_T, r_T, \kappa_{T-1} \right] = \sum_{k=0}^{K-1} \hat{a}^{(2,T)}_k e_k(A_T, r_T, \kappa_{T-1})
\]

- Calculate the conditional premium \( \pi^S(S_T, \kappa_{T-\Delta t}) \),

\[
\pi^S(A_T, r_T, \kappa_{T-1}) = \hat{E}^P \left[ f^{(h)}_T \right] + \beta \sqrt{\hat{E}^P \left[ (f^{(h)}_T)^2 \right] - \left( \hat{E}^P \left[ f^{(h)}_T \right] \right)^2}.
\]

**Second/Outer Step Regression**

\[
\hat{E}^Q \left[ \pi^S(A_T, r_T, \kappa_{T-1}) \mid A_{T-1}, r_{t-1}, \kappa_{T-1} \right] = \sum_{k=0}^{K-1} b^T_k e^{\pi^S}(A_{T-1}, r_{T-1}, \kappa_{T-1}).
\]
**TC Two-step Std-Dev Price vs Expected Value**

Figure: Time-consistent and market-consistent Standard-Deviation actuarial price vs. discounted expected value of the participating contract with 95% confidence interval and different maturities $T = 1, 2, ..., 30$. Parameter set: $A_0 = P_0 = 100$, $\sigma_A = 15\%$, $\sigma_r = 1\%$, $\rho_{A,r} = 0.25$, $r_G = 2\%$, $\alpha = 0.3$, $\gamma = 0.25$, $n = 1,000$, $N = 100$. 
## Proportion of the Risk-loadings in MC Price

### Expected Value < 1-period Std-Dev Price < TC Std-Dev Price

**Table:** Values of the participating contract with different maturities and initial cohort of $N_{40}(0) = 1,000$ and the ratio of one-period risk loading and time-consistency risk premium on top of the expected value of the contract. Parameter set: $A_0 = P_0 = 100$, $\sigma_A = 15\%$, $\sigma_r = 1\%$, $\rho_{A,r} = 0.25$, $r_G = 2\%$, $\alpha = 0.3$, $\gamma = 0.25$, $n = 1,000$, $N = 100$.

<table>
<thead>
<tr>
<th>T</th>
<th>Two-step Price</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expected-value</td>
<td>95,558.6</td>
<td>92,088.3</td>
<td>89,138.6</td>
<td>86,328.2</td>
<td>83,600.2</td>
<td>79,284.6</td>
</tr>
<tr>
<td></td>
<td>One-period Std-Dev</td>
<td>95,574.6</td>
<td>92,123.6</td>
<td>89,198.5</td>
<td>86,422.7</td>
<td>83,746.6</td>
<td>79,513.5</td>
</tr>
<tr>
<td></td>
<td>Time-Consistent Std-Dev</td>
<td>95,752.7</td>
<td>93,072.1</td>
<td>91,342.2</td>
<td>89,802.1</td>
<td>88,761.6</td>
<td>86,365.5</td>
</tr>
<tr>
<td>One-period Risk-loading</td>
<td>0.02%</td>
<td>0.04%</td>
<td>0.07%</td>
<td>0.11%</td>
<td>0.19%</td>
<td>0.31%</td>
<td></td>
</tr>
<tr>
<td>Time-consistency Premium</td>
<td>0.19%</td>
<td>1.03%</td>
<td>2.40%</td>
<td>3.91%</td>
<td>5.99%</td>
<td>8.62%</td>
<td></td>
</tr>
<tr>
<td>Total Risk-loading</td>
<td>0.20%</td>
<td>1.07%</td>
<td>2.47%</td>
<td>4.02%</td>
<td>6.17%</td>
<td>8.93%</td>
<td></td>
</tr>
<tr>
<td>Ratio of TC Premium</td>
<td>91.8%</td>
<td>96.4%</td>
<td>97.2%</td>
<td>97.2%</td>
<td>97.0%</td>
<td>96.5%</td>
<td></td>
</tr>
</tbody>
</table>
