Consistent modeling and efficient pricing of VIX options

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Introduction

- The Volatility Index (VIX) is a measure of market volatility solely based on OTM option prices on the S&P 500.
- It is determined as the square root of the value of a portfolio that replicates the "fair" price of a 30-days ($=\tau$) variance swap.
- The continuous time version of the VIX is the square root of the log-price conditional annualized quadratic variation

$$\begin{aligned}\mathrm{VIX}_t^2 &= \frac{1}{\tau} E_t \left[[X]_t^{t+\tau} \right] \\ &:= \frac{1}{\tau} \lim_{N \to +\infty} E_t \left[\sum_{n=0}^{N} \left(\log \left(S_{t+n\frac{\tau}{N}} / S_{t+(n-1)\frac{\tau}{N}} \right) \right)^2 \right].\end{aligned}$$

Modeling VIX under stochastic volatility

In our setting today's price of a VIX call option with strike K and maturity T is always

$$C(K, T) = e^{-rT} E\left[(\mathrm{VIX}_T - K)^+\right],$$

therefore we model the risk-neutral dynamics of the VIX.

- The VIX has no explicit dependence on the stock price, but they can be linked through the instantaneous variance v.
- A starting point is the Heston model

$$\begin{aligned} d\log(S_t) &= \left(r - \frac{v_t}{2}\right) dt + \sqrt{v_t}\, dZ_t \\ dv_t &= \kappa(\theta - v_t)\, dt + \epsilon\sqrt{v_t}\, dW_t, \\ d\langle Z, W\rangle_t &= \rho\, dt. \end{aligned}$$

Advantages of the Heston model

- We can express VIX_t^2 as an affine function of v_t

$$\mathrm{VIX}_t = 100 \cdot \sqrt{a v_t + b}.$$

- The density of v_t is available in closed-form.

A pitfall of the Heston model

It is known that the Heston model fails in matching the upward sloping skew of VIX implied volatilities.

Possible generalizations

- Adding jumps in stock and volatility (AJD framework).
  - The affine relation between VIX^2 and v is preserved.
  - The Laplace transform of v is known in semi-closed form.
  - The moments of v can be determined very efficiently by the matrix-based technique of Cuchiero et al. (2010).
- Introducing additional volatility factors (double Heston, double mean reverting, etc.).
  - The affine relation between VIX^2 and v is preserved as long as all the volatility factors have linear drift.
  - Remaining in pure-diffusion allows for passing through PDEs.

SUBPROJECT 1 - PRICING VIX UNDER AJD

THE AJD FRAMEWORK

Under the affine jump-diffusion (AJD) framework we have

$$
\begin{aligned}
d\log(S_t) &= \left(r - \lambda\bar{\mu} - \frac{1}{2}v_t\right)dt + \sqrt{v_t}\,dZ_t + dJ_t^X, \\
dv_t &= \kappa(\theta - v_t)dt + \epsilon\sqrt{v_t}\,dW_t + dJ_t^v, \\
d\langle Z, W\rangle_t &= \rho\, dt,
\end{aligned}
$$

where $J = (J^X, J^v)$ is a pure jump process taking values in $\mathbb{R} \times \mathbb{R}^+$ and $\bar{\mu}$ is the compensator of J^X.

Orthogonal polynomial expansions

Denoting by f the risk-neutral density related to v we approximate f as follows

$$f(x) \approx \phi(x)\left(1 + \sum_{k=0}^{n} c_k h_k(x)\right)$$

where ϕ (kernel) represents a "tractable initial guess" for f and

- h_k are polynomials that only depend on ϕ,
- c_k are corrective factors embedding all the information on f.

The polynomials $(h_k)_{n\in\mathbb{N}}$ are chosen to be orthogonal with respect to ϕ, i.e.

(i) $\mu_k := \int_{-\infty}^{+\infty} x^k \phi(x)\,dx < +\infty, \quad \forall k \in \mathbb{N}$

(ii) $\int_{-\infty}^{+\infty} h_k(x) h_\ell(x) \phi(x)\,dx = 0, \quad \forall k \neq \ell$

Why using orthogonal polynomials?

If $(h_k)_{k\in\mathbb{N}}$ are orthogonal to ϕ then c_k are linear combinations of moments of f

$$c_k = \frac{1}{C_k} \sum_{i=0}^{k} w_{ki} \int_{-\infty}^{+\infty} x^i f(x)\,dx.$$

with $h_k(x) = \sum_{i=0}^{k} w_{ki} x^i$ and $C_k = \sqrt{\int_{-\infty}^{+\infty} h_k^2(x)\phi(x)\,dx}$

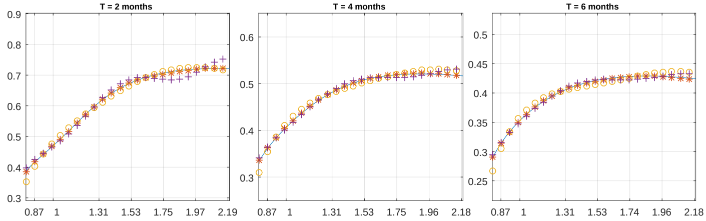
The eGIG kernel

We consider an enriched generalized inverse Gaussian (eGIG) kernel

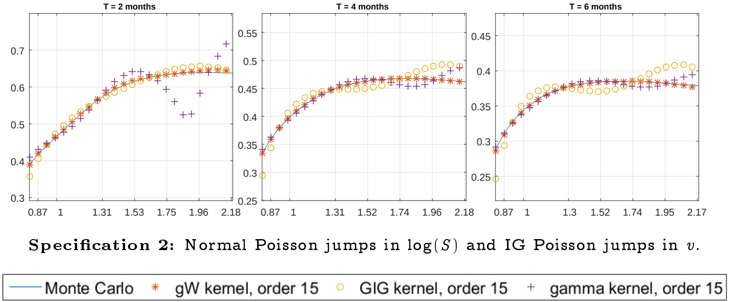
$$\phi(x) = x^{\alpha-1} e^{-\left(\beta x^{p} + \gamma x^{-1}\right)} 1_{[0,+\infty)}(x), \quad \alpha, \beta, \gamma > 0,\ p \in (\frac{1}{2}, 1].$$

We focus on three subcases:

- the gamma ($p = 1, \gamma = 0$) kernel
- the GIG ($p = 1$) kernel
- the gW ($\gamma = 0$) kernel



Specification 1: Normal Poisson jumps in $\log(S)$ and exponential Poisson jumps in v.



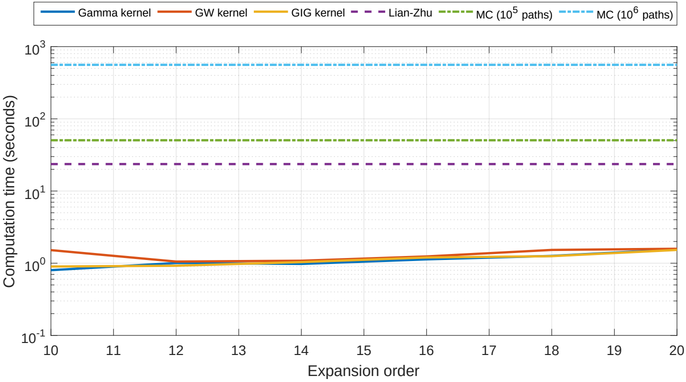
Specification 2: Normal Poisson jumps in $\log(S)$ and IG Poisson jumps in v.

Figure: VIX implied volatilities/moneyness under AJD with Poisson jumps.

Efficiency

Under Specification 2 we compare

- Our technique (orders ranging from 10 to 20).
- Laplace integration optimized formula of Lian and Zhu (2013).
- Monte Carlo with Euler scheme on a time grid of 10^3 points/month.



Subproject 2 - Pricing VIX under multi-factor pure diffusion

We consider the following multi-factor SV model

$$d\log(S_t) = \left(r - \frac{1}{2}\sum_{i=1}^{q} v_t^i\right) dt + \sum_{i=1}^{q} \sqrt{v_t^i}\, dZ_t^i,$$
$$dv_t = (\alpha v_t + \beta)\, dt + \eta(t, v_t)\, dW_t, \quad v_t \in \mathbb{R}^m, \ m \geq q,$$

where $\alpha \in \mathbb{R}^{q\times q}$, $\beta \in \mathbb{R}_+^q$, and $\eta = \mathrm{diag}(\eta_1, \ldots, \eta_q) \in \mathbb{R}^{q\times q}$ is a diagonal matrix of functions.

If Z have no correlation we have

$$\mathrm{VIX}_t = 100 \cdot \sqrt{\langle a, v_t\rangle + b}$$

for some $a \in \mathbb{R}^m$, $b \in \mathbb{R}$.

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Some history:

- Hagan and Woodward (1999) found asymptotics for the implied volatilities under a CEV model based on a PDE-technique.
- Lorig et al. (2015) have recently extended this technique to a broader pure-diffusion setting.
- Based on this, Pagliarani and Pascucci (2016) have extended these results to include the implied volatility sensitivities.
- We extend these results to a context where the underlying (the VIX futures) dynamics are not explicit.
- All these results apply to short-time and near ATM options.

Examples

Introducing an elasticity coefficient in the vol-of-vol

$$d\log(S_t) = -\frac{1}{2}v_t\,dt + \sqrt{v_t}\,dW_t^X$$
$$dv_t = k\left(v_t - \theta^h\right)dt + \epsilon^h v_t^\delta\,dW_t^Y$$

We have

$$\lim_{\substack{K\to \mathrm{VIX}_0\\ T\to 0}} \frac{\partial}{\partial K}\sigma^{\mathrm{imp}}(T,K) = \frac{\epsilon v_0^{\delta-\frac{1}{2}}\left(2\delta \mathrm{VIX}_0^2 - 2av_0\right)}{4\mathrm{VIX}_0^2\sqrt{v_0}}.$$

The slope is controlled by

$$2\delta\mathrm{VIX}_0^2 - 2av_0 = 2(\delta-1)av_0 + 2\delta b$$

Randomizing the vol-of-vol size (SVV model)

$$
\begin{aligned}
d\log(S_t) &= -\frac{1}{2} v_t^1\, dt + \sqrt{v_t^1}\, dZ_t \\
dv_t^1 &= \kappa_1 \left(v_t^1 - \theta_1\right) dt + v_t^2 \sqrt{v_t^1}\, dW_t^1 \\
dv_t^2 &= \kappa_2 \left(v_t^2 - \theta_2\right) dt + \eta\left(v_t^2\right) dW_t^2 \\
d\left\langle W^1, W^2 \right\rangle_t &= \rho\, dt
\end{aligned}
$$

We have

$$
\lim_{\substack{K \to \mathrm{VIX}_0 \\ T \to 0}} \frac{\partial}{\partial K} \sigma^{\mathrm{imp}}(T, K) = \frac{v_0^2 \left(\mathrm{VIX}_0^2 - 2 a_1 v_0^1\right)}{4\sqrt{v_0^1}\,\mathrm{VIX}_0^2} + \rho \frac{\eta(v_0^2)}{2 v_0^2}.
$$

Thank you for your attention.