Methods for modelling and calibrating to volatility surfaces

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Before starting...

- **PhD:** 01.2013-31.2015
- **My Supervisors** prof. Elisa Nicolato, prof. Peter Løchte Jørgensen
- **Institution:** Aarhus University
- **Research staying abroad:** TUT, Maastricht University, Malaga University, Imperial College
- **Currently:** Market Risk Models and Measures, Nordea in Copenhagen

Overview of my PhD thesis

- **Simple Simulation schemes for CIR and Wishart processes:** Simulation method with applications to interest rate, stochastic volatility and stochastic correlation models
- **The Impact of Jump Distributions on the Implied Volatility of Variance:** Modelling analysis with application to VIX and realized variance options
- **The Multivariate Mixture Dynamics Model: Shifted dynamics and correlation skew:** Modelling and Calibration study with applications to FX

Paper 1

- **Simple Simulation schemes for CIR and Wishart processes**
- Joint work with Paolo Baldi (Tor Vergata University, Rome)
- Published on the *International Journal of Theoretical and Applied Finance*

The Wishart process

The Wishart process (Bru, 1991) is a matrix valued process ($d \times d$), solution of:

$$d\Sigma_t = \left(\alpha\, a^T a + b\Sigma_t + \Sigma_t b^T\right) dt + \left(\sqrt{\Sigma_t} dW_t a + a^T dW_t^T \sqrt{\Sigma_t}\right)$$

where W_t is a $d \times d$ Brownian motion, $\alpha \geq d-1$, a, $b \in \mathcal{M}_{d\times d}(\mathbb{R})$ and Σ_0 is a symmetric positive semidefinite matrix. b is usually assumed to be negative semi-definite to grant some mean-reverting property

Case $d = 1 \Rightarrow$ square root (CIR) process

$$dv_t = (a - kv_t)dt + \sigma\sqrt{v_t}dW_t$$

How to simulate?

- Exact simulation for the Wishart process is possible but very time consuming
- Euler-Maruyama scheme tricky because of the presence of the square root

$$\hat{v}_{t_{i+1}} = \hat{v}_{t_i} + \frac{T}{n}(a - k\hat{v}_{t_i}) + \sigma\sqrt{\hat{v}_{t_i}}\left(W_{t_{i+1}} - W_{t_i}\right)$$

- Possible alternative: simulation through the schemes in Baldi and Pisani (2013)

Simulation scheme in (Baldi and Pisani 2013)

- Affine processes: it is possible to associate an infinitesimal generator characterizing the sde
- Assuming $\alpha \geq d$ we can split the infinitesimal generator of a Wishart process as:

  $$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$$

  where $\mathcal{L}_1$ is the generator of a deterministic process and $\mathcal{L}_2$ is the generator of the square of a multidimensional Ornstein-Uhlenbeck process.

- process associated to $\mathcal{L}_1$: easy to simulate
- process associated to $\mathcal{L}_2$: can be simulated through Gaussian variables
- Combining both we obtain a simulation scheme for the Wishart

Simulation scheme in (Baldi and Pisani 2013)

- **Advantages**
  - It works better than a simple Euler-Maruyama scheme
  - In the Wishart case: other methods are difficult to implement, this method involves only simulation of Gaussian variables and some matrix manipulations
- **Drawbacks**
  - Convergence granted only under the condition $\alpha \geq d$ ($\kappa\theta \geq \frac{\sigma^2}{4}$ in the Heston model).
  - However 1) this condition is less restrictive than the Feller condition 2) is it really restrictive?

Paper 2

- **The Impact of Jump Distributions on the Implied Volatility of Variance**

- Joint work with Elisa Nicolato (Aarhus University) and David Sloth (Danske Bank)
- Submitted

Overview

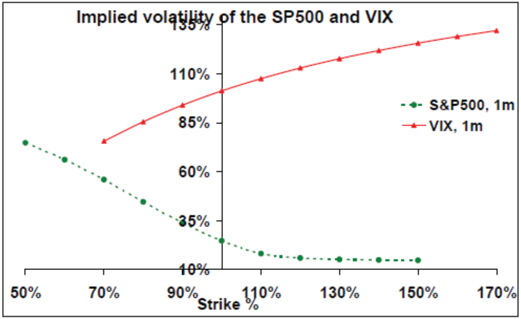
- We consider the SVJ-v model augmenting the Heston model with jumps in the instantaneous variance

$$
\begin{aligned}
dX_t &= -\tfrac{1}{2} v_t \, dt + \sqrt{v_t} dW_t \\
dv_t &= \lambda(\eta - v_t) dt + \varepsilon \sqrt{v_t} dB_t + dJ_t^v \; .
\end{aligned}
$$

  where λ,η, ε are positive constants, $d\langle W, B \rangle = \rho dt$ and J^v is a Lévy process with positive jumps

- and we analyze the impact of the **distribution** of the jumps on the wings of the implied volatility from realized variance options and VIX options

The implied volatility skew from volatility derivatives



- **Equity Derivatives**: downward sloping
- **Volatility Derivatives** (realized variance options, VIX options): upward sloping

RV and VIX under the SVJ-v model

1.The continuous time version of the realized variance (the quadratic variation of S_t) coincides with the integrated variance

$$RV_T = \int_0^T v_t dt$$

2. The square of the VIX is an affine tranformation of the instantaneous variance

$$\begin{aligned} VIX_t &= \sqrt{Av_t + B} \\ a &= \frac{1}{\lambda\tau}(1 - e^{-\lambda\tau}) \\ b &= (\frac{\mathbb{E}[J_1]}{\lambda} + \theta)(1 - a) \\ \tau &= 30/365 \end{aligned}$$

Main advantages of the SVJ-v model

- The Laplace transforms of RV and v_t are available in closed form
- This allow us for an handy study of the smile-wings behaviour for realized variance options and VIX options (number of finite positive and negative moments, link with the behaviour of the tail function)
- To sum up, the SVJ-v model allow for an easy study of the effect of variance-jumps distributions on the implied volatility from realized variance options and VIX options

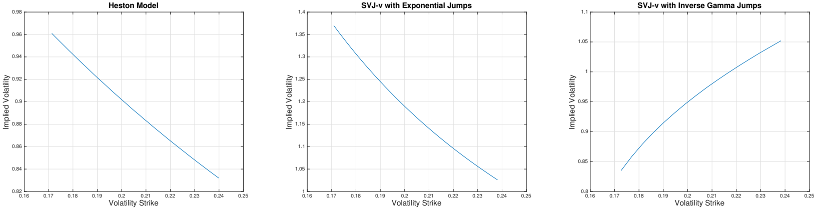
Conclusion

Previous literature

- Equity Derivatives: The particular distribution of the jump size does not have a big impact on the qualitative shape of the corresponding implied volatility

Our main findings

- Volatility Derivatives: the particular **distribution** of the jump does have an impact
- The commonly used exponential distribution may not be the most situable in reproducing the upward sloping volatility skew from volatility derivatives



- **Left:** Heston model. **Middle:** SVJ-v with Exponential jumps **Right:** SVJ-v with Inverse Gamma jumps
- Parameters in the Heston model from *Bakshi et al.* obtained by calibration to out-of-the-money options on the S&P 500
- The inverse gamma parameters obtained by moment-matching the exponential jump size

Paper 3

- **The Multivariate Mixture Dynamics Model: Shifted dynamics and correlation skew**
- Joint work with Damiano Brigo (Imperial College) and Francesco Rapisarda (Bloomberg)
- About to be submitted

The Lognormal Mixture Dynamics model

- Given N **instrumental processes** (following a Geometric Brownian motion)

$$dY^i(t) = \mu(t)Y^i(t)dt + \sigma^i(t)Y^i(t)dW(t)$$

with marginal densities p_t^i, we look for a process

$$dX(t) = \mu(t)X(t)dt + \nu(t, X(t))X(t)dW(t) \tag{1}$$

whose marginal density is a *mixture* of the single densities p_t^i:

$$p_t(x) = \sum_{i=1}^{N} \lambda_i p_t^i(x),\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1.$$

- Define

$$\nu(t, x) = \sqrt{\frac{\sum_{i=1}^N \lambda^i \sigma^i(t)^2 p_t^i(x)}{\sum_{i=1}^N \lambda^i p_t^i(x)}}.$$

Under few assumptions there is a unique solution of the SDE (1)

The Lognormal Mixture Dynamics model

- Prices of European options are combination of Black and Scholes prices
- The same linear combination holds for the Greeks

Introducing a shift

- However volatility smiles with a minimum exactly at the ATM forward are the only possible
- In order to gain more flexibility we can add a *shift*

$$S_t = X_t + \alpha(t)$$

- Shifts gives more flexibility: they allow to move the smile minimum point from the ATM forward
- preserving analytical tractability: for $\alpha(T) \leq K$,

$$C(K, T) = e^{-rT}\mathbb{E}[(S_T - K)^+] = e^{-rT}\mathbb{E}[(X_T - (K - \alpha(T)))^+]$$

The multidimensional framework: the Shifted SCMD model

- Imagine we have calibrated S_1, S_2 each to a shifted LMD model
- We want to reconnect their dynamics into a bi-dimensional model. How can we do that?

Classical approach:

- Introducing correlation between the Brownian motions driving the LMD models

$$\begin{aligned} dS_1(t) &= \mu S_1(t)dt + \nu_1(t, S_1(t))S_1(t)dW_1(t), \\ dS_2(t) &= \mu S_2(t)dt + \nu_2(t, S_2(t))S_2(t)dW_2(t) \\ d\langle W_1, W_2 \rangle &= \rho dt \end{aligned}$$

An alternative approach: the Shifted MVMD model

Alternative approach:

- Introducing correlation between the instrumental processes Y^i

$$
\begin{aligned}
dY_1^i(t) =& \mu(t) Y_1^i(t) dt + \nu^i(t, Y_1^i(t)) Y_1^i(t) dW_1(t) \\
dY_2^j(t) =& \mu(t) Y_2^j(t) dt + \nu^j(t, Y_2^j(t)) Y_2^j(t) dW_2(t) \\
d\langle W_1, W_2 \rangle =& \rho dt
\end{aligned}
$$

- and mix the component densities in all the possible ways

$$
p_{\underline{S}(t)}(\underline{x}) = \sum_{i,j=1}^{N} \lambda_1^i \lambda_2^j p_t^{i,j}(\underline{x}), \;\; p_t^{i,j}(\underline{x}) = p_{[Y_1^i(t), Y_2^j(t)]^T}(\underline{x}).
$$

- Mixture of multidimensional (bidimensional) processes
- This is consitent with the univariate smiles - the dynamics of the individual assets are exactly reproduced

Shifted SCMD vs. Shifted MVMD

Advantages of the (shifted) MVMD model:

- Analitical tractability
- Description of dependence structure
- What about fit to market data?

Numerical case study: FX cross rates

- We calibrate S_1 to an LMD model, using market prices on USD/EUR
- We calibrate S_2 to an LMD model, using market prices on EUR/JPY
- In order to calibrate the bi-dimensional model (S_1, S_2), we look at options with payoff $(S_1 S_2 - K)^+$
- These are call options on the FX exchange rate $S_3 = S_1 S_2 = USD/JPY$
- From these, we can find ρ

Meaning of ρ

- Under shifted SCMD dynamics, the instantaneous local correlation is ρ
- Under shifted MVMD dynamics, the instantenous local correlation has a more complicated expression than just ρ:

$$\frac{\rho \sum_{i,j=1}^{N} {\lambda_1}^i {\lambda_2}^j \sigma_1^{(i)} \sigma_2^{(j)} \tilde{\ell}_t^{(ij)}(x_1, x_2)}{\sqrt{\left(\sum_{i,j=1}^{N} {\lambda_1}^i {\lambda_2}^j \sigma_1^{(i)2} \tilde{\ell}_t^{(ij)}(x_1, x_2)\right)\left(\sum_{i,j=1}^{N} \lambda_1 i {\lambda_2}^j \sigma_2^{(j)2} \tilde{\ell}_t^{(ij)}(x_1, x_2)\right)}}$$

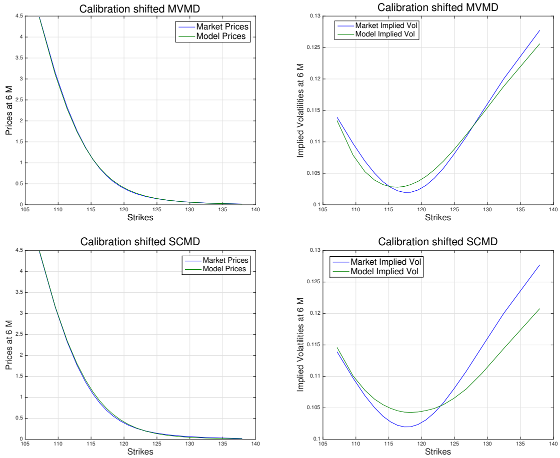


Figure: *Calibration on 6 months options, from 19 February 2015. The correlation parameter is $\rho = -0.5992$ for the MVMD model (top) and $\rho = -0.5472$ for the SCMD model (bottom).*

Introducing multiple correlation parameters

- In some cases calibration on the shifted MVMD model, though performing better than on the shifted SCMD model, is not that good
- **Idea:** generalize the model by

$$\frac{\sum_{i,j=1}^{N} {\lambda_1}^i {\lambda_2}^j \rho^{i,j} \sigma_1^{(i)} \sigma_2^{(j)} \tilde{\ell}_t^{(ij)}(x_1, x_2)}{\sqrt{\left(\sum_{i,j'=1}^{N} {\lambda_1}^i {\lambda_2}^j \sigma_1^{(i)2} \tilde{\ell}_t^{(ij)}(x_1, x_2)\right)\left(\sum_{i,j=1}^{N} {\lambda_1}^i {\lambda_2}^j \sigma_2^{(j)2} \tilde{\ell}_t^{(ij)}(x_1, x_2)\right)}}$$

- N^2 (4 in our experiments) different values for $\rho \implies$ more flexibility

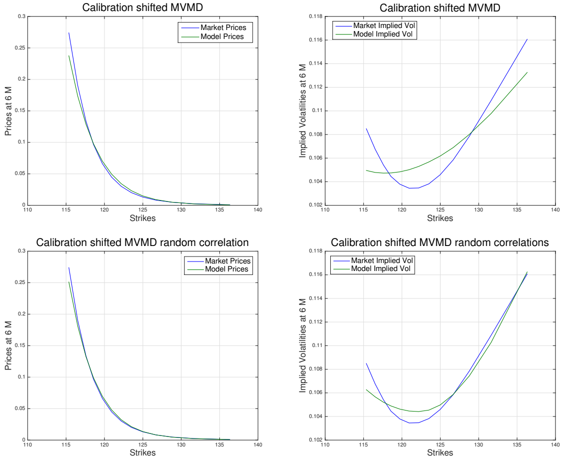


Figure: *Calibration on 6 months options, relative to 7th September 2015 under the MVMD model (top) and the MVMD model with multiple correlation parameters. In the first case* $\rho = -0.6147$, *in the second one* $\rho^{1,1} = -0.8717$, $\rho^{1,2} = -0.1762$, $\rho^{2,1} = -0.6591$, $\rho^{2,2} = -0.2269$.

Thank you for your attention!