Project Report RP1 - Deliverable 1.3

Report on Validation and Comparison of Developed and Implemented Risk Minimization Framework

Project Title: A Unified Risk Minimization Framework.

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The main purpose of this document is to report the research progress of the Research Project 1 (RP1) – Working package 1, according to Tasks and Deliverables (D.1.3) of the HPCFinance. The research leading to this report has received funding from the European Union's Seventh Framework Programme FP7/2007-2013 under grant agreement No. 289032. This report contains three working papers as follows:

Title: Effective Methods for Pricing Variable Annuities with Guaranteed Minimum Withdrawal Benefits

Abstract: In this paper, we review a general framework for pricing the special form of variable annuity called the Guaranteed Minimum Withdrawal Benefits (GMWB). This type of variable annuity guarantees a minimum annual or periodic withdrawal amount for an extended time period. In addition, the policyholder has a right to deduct the guaranteed amount from the policy's account until the maturity, regardless of whether the policy's account value drops to zero. We used the insurer's perspective approach, stating the net profit the insurance company will earn after paying the withdrawal benefits to the policyholder when the account value depleted to zero. We presented the two different approaches for the GMWB valuation. In particular, we analyze the pricing of GMWB guarantee with the ratchet designs and the impact of frequent ratchets on the value of GMWB guarantee. Moreover, we also analyzed the fair insurance fees associated with the distinct designs of GMWB. Furthermore, this paper provides a study on the effect of a various financial parameters on the GMWB guarantee's value and the fair insurance fee.

Title: Pricing and Hedging of the European Option Linked to Target Volatility Portfolio

Abstract: Advanced risk management strategies have become popular since they protect the investor's investment from market crashes. The latest risk management strategy is called the target volatility strategy. The target volatility strategy is used in order to maintain a stable realized volatility of a portfolio. The strategy re-balances the allocation of risky asset to non-risky asset in order to protect the portfolio from equity market crashes. This paper provides an extensive analysis of the target volatility portfolios and financial derivatives linked to target volatility portfolios. We demonstrate the performance of target volatility portfolios under different financial models. We also examine the effects of re-balancing frequencies on target volatility portfolios. We focus on the pricing and hedging of the European option linked to target volatility portfolios. In particular, we investigate the impact of stochastic equity asset volatility on the pricing and hedging of the European option linked to target volatility portfolios. We also consider different dynamic hedging strategies and compare their performance.

Title: An Analysis of Guaranteed Lifetime Withdrawal Benefits Linked to Target Volatility Portfolio

Abstract: GLWB is currently the most popular type of a variable annuity. GWLB promises the policyholder to annually withdraw a fixed amount from his investment account for the rest of his life, even if the value of the investment account drops to zero. However, from the insurer's point of view, GLWB contains several types of risk such as mortality risk, interest rate risk and financial risk. Target volatility strategy is used to create a dynamic re-balancing portfolio such that the overall volatility of portfolio maintains a stable level at all time. The strategy shifts the allocation of equity asset to non-risky asset in order to protect the portfolio from equity market crashes. Many insurance companies offer or consider GLWB linked to target volatility portfolios. This paper provides an extensive analysis of the GLWB linked to target volatility portfolio. In particular, we investigate the impact of stochastic volatility and mortality intensity on the pricing of the GLWB linked to target volatility portfolio. Moreover, we examine the lifetime probability of ruin for the target volatility portfolios.

Working Paper One

Effective Methods for Pricing Variable Annuities with Guaranteed Minimum Withdrawal Benefits

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Abstract

In this paper, we review a general framework for pricing the special form of variable annuity called the Guaranteed Minimum Withdrawal Benefits (GMWB). This type of variable annuity guarantees a minimum annual or periodic withdrawal amount for an extended time period. In addition, the policyholder has a right to deduct the guaranteed amount from the policy's account until the maturity, regardless of whether the policy's account value drops to zero. We used the insurer's perspective approach, stating the net profit the insurance company will earn after paying the withdrawal benefits to the policyholder when the account value depleted to zero. We presented the two different approaches for the GMWB valuation. In particular, we analyze the pricing of GMWB guarantee with the ratchet designs and the impact of frequent ratchets on the value of GMWB guarantee. Moreover, we also analyzed the fair insurance fees associated with the distinct designs of GMWB. Furthermore, this paper provides a study on the effect of a various financial parameters on the GMWB guarantee's value and the fair insurance fee.

Key Words: Variable annuity, fair insurance fee, guaranteed minimum withdrawal benefit, ratchet feature.

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1 Introduction

Variable annuities are among the most important types of equity-linked pension products and becoming popular in Europe as well as in the rest of the world. The term variable annuity refers to a broad range of retirement saving products. In addition, the benefits received by the policyholder depends on the performance of the investment fund. Insurance companies usually offer a vast range of investment choices to the policyholders. Financial markets are uncertain, this creates investor demand for guaranteed returns. To meet this demand, insurers provide minimum guarantees to protect the policyholder against financial risk. Under a variable annuity policy, the policyholder pays a lump sum premium (payment) or periodic premiums to the insurance company. This premium is invested in mutual funds chosen by the policyholder (this stage is called the accumulation phase). However, during the accumulation phase, the account value changes with respect to the performance of underlying fund. The policyholder may have to pay further premiums or can make partial withdrawals depending on the investment returns. The accumulation period ends when the policyholder decides to receive their investment as a lump sum benefit or as periodic payments (de-accumulation phase).

In addition, the policyholder can choose various types of guarantees in a variable annuity. Insurance companies use different names for these guarantees, see Hardy (2003). These guarantees classified in five main types:

- Guaranteed Minimum Death Benefits (GMDB)
- Guaranteed Minimum Accumulation Benefits (GMAB)
- Guaranteed Minimum Income Benefits (GMIB)
- Guaranteed Minimum Withdrawal Benefits (GMWB)
- Guaranteed Lifetime Withdrawal Benefits (GLWB)

General information about the different types of the guarantees in variable annuities and the market history of variable annuities can be found in Ledlie et al. (2008). Since the early 1990s, the market for variable annuities has grown rapidly in the USA and in various other parts of the world. However sales of variable annuities slumped in the recent 2007 financial crisis. Because of the high volatility during 2007 crisis, many VA writers were trapped in the need for extra capital for hedging programs due to the under-pricing of variable annuities. As a result, several insurers quitted the business or re-structured their products. The crisis highlighted the need for proper risk management techniques and effective methods for pricing these products.

The Guaranteed Minimum Withdrawal Benefits is a most popular guaranteed type among variable annuity customers. In the present paper, we look at variable annuities containing a GMWB that guarantees the periodic withdrawal amount for an extended period of time regardless of the performance of the investment fund. Furthermore, after the maturity of the contract, any remaining amount in the account will be refunded to the policyholder or policyholder's beneficiaries. We use the no-arbitrage model Black and Scholes (see, Black and Scholes (1973)) and the principles of mathematical finance in line with the traditional work of Boyle and Schwartz (1977). Our approach follows recent literature on the pricing of VAs: Bauer et al. (2008), Kolkiewicz and Liu (2012), Holz et al. (2012), Kling et al. (2011), Milevsky and Salisbury (2006), Piscopo and Haberman (2011), and Funga et al. (2013). Kling et al. (2011) investigate the impact of stochastic equity volatility on guarantee withdrawal rates and the hedging of GLWB. The Piscopo and Haberman (2011) approach based on the decomposition of the GLWB's value into death and living benefits and makes an analysis of the impact of the mortality risk on the price of GLWB. Our paper provides further detailed analysis on the GMWB contract. We review the modeling framework presented in Milevsky and Salisbury (2006) and Kolkiewicz and Liu (2012). In particular we analysis the GMWB with ratchet designs, and the impact of frequent ratchet on the fair insurance rates. In addition, we also analysis the impact of various financial parameters on GMWB guarantee value and fair insurance rates.

The valuation of GMWB guarantees is important to the insurer. The insurer guarantees the periodic withdrawal up till the maturity of contract, even if the investment account is depleted to zero. The withdrawal payments paid by the insurer after the policyholder's account value hit zero is known as GMWB guarantee.

We consider the general pricing framework for Guaranteed Minimum Withdrawal Benefits that is consistent with the no-arbitrage principle. Additionally, the paper contributes to the study of a different withdrawal design and especially the effect of ratchet design on a GMWB guarantee. Furthermore, we present two approaches for Guaranteed Minimum Withdrawal Benefits valuation.

The rest of the paper is organized as follows, in the next section, we give the detailed description of GMWB with different product designs. Also, we give an example of a simple GMWB contract and the dynamics of a policyholder's account. Section 3 introduces two approaches for valuation of GMWB. We generalize and review the pricing framework of Kolkiewicz and Liu (2012) and Milevsky and Salisbury (2006). There is no closed-form solution for the valuation of the GMWB guarantee due to their complexity. Hence, we present the numerical scheme in section 5. The numerical results are provided in section 6. Also, we present an illustration of how to determine the fair insurance rates and the impact of various financial parameters on a GMWB guarantees and fair insurance rates. Conclusions and suggestions for future research are presented in the last section.

2 The Contract Design

In this section, we introduce and explain a variable annuity contract with Guaranteed Minimum Withdrawal Benefits (GMWB) with different designs of guaranteed withdrawal amount. Suppose that at the beginning of the contract, the policyholder pays a lump sum amount to the insurer and this amount is invested in a fund. The policyholder can choose between different types of funds which can be combination of various stocks, bonds, real-estates, etc. We assume that the policyholder does not change the fund type after the inception of the contract. Furthermore, we consider that there is no accumulation phase and the policyholder will receive withdrawal amount from the inception of the contract. Moreover, the contract matures at time T. Let us suppose that the length of the time interval between withdrawal dates is denoted by $h > 0$, and the total number of policy anniversaries $N \in \mathbb{Z}^+$ is expressed as:

$$N = \frac{T}{h}, \tag{1}$$

Further, we assume that the withdrawal amount is paid to the policyholder at the end of each policy anniversary $0 < t_1 < t_2 < t_3 \cdots < t_N \leq T$, $i = 1, 2, 3, \cdots, N$, and these withdrawal amounts depend upon the guarantee types and a performance of fund. Note that an insurance fee is subtracted at each policy anniversary and insurance fee is charged on policyholder's account value. Let us suppose that $S_{t_i} \geq 0$ denotes the price of the fund or equity index at a time $t_i \in [0, T]$ and the single premium P invested in the fund. We also assume that the risk free rate is constant and the dividend yield is zero. The policyholder's sub-account value is denoted by A_{t_i} and the initial value of the sub-account is $A_{t_0} = P = S_{t_0}$. We ignore any up-front commissions and the initial guaranteed withdrawal amount is expressed as

$$WD_0^g = c_p \cdot P, \tag{2}$$

where c_p is the guaranteed rate. In addition, we consider that at the maturity of the contract, the remaining value of the policyholder's account will be paid to the policyholder or policyholder's beneficiary. This remaining value at maturity is called beneficial amount. The insurer charges the insurance fee for GMWB guarantee.

2.1 Design of the Guaranteed Withdrawal

In this section, we introduce the two different designs for guaranteed withdrawal amount and their corresponding functions.

Plain Design

The plain design is the simplest form of guaranteed withdrawal and it is called as minimum guaranteed withdrawal. In the plain design, the guaranteed withdrawal amount is constant and the withdrawal amount does not depend on financial market fluctuations. The amount of the plain guarantee is determined by a guarantee rate c_p of the initial premium P regardless of the fund value.

$$WD^g_{t_i} = c_p \cdot P,\ i = 1, 2, ..., N. \tag{3}$$

Ratchet/Step-up Design

The ratchet design updates the guaranteed withdrawal amount on each policy anniversary. The ratchet is defined as the greater value of previous guaranteed withdrawal amount and the policyholder's account value $A^-_{t_i}$ multiplied by c_p on the policy date (note: $A^-_{t_i}$ is defined in next section). Moreover, the guaranteed withdrawal amount may increase but never decrease. The ratchet feature allows the policyholder to receive the higher guaranteed withdrawal amounts when the markets are performing very well. This type of guarantee design is attractive to policyholders especially in volatile markets. But the drawback of ratchet feature is that it is more expensive then plain design. In this case, we have

$$WD^g_0 = c_p \cdot P, \tag{4}$$

$$WD^g_{t_i} = \max\left(WD^g_{t_{i-1}}, c_p \cdot A^-_{t_i}\right), i = 1, 2, \cdots, N. \tag{5}$$

2.2 The Policyholder's Account

We presume that administration fees are ignored and the insurance fee is denoted as ϕ^{fee}. We also assume there are no surrenders and the withdrawal amounts do not exceed the defined guaranteed $WD^g_{t_i}$ amount. In reality, the policyholder allowed to withdraw a higher amount than the guaranteed $WD^g_{t_i}$ amount, but he has to pay a penalty fee for this higher amount and in the current article, we do not consider this scenario.

Further, we have to distinguish between the values of the policyholder's sub-accounts $(\cdot)^-_t$ immediately before and $(\cdot)^+_t$ after withdrawals. The state vector $Y_{t_i} = (A_{t_i}, WD^g_{t_i})$ at time t_i incorporates all information about the contract, and I_{t_i} denotes the returns in each period,

$$I_{t_i} := \frac{S_{t_i}}{S_{t_{i-1}}},\ t_i = i \cdot h,\ i = 1, 2, \cdots, N. \tag{6}$$

Moreover, the total number of policy anniversaries is determined by $N = \frac{T}{h}$. Also, the policyholder is limited to withdrawal $WD^g_{t_i}$ per annual and if the policyholder wants to withdrawal more then once a year, then guaranteed withdrawal $WD^g_{t_i}$ is deducted periodically with respect to the length of each policy anniversary h. Hence, at every anniversary the guaranteed withdrawal amount is equal to $WD^g_{t_i} \cdot h$ and length of policy anniversary can be monthly, quarterly or yearly. When the policyholder's account value A_{t_i} depletes to zero it will remain zero and a negative value is not allowed. In this setup, we have

$$A^-_{t_i} = A^+_{t_{i-1}} \cdot I_{t_i} \cdot \exp(-h \cdot \phi^{fee}),\ i = 1, 2, \cdots, N. \tag{7}$$

$$A^+_{t_i} = \max\left(A^-_{t_i} - WD^g_{t_i} \cdot h, 0\right) \tag{8}$$

We introduce the shadow account $SA^+_{t_i}$ which keeps track of fund performance until maturity regardless of whether real account value $A^+_{t_i}$ has reached zero. The main purpose of shadow account is to track the difference between the remaining guaranteed withdrawal amount $WD^g_{t_i}$ and the actual policyholder's account value $A^-_{t_i}$ which the insurer had to finance at the time of depletion.

$$SA^+_{t_i} = A^-_{t_{i-1}} - WD^g_{t_i} \cdot h,\ i = 1, 2, 3, \cdots, N. \tag{9}$$

Obviously, the relationship between the sub-account $A^+_{t_i} > 0$ and the shadow account is expressed as:

$$A^+_{t_i} = \max(SA^+_{t_i}, 0). \tag{10}$$

2.3 Numerical Example of Product Design

To illustrate the GMWB policy, we use example of a simple contract with semi-annual withdrawals. We assume that the GMWB policy begins with an initial premium of 100\$ with a semi-annual anniversary length and the guaranteed withdrawal of 7\$ per annum. Furthermore, we assume that the contract matures at time $T = 10$. Under the setup, the policyholder withdraws the guaranteed amount of 3.5\$ semi-annually $\left(h = \frac{1}{2}\right)$ and the total numbers of payments are 20. Table 1 shows the set of semi-annual investment returns, how the sub-account balance changes and the structure of the GMWB policy with a plain design. Also we have chosen a set of randomly generated scenarios and set insurance fee $\phi^{fee} = 0$.

Further, at the end of every six months, we record an investment return and then multiplied with the previous account balance and deduct guaranteed withdrawal. For example, the investment return is -8.39% after the first semi-annual anniversary. Hence the sub-account balance before the withdrawal drops to 91.61\$, and then 3.5\$ is withdrawn. During the third year, there is a positive investment return of 6.5% and the account value before withdrawal grows to 56.12\$, etc.

The performance of the financial markets strongly affects the GMWB policy and the policyholder's account. If the market performs well, the policyholder's account can end with a positive balance which will be paid to the policyholder at maturity. When returns are poor, then the policyholder's account can be depleted to zero.

However, in this case, the policyholder will get the guaranteed withdrawal amount from the insurer, and the insurer is responsible for the remaining amount. The guaranteed withdrawal amount can be decomposed into account withdrawals plus the GMWB guarantee that steps in when the account value $A_{t_i}^+$ exhausted (note: we will discuss the GMWB guarantee payoff in the next section).

The GMWB policy protects the policyholder's investment from the market fluctuations. For example, in Table 1, we assume a poor yield in some of the years therefore the account value $A_{t_i}^+$ becomes zero at the start of the year 8. At the eighth year, the account balance is 1.64\$ before withdrawing the guarantee amount which is not sufficient to finance the guaranteed withdrawal of 3.5\$. Hence the GMWB guarantee kicks in and continues to provide the remaining withdrawal amounts for the next two years.

We have introduced the shadow account to track the difference between the remaining guaranteed withdrawal amount and the actual account value. In this case, it is -1.85\$, which the insurer had to finance at the time of depletion of the policyholder's account $A_{t_i}^+$. Also, the remaining amount 14\$ will be paid over the course of two years. In our example, the GMWB guarantee payoff starts in the eighth year but the trigger time of the GMWB guarantee is generally random.

Time Period	Investment Returns	Account before Withdrawal	Plain Design Withdrawal	Account after Withdrawal	Shadow Account
0.5	-8.39%	91.61	3.5	88.11	88.11
1	-14.02%	75.73	3.5	72.23	72.23
1.5	-5.26%	68.43	3.5	64.93	64.93
2	-0.9%	64.34	3.5	60.84	60.84
2.5	-6.7%	56.74	3.5	53.24	53.24
3	6.5%	56.12	3.5	52.62	52.62
3.5	-15%	44.29	3.5	40.79	40.79
4	-20%	32.65	3.5	28.94	28.94
4.5	0.3%	29.03	3.5	25.53	25.53
5	-13%	22.20	3.5	18.70	18.70
5.5	2%	19.24	3.5	15.74	15.74
6	5%	16.59	3.5	13.09	13.09
6.5	-13%	11.35	3.5	7.85	7.85
7	15%	9.05	3.5	5.55	5.55
7.5	-3%	5.40	3.5	1.90	1.90
8	-13%	1.65	3.5	0	-1.85
8.5	n/a	0	3.5	0	-3.5
9	n/a	0	3.5	0	-3.5
9.5	n/a	0	3.5	0	-3.5
10	n/a	0	3.5	0	-3.5

Table 1 Example of the GMWB with plain design, random returns and the account value.

Some guaranteed withdrawals are design with the ratchet feature that allows the policyholder to reset the amount of the guaranteed withdrawal to the higher value when the market are performing strongly. Insurance companies offer different types of exercise times for ratchet options and a GMWB policy with the ratchet features usually costs extra 20 to 40 basis points as compared to plain design. In our second example, we assume that a ratchet feature is available semi-annually $\left(h = \frac{1}{2}\right)$, the initial premium is 100$ with a semi-annual anniversary length and the guaranteed withdrawal of 7$ per annum. Moreover, the contract matures after 10 years. The initially guaranteed withdrawal amount is 3.5$ semi-annually. The ratchet is calculated by using equation (5) on each anniversary. After the first semi-annual anniversary, for instance, the investment return is positive (4%). Then the ratchet steps up the minimum guaranteed withdrawal from 3.5$ to 3.63$ and 3.63$ will be withdrawn from

the account value.
Again, after the second semi-annual anniversary because of positive returns (39%), the ratchet steps up minimum guaranteed withdrawal from 3.63$ to 4.87$. Moreover once the value of the minimum guaranteed withdrawal amount has increased it will not decrease. The table 2 shows, the withdrawal amounts and the account value with the ratchet feature.

Time Period	Investment Returns	Account before Withdrawal	Ratchet Design Withdrawal	Account after Withdrawal	Shadow Account
0.5	4%	104.09	3.63	100.12	100.12
1	39%	139.67	4.87	134.35	134.35
1.5	−33%	90.50	4.87	85.34	85.34
2	−27%	62.36	4.87	57.28	57.28
2.5	12%	64.56	4.87	59.48	59.48
3	−5%	56.95	4.87	51.90	51.90
3.5	−4%	50.12	4.87	45.09	45.09
4	0.1%	45.59	4.87	40.56	40.56
4.5	62%	65.84	4.87	60.75	60.75
5	−33%	41.01	4.87	36.01	36.01
5.5	−13%	31.62	4.87	26.65	26.65
6	−26%	19.80	4.87	14.86	14.86
6.5	28%	19.13	4.87	14.20	14.20
7	27%	18.05	4.87	13.12	13.12
7.5	38%	18.19	4.87	13.25	13.25
8	23%	16.31	4.87	11.38	11.38
8.5	−18%	9.35	4.87	4.45	4.45
9	−7%	4.15	4.87	0	−0.72
9.5	n/a	0	4.87	0	−4.87
10	n/a	0	4.87	0	−4.87

Table 2 Example of the GMWB with ratchet design, random returns and the account value.

3 Valuation Perspectives of the GMWB

There are two perspectives for pricing GMWB contract. The policyholder is interested in the entire sum of the payments which he receives over the duration and at the maturity of the contract. On the other hand the insurer is concerned about the total amount of insurance fees and the withdrawal amount paid after the policyholder's account value hit zero. We assume that the financial market is frictionless with continuous trading with no transaction cost. Also, the financial market is arbitrage free which implies the existence of a risk-neutral measure $\mathbb{Q}$. Under this assumption, all the future guaranteed withdrawal amount can be evaluated as expected discounted values, see Holz et al. (2012). Let us denote $E^{\mathbb{Q}}[\cdot]$ the expectation which is evaluated under the risk-neutral probability measure $\mathbb{Q}$. Let us denote the filtration $\mathcal{F}_{t_i}$ which contains all the information about the financial market at time t_i.

3.1 Insurer Valuation Approach

From the insurer's perspective, the value of the GMWB can be calculated as the expected discounted insurance fee minus the expected discounted withdrawal benefits paid by the insurer after the policy's account hit zero. This states the net profit that the insurance company will earn after paying the withdrawal benefits to the policyholder when a policyholder's account value has depleted to zero. let $k \leq N$ be a random variable defined by

$$k^* = \inf \left\{ k \geq 0 \mid A^+_{t_k} = 0 \right\}, \tag{11}$$

is the time when the GMWB guarantee payoff is triggered and the policyholder's sub-account $A^+_{t_{k^*}}$ at time $t_{k^*} \in (0, T]$ depletes to zero.

The policyholder will withdraw the guaranteed amount $WD^g_{t_i} \cdot h$ at the end of each policy anniversary $t_i = t + i \cdot h$, $i = 1, 2, \cdots, N$, where $N = \frac{T-t}{h}$. Moreover, when the account value $A^+_{t_{k^*}}$ depletes to zero, the policyholder still receives the guaranteed amount $WD^g_{t_i} \cdot h$ until T. Let H denote the $t-$value of all expected total withdrawal benefits discounted with the constant risk free rate:

$$H_t(r, c_p) = E^{\mathbb{Q}}_t \left[\sum_{i=1}^{N} WD^g_{t_i} \cdot h \cdot e^{-\int_t^{t_i} \cdot r \cdot ds} \right], \tag{12}$$

as we assume that the risk free rate r is constant,

$$H_t(r, c_p) = E^{\mathbb{Q}}_t \left[\sum_{i=1}^{N} WD^g_{t_i} \cdot h \cdot e^{-r \cdot (t_i - t)} \right], \tag{13}$$

in particular the present value of expected total guaranteed withdrawal amounts is at time $t_0 = 0$ is expressed as:

$$H_0(r, c_p) = E^{\mathbb{Q}}_0 \left[\sum_{i=1}^{N} WD^g_{t_i} \cdot h \cdot e^{-r \cdot t_i} \right], \tag{14}$$

where $WD^g_{t_i}$ is defined according to different product designs which are mentioned in section (2.1). In the case of plain design the equation (14) is not a random variable. However, in the case of the ratchet design, equation (14) is a random variable whose depends upon the revolution of the policyholder's account value.
The remaining guaranteed withdrawals after the after the sub-account value $A^+_{t_{k^*}}$ depletes to zero at time t_{k^*} is given as:

$$H^*_{t_{k^*}}(r, c_p) = E^{\mathbb{Q}}_{t_{k^*}}\left[\sum_{i=1}^{N} WD^g_{t_{k^*}} \cdot h \cdot e^{-r\cdot(t_i - t_{k^*})}\right], \tag{15}$$

In particular,the guaranteed withdrawal amount in the ratchet design is defined as:

$$WD^g_{t_{k^*}} = \max\left(WD^g_{t_{k^*}-1}, c_p \cdot A^-_{t_i}\right).$$

According to Kolkiewicz and Liu (2012), the present value of the GMWB guarantee is seen as a put option at random time t_{k^*}, where the strike is the value of remaining guaranteed benefit $H^*_{t_{k^*}}(r, c_p)$ and the shadow account is our underlying fund and it is given as:

$$V^g_0(P, \phi^{fee}, c_p) = E^{\mathbb{Q}}_0\left[e^{-r\cdot t_{k^*}} \cdot \max(H^*_{t_{k^*}}(r, c_p) - SA^+_{t_{k^*}}, 0)\right]. \tag{16}$$

We assume that all insurance fees are deducted at the end of the policy anniversaries. The present value of all expected total insurance fees deducted from the account value is expressed as:

$$V^f_0(P, \phi^{fee}, c_p) = E^{\mathbb{Q}}_0\left[\sum_{i=1}^{k^*} e^{-r\cdot t_i} \cdot A^-_{t_i} \cdot (1 - e^{-h\cdot\phi^{fee}})\right]. \tag{17}$$

We denoted $V_0(P, \phi^{fee}, c_p)$ as the insurer's net profit/loss at time zero and the insurer's net present value of GMWB liabilities at time zero is defined as:

$$V_0(P, \phi^{fee}, c_p) = V^f_0(P, \phi^{fee}, c_p) - V^g_0(P, \phi^{fee}, c_p). \tag{18}$$

To price the GMWB contract $fair$ from an insurers's perspective, we use the equivalence principle under measure $\mathbb{Q}$ (see Aase and Persson (1994)) which states that the expected present values of insurance fees paid by the insured and the expected present value of the benefits paid by insurer should be equal. Thus, the *fair insurance fee* rate is the fee $\phi^{fee^*} \geq 0$ such that:

$$V_0(P, \phi^{fee^*}, c_p) = 0. \tag{19}$$

Equation (19) does not have a closed form solution and we will use a numerical methods to find ϕ^{fee^*}. In order to price the GMWB according to the equivalence principle under the risk-neutral measure $\mathbb{Q}$ and to obtain the fair insurance fee, we need to find the present value of total insurance fees at the fair rate ϕ^{fee^*}. Thus, fair rate ϕ^{fee^*} should be set exactly such that the total insurance fees cover the GMWB guarantee, see Aase and Persson (1994).

3.2 Policyholder Valuation Approach

We know from equation (8) that the policyholder's account cannot be negative and any remaining amount in the policyholder's account at maturity T will be returned to the policyholder and this amount is also know as return benefits. This cash inflow is regarded as a call option from "the policyholder perspective". The present value of this call option at time 0 can be expressed as:

$$V_0^C(P, \phi^{fee}, c_p) = E_0^{\mathbb{Q}} \left[e^{-r \cdot T} \cdot \max(A_T^+, 0) \right]. \tag{20}$$

Using the second approach, the GMWB contract should be priced fairly such that the initial premium must equal the future guaranteed payments plus the return benefits, see Milevsky and Salisbury (2006). We view equation (14) and (20) as total cash inflows for the policyholder and the fair insurance fee ϕ^{fee^*}must satisfy:

$$\begin{aligned} H_0(r, c_p) + V_0^C(P, \phi^{fee^*}, c_p) &= P \\ P - H_0(r, c_p) - V_0^C(P, \phi^{fee^*}, c_p) &= 0 \end{aligned} \tag{21}$$

4 Black-Scholes-Merton Model

We use the Black-Scholes-Merton model for pricing variable annuities with the GMWB. In the Black-Scholes-Merton model, the risky asset volatility is deterministic and constant over time, see Black and Scholes (1973).

Let $\{S_t\}_{t \geq 0}$ be the price process of the fund and $\{B_t\}_{t \geq 0}$ is the price process of the money market account defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We assume that the financial market is arbitrage free. This implies that there exists the martingale measure $\mathbb{Q}$. The price process S_t follows the geometric Brownian motion whose dynamics is given by the following stochastic differential equation:

$$dS_t = r \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dW_t, \;\; S_0 \geq 0, \tag{22}$$

where r is the constant interest rate, σ is the constant volatility and W_t is a $\mathbb{Q}-$Wiener process. In this case:

$$S_t = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\cdot t + \sigma\cdot W_t\right\},\ S_0 \geq 0. \tag{23}$$

The money market account B_t is defined as:

$$dB_t = r\cdot B_t\cdot dt,\ B_0 = 1. \tag{24}$$

5 Valuation by the Monte-Carlo Method

There is no close-form solutions for formulas defined in Section 3. Therefore, we used the Monte-Carlo simulation to carry for the valuation of the GMWB policy. Moreover, we fixed a discretization $D = \{0 \leq t_0, t_1, \cdots, t_l = T\}$, where $k = 1, 2, \cdots, l$, $\Delta_k = t_k - t_{k-1}$, and l is the total number of the time steps, see Kienitz and Wetterau (2012). We simulated the paths of the fund process by:

$$\widehat{S}^j_{t_k} = \widehat{S}^j_{t_{k-1}} \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\cdot \Delta_k + \sigma\cdot\sqrt{\Delta_k}\cdot Z^j_k\right\}, \tag{25}$$

$$\widehat{S}^j_0 = S_0,$$

where Z^j_k denotes the standard normal independently distributed random variables and $j = 1, 2, 3, \cdots, M$ denotes the number of simulations. For any financial market developments and the policyholder's account, the evolution of all the accounts and the process will follow the rules given in Section 2. Moreover, we gave a example of the policyholder's account with plain design and with the ratchet design in Appendix A.1 & A.2, respectively. The approximation of guaranteed amount $H_0(r, c_p)$ and the insurance fees $V^f_0(P, \phi^{fee^*}, c_p)$ are defined as:

$$F_1(\widehat{S}) \approx \sum_{i=1}^{N} WD^g_{t_i}\cdot e^{-r\cdot t_i},$$

$$\widehat{H}_0(r, c_p) \approx \frac{1}{M}\sum_{j=1}^{M} F_1(\widehat{S}^j). \tag{26}$$

$$F_2(\widehat{S}) \approx \sum_{i=1}^{k^*} e^{-r\cdot t_i}\cdot A^-_{t_i}\cdot (1 - e^{-h\cdot\phi^{fee}}),$$

$$\widehat{V}_0^f(P, \phi^{fee}, c_p) \approx \frac{1}{M}\sum_{j=1}^{M} F_2(\widehat{S}^j). \tag{27}$$

Further, the approximation of the GMWB guarantee value $V_0^g(P, \phi^{fee}, c_p)$ and the call option $V_0^c(P, \phi^{fee}, c_p)$ are expressed as:

$$F_3(\widehat{S}) \approx e^{-r \cdot t_{k^*}} \cdot \max(H^*_{t_{k^*}}(r, c_p) - SA^+_{t_{k^*}}, 0),$$

$$\widehat{V}_0^g(P, \phi^{fee}, c_p) \approx \frac{1}{M}\sum_{j=1}^{M} F_3(\widehat{S}^j). \tag{28}$$

$$F_4(\widehat{S}) \approx e^{-r.T}. \max(A^+_T, 0),$$

$$\widehat{V}_0^c(P, \phi^{fee}, c_p) \approx \frac{1}{M}\sum_{j=1}^{M} F_4(\widehat{S}^j), \tag{29}$$

where $\widehat{S}^j = \left(\widehat{S}^j_{t_k}\right)_{1 \le k \le N}$ is a one path of the underlying fund process.

6 Results

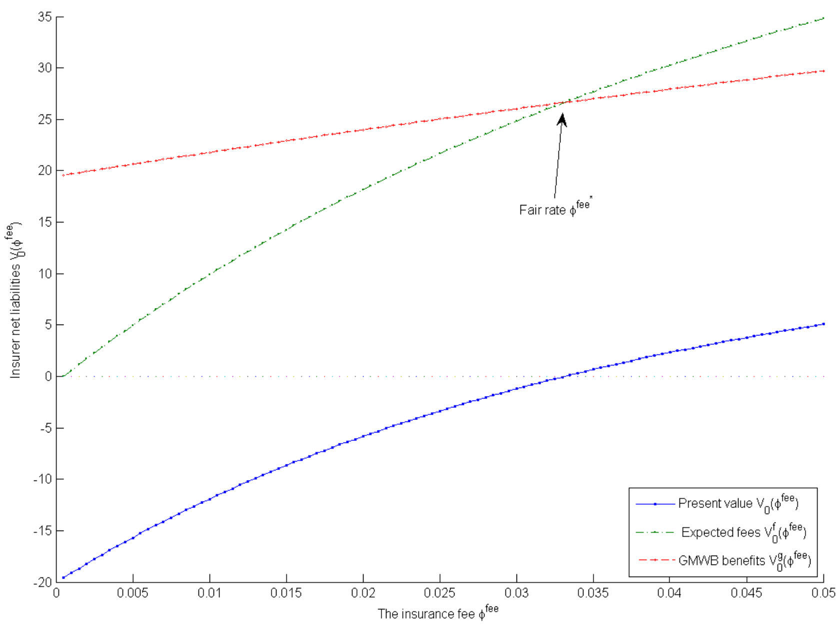

Figure 6: The expected GMWB liability with quarterly ratchet feature and GMWB guarantee vs the expected future insurance fee, $S_0 = 100$€, $r = 5\%$, $\sigma = 20\%$, $T = 30$.

We assume a constant risk-free rate $r = 5\%$, volatility $\sigma = 20\%$, a single premium $P = S_0 = 100\$$, and $T = 30$. Further, there is no accumulation period, the withdrawal of the guaranteed amount begins from the first year, and ratchet is provided every fourth month. We use the "insurer's perspective" approach in which the insurer's net present value of the GMWB liabilities must be zero and satisfying

$$V_0(P, \phi^{fee^*}, c_p) = V_0^f(P, \phi^{fee^*}, c_p) - V_0^g(P, \phi^{fee^*}, c_p) \approx 0.$$

To illustrate, how fair insurance fees can be calculated in our framework, we analyze their influence on the net value of the insurer's liability. The contract contains a GMWB with quarterly ratchet feature and the initial minimum annual guaranteed withdrawal of $WD_0^g = c_p \cdot P = 0.05 \cdot 100 = 5\$$.

Figure 6 shows the insurer's net liability as a function of the annual insurance fee. This contract is more valuable due to the potential of rise in the minimum withdrawal guaranteed and a higher fair insurance fee needed to make this contract fair. When a higher insurance fee ϕ^{fee^*} is deducted from the policyholder's account, it also entails the risk that the policyholder's account gets exhausted more rapidly. As a resulting it increases the GMWB guarantee value. A fair insurance fee of 338 bps is needed to make the present value of the insurer's liability equal to zero and

$$V_0^f(P, \phi^{fee^*}, c_p) \approx V_0^g(P, \phi^{fee^*}, c_p).$$

To compute the fair insurance fee, we use the Secant method to find the root of

$$V_0(P, \phi^{fee^*}, c_p) \approx 0.$$

Moreover, the same random numbers used for $V_0^f(P, \phi^{fee^*}, c_p)$ and $V_0^g(P, \phi^{fee^*}, c_p)$. The Secant method requires two initial guesses $x_0 = \phi_0^{fee}$ and $x_1 = \phi_1^{fee}$ with different signs of the functions $y_0 = V_0(P, x_0, c_p)$ and $y_1 = V_0(P, x_1, c_p)$, and the method proceeds as follows:

$$x_{i+1} = x_i - y_i \cdot \frac{x_i - x_{i-1}}{y_i - y_{i-1}}.$$

We choose a tolerance of 0.0001 as the stopping criteria.

6.1 Fair Insurance Fee for Different Contract Designs

In this section, we analyze GMWB contracts with different withdrawal designs. We start with GMWB contracts with plain withdrawal and then move on to the ratchet/step-up withdrawals. We simulate 100000 fund paths and consider a risk free rate $r = 5\%$, volatility $\sigma = 20\%$, a single premium $P = S_0 = 100$, and a maturity of 20 years.

6.1.1 GMWB with Plain Design

c_p	Contract Type		
	Annual withdrawal	Semi-annual withdrawal	Quarterly withdrawal
4%	9	9.3	9.3
4.5%	17	17	17
5%	27	28	28.3

Table 3 Fair insurance fee (bps) for GMWB contracts with plain design and $T = 20$ (Standard deviation error $0.05 - 0.07$).

We analyze and compare the three distinct types of withdrawal plans with various annual guaranteed rates. Table 3 shows that fair insurance fee rates for three different contract types ($h = 1, \frac{1}{2}, \frac{1}{3}$). Table 4 shows the GMWB guarantees values and total withdrawal amount.

The results show no major rise in the fair insurance fees between the annual withdrawal and quarterly withdrawal schemes. The analysis also shows that when we increase guaranteed rates c_p, it increase fair insurance fees. The numerical results for plain design GMWB are consistent with the work of Kolkiewicz and Liu (2012).

c_p	GMWB guarantee V_0^g	Withdrawal benefits H_0
4%	1.30	49.31
4.5%	2.20	55.48
5%	3.55	61.64

Table 4 GMWB guarantees and total withdrawal benefits with plain design.

In table 4, we use the fair insurance rates ϕ^{fee^*} for corresponding guarantee rates c_p to calculate the value of GMWB guarantees and total withdrawal benefits H_0 for GMWB contracts with plain design ($h = 1$). We find that an additional 10 bps fair insurance fee needed to finance the increasing guaranteed rate c_p and withdrawal benefit increases with higher annual guaranteed rates.

6.1.2 Fair Insurance Fee for GMWB with Ratchet Design

c_p	Contract Type		
	Annual step-up	Semi-annual step-up	Quarterly step-up
4%	18	20	21.2
4.5%	35	38	41
5%	64	69	72

Table 5 Fair insurance fee (bps) for GMWB contracts with ratchet design and $T = 20$ (Standard deviation error 0.05-0.07).

For analysis of the GMWB contract with ratchet features, we compare the three GMWB contracts with different guaranteed rates and ratchet frequencies. Contract 1 has an annual step-up, contract 2 has a semi-annual step-up, and contract 3 has a quarterly step-up. Table 5 shows fair insurance rates for different guaranteed rates c_p.

c_p	GMWB guarantee V_0^g	Withdrawal benefits H_0
4%	2.23	72.59
4.5%	3.96	78.41
5%	6.59	84.25

Table 6 GMWB guarantee and withdrawal benefits for GMWB contracts with annual ratchet design.

The ratchet feature provides an additional benefit to the policyholder when the financial markets are performing well. Furthermore, once the minimum guaranteed withdrawal amount is stepped-up, it cannot be reduced. Our analysis shows that the fair insurance fee rises as the ratchets get more frequent. The is caused by the fact that the minimum guaranteed withdrawal amount is likely to increase with increased ratchet frequency. The additional insurance fee of $3-5$ bps is needed to finance the semi-annual and quarterly ratchet.

6.2 Sensitivity Analysis

In current sub-section, we analyze the impact of parameter risk on GMWB contracts. We study the effect on the GMWB guarantees price and fair insurance fees with respect to different parameters, maturity time T, volatility of an underlying fund, and interest rate level.

6.2.1 Sensitivity Analysis with Respect to Maturity Time

We analyze the influence of different maturity times on GMWB guarantee and fair insurance fees. Moreover, we consider a single premium $P = 100$, risk free rate $r = 5\%$ and volatility is set at σ=20%. As well as, the maturity of the contract is 25 and 30 years, respectively. Tables 7 & 8 show an example of the GMWB contracts with plain designs. Note that the fair rates are increased by $10-20$ bps with respect to longer maturity and higher guaranteed rates c_p. The values of the guarantee rates c_p have a significant impact on the fair insurance fee. When c_p increases from 4.5% to 5% for maturity of 30 years then fair insurance fees increase from 45 to 74 bps. It is therefore, reasonable for the insurance company to focus on providing lesser guaranteed rate rather than charging higher fair insurance fees. Further, tables 9 & 10 show fair insurance fees are high in the case of the ratchet feature and especially when we consider semi-annual step-up and quarterly step-up. In this case, the insurer should again focus on reduced guaranteed rates c_p for longer maturity and only provide an annual step-up.

c_p	Maturity T	Contract Type: Annual withdrawal	Semi-annual withdrawal	Quarterly withdrawal
	25			
4%		19	18	17
4.5%		32	32	32
5%		52	53	54
	30			
4%		28	27	28
4.5%		45	46	47
5%		74	76	75

Table 7 Fair insurance fee (bps) for GMWB contracts with maturity 25 and 30 years with plain design (Standard deviation error $0.05 - 0.07$).

	GMWB guarantee V_0^g			Withdrawal benefits H_0		
c_p $T = 25$	Annual withdrawal	Semi-annual withdrawal	Quarterly withdrawal	Annual withdrawal	Semi-annual withdrawal	Quarterly withdrawal
4%	2.94	2.85	2.83	55.66	56.36	56.72
4.5%	4.73	4.67	4.67	62.62	63.41	63.81
5%	7.02	7.19	7.24	69.58	70.46	70.90
$T = 30$						
4%	4.86	4.72	4.89	60.60	61.37	61.76
4.5%	7.24	7.33	7.55	68.18	69.04	69.48
5%	10.58	10.91	10.76	75.76	76.71	77.20

Table 8 GMWB guarantee and withdrawals amount for GMWB contracts with plain design.

c_p	Maturity T	Contract Type: Annual step-up	Semi-annual step-up	Quarterly step-up
	25			
4%		44	44	51
4.5%		82	88	94
5%		145	155	168
	30			
4%		70	84	91
4.5%		142	157	170
5%		258	291	322

Table 9 Fair insurance fee for GMWB contracts for maturity 25 and 30 years with ratchet design (Standard deviation error 0.05 − 0.07).

c_p	GMWB guarantee V_0^g			Withdrawal benefits H_0		
$T = 25$	*Annual step-up*	*Semi-annual step-up*	*Quarterly step-up*	*Annual step-up*	*Semi-annual step-up*	*Quarterly step-up*
4%	5.76	6.12	6.47	83.07	83.99	84.79
4.5%	9.53	9.91	10.42	89.34	89.84	90.60
5%	14.38	15.09	15.90	93.86	94.71	95.33
$T = 30$						
4%	10.20	10.74	11.73	90.11	91.10	91.85
4.5%	15.96	16.91	17.78	95.40	96.05	96.62
5%	23.66	25.12	26.46	98.57	98.99	99.25

Table 10 GMWB guarantees and withdrawal amount for GMWB contracts with ratchet design.

6.2.2 Sensitivity Analysis with Respect to the Risk Free Rate

We analyze the influence of the rate of interest on the GMWB guarantee. We set the parameters equal to $\sigma = 20\%$, $T = 25$, and $\phi^{fee} = 0$. We have considered plain design GMWB contracts and annual ratchet design GMWB. Moreover, we have varied the values of the interest rate r from zero to 10%. Figure 6.2.2 shows the GMWB guarantees for different values of the interest rate r. As expected, the increasing interest rate reduces the value of the GMWB guarantees and as a result, a lower fair insurance fee is required to finance the GMWB contract with increasing interest rates.

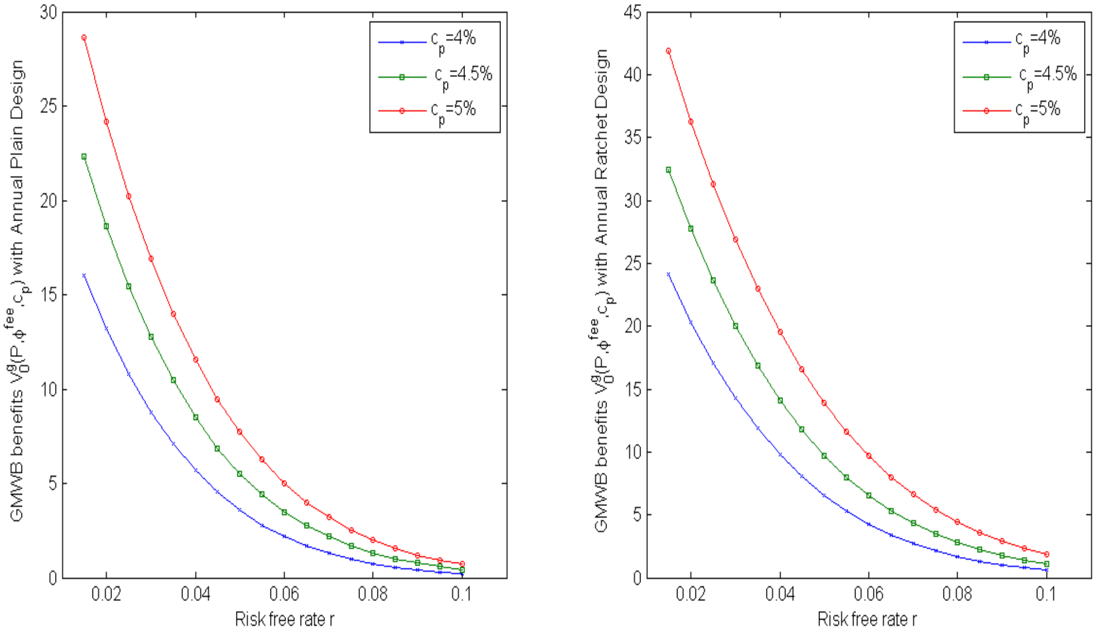

Figure 6.2.2: Sensitivity of $V_0^g(P,\phi^{fee},c_p)$ with respect to interest rate r and $\phi^{fee} = 0$.

6.2.3 Sensitivity Analysis with Respect to the Volatility

In this segment, we analyze the sensitivity of the GMWB guarantees and the fair insurance fees with respect to the fund volatility. Figure 6.2.3 shows the GMWB guarantees as a function of the varying volatility. It clearly shows that increasing volatility level has a significant impact on the value of the GMWB guarantees.

Figure 6.2.3 shows the fair insurance fees for different volatility levels. The fair insurance rates and the GMWB guarantee value appear to be very sensitive with respect to fund volatility. Accordingly, it is consistent with a Financial theory that suggests options are more expensive when volatility is high. There is a positive relationship between the fair insurance rates and the volatility. Thus, the insurer can influence the volatility by choosing funds with low volatility. An important risk management tool for insurers who offer GMWB is to restrict and control the fund volatility by offering a different mix of stock, bonds and etc.

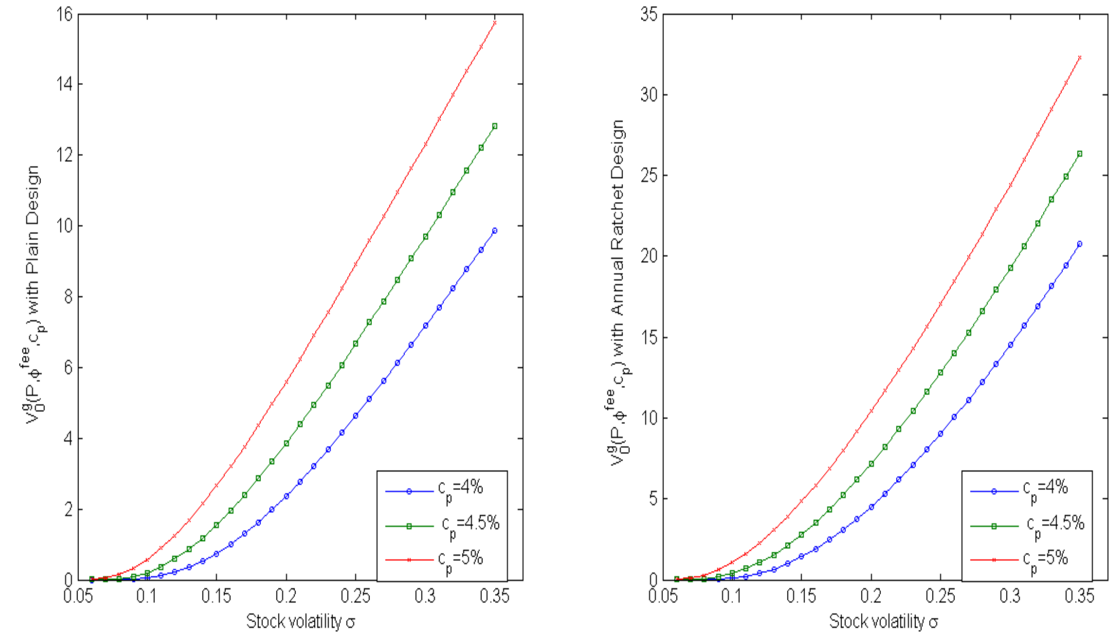

Figure 6.2.3: Sensitivity of the GMWB guarantees $V_0^g(P,\phi^{fee},c_p)$ with respect to volatility σ. Parameters $P = 100$, $r = 5\%$, $T = 25$, and $\phi^{fee} = 0$.

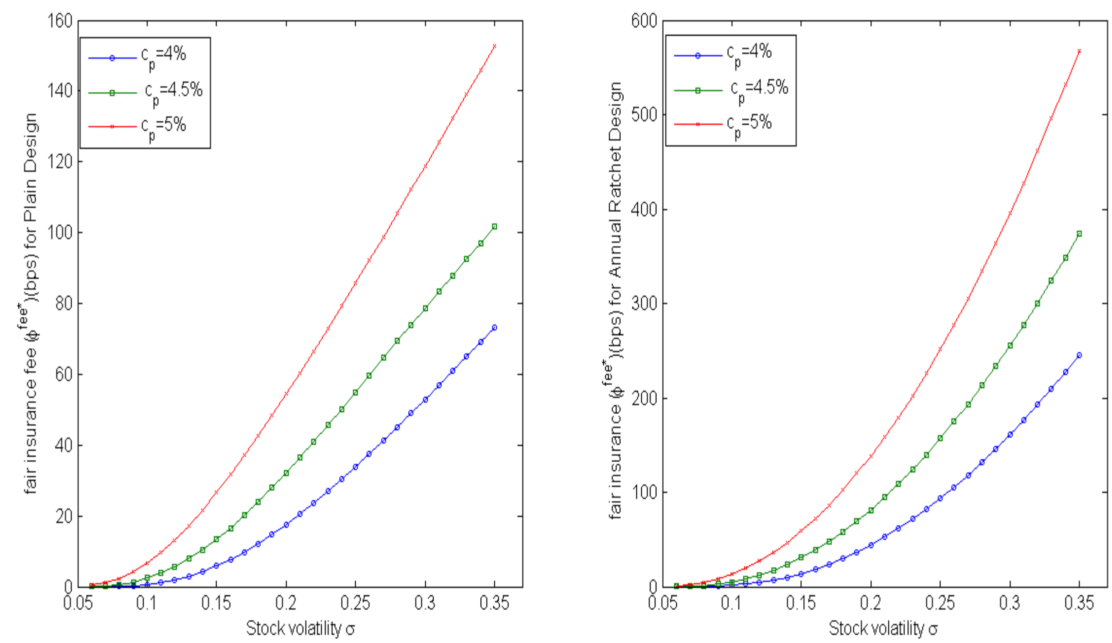

Figure 6.2.3: Sensitivity of fair insurance fee ϕ^{fee^*} with respect to volatility. Parameters $P = 100$, $r = 5\%$, and $T = 25$.

7 Conclusion

In this paper, we have described and analyzed Guaranteed Minimum Withdrawal Benefits (GMWB). We have discussed the difficulties involved in GMWB valuation and presented a generalized pricing framework for GMWB. The paper provides a detailed description of different product designs, numerical results, and sensitivity analyses. In addition, we have shown how the value of GMWB guarantee changes with respect to different parameters e.g. risk-free rate, fund's volatility, and guarantee rate.

We have considered the plain design of the GMWB as well as more sophisticated withdrawal designs such as the ratchet/step-up feature. The results indicate that product designs with a ratchet feature increases the fair insurance rates and we find that a fair insurance fee should be a higher for semi-annual and quarterly step-ups. Furthermore, fair insurance fees are highly sensitive to the volatility of an underlying fund. The value of the GMWB guarantee increases exponentially with increasing volatility of the investment account, and it is also highly sensitive to low interest rates.

In the light of the analysis conducted in this paper, future research could aim at extending the capital market model with stochastic interest rates, stochastic volatility, or Levy-type framework. Further, we have disregarded mortality in this paper, but it would be worthwhile to extend the modelling framework to include mortality and implementation of hedging strategies. Finally, it would be interesting to analyze how the insurer can reduce the financial risk by offering the dynamic funds, which are controlled by volatility targets.

8 Appendix

8.1 A.1 Example of Paths of Policyholder's Account under Plain Design

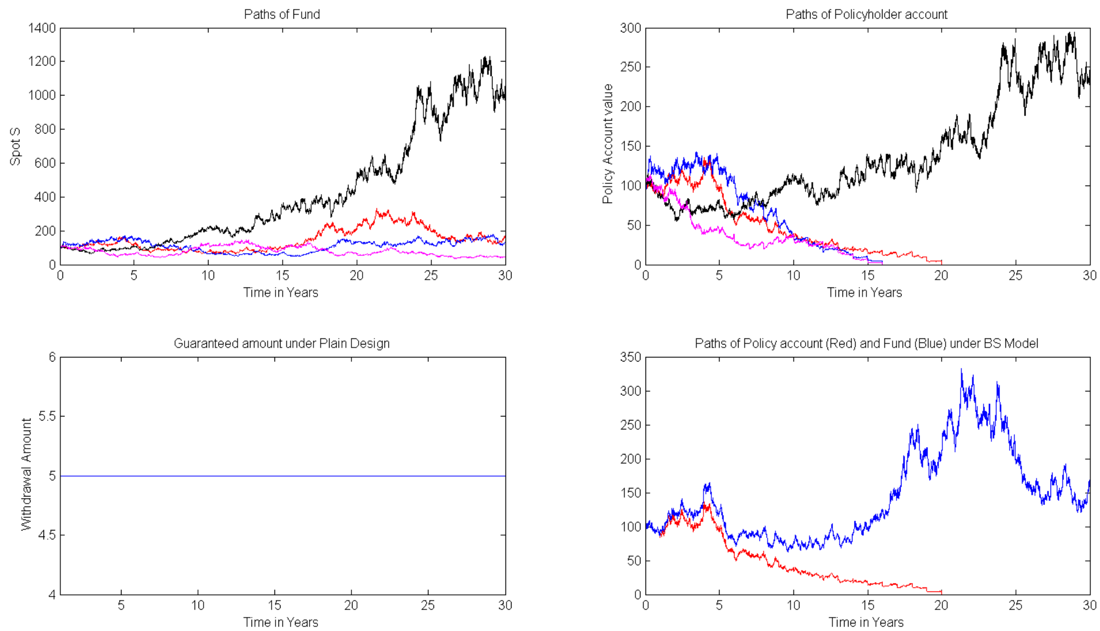

Figure8.1: Paths of policyholder's account value with Plain design withdrawal. $S_0 = 100€$, $A_0 = 100€$, $r = 5\%$, $\sigma = 20\%$, $c_p = 5\%$, $T = 30$.

8.2 A.2 Example of Paths of Policyholder's Account under Ratchet Design

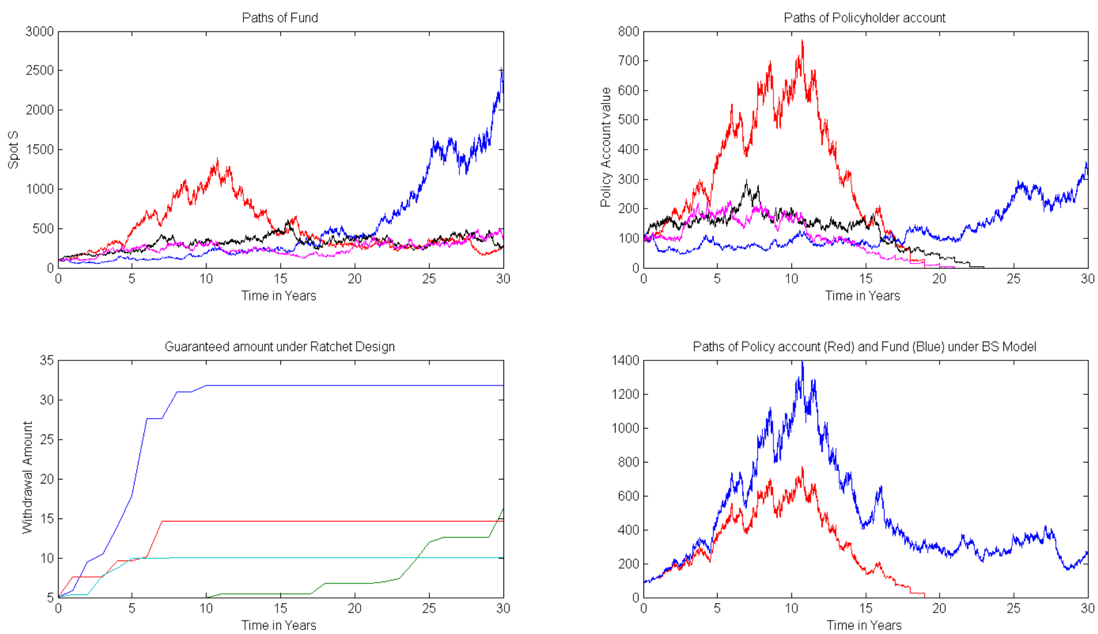

Figure 8.2: Paths of policyholder's account value with Annual Ratchet design.

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Working Paper Two

Pricing and Hedging of the European Option Linked to Target Volatility Portfolio

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Abstract

Advanced risk management strategies have become popular since they protect the investor's investment from market crashes. The latest risk management strategy is called the target volatility strategy. The target volatility strategy is used in order to maintain a stable realized volatility of a portfolio. The strategy re-balances the allocation of risky asset to non-risky asset in order to protect the portfolio from equity market crashes. This paper provides an extensive analysis of the target volatility portfolios and financial derivatives linked to target volatility portfolios. We demonstrate the performance of target volatility portfolios under different financial models. We also examine the effects of re-balancing frequencies on target volatility portfolios. We focus on the pricing and hedging of the European option linked to target volatility portfolios. In particular, we investigate the impact of stochastic equity asset volatility on the pricing and hedging of the European option linked to target volatility portfolios. We also consider different dynamic hedging strategies and compare their performance.

Keywords: Target volatility portfolio, stochastic volatility, re-balance frequency, EWMA method

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1 Introduction

It is well known that insurance companies suffered unexpectedly large losses during the global financial crises in 2007-2008 due to a rise in the value of the equity-linked pension products. The implied and realized equity volatility rose during the global financial crises. Also, high volatility increased the value of financial derivatives linked to equity indices. Consequently, the value of most equity-linked pension products increased. Insurance companies who chose poor risk management strategies suffered huge losses due to the increase in the value of equity-linked products. Several insurance companies have been seeking new risk management strategies to reduce their cost and offer attractive products to investors. Also, investors became more aware of the risk and realized the importance of risk management strategies. One of the popular risk management strategy is called the target volatility strategy. There are various new market indices based on the concept of target volatility strategy, e.g. EuroStoxx 50 Risk Control Index, S&P 500 Risk Control Index, Dow Jones Volatility Control Index, etc.

The main purpose of using the target volatility strategy is to keep the realized volatility of the portfolio under control and preferably under a certain level. The idea behind the strategy is to shift the allocation of the risky asset to non-risky asset during the high volatility of risky asset and vice versa. The dynamic allocation is determined by the realized volatility of the risky asset and a pre-defined volatility target level. When the realized volatility of the risky asset is high, the strategy invests partially in the non-risky asset to protect investors. While during the low realized volatility, the strategy invests completely in to risky asset. Moreover, the amount invested in the non-risky asset also depends on a volatility target level. Hence, the overall realize volatility of the portfolio is kept under control. As a result, the portfolio is less affected by falling markets than a pure risky investment. Recently, several investment banks and pension companies are offering financial derivatives linked to target volatility portfolios (TVP). The target volatility strategy significantly decreases the economic value of financial derivatives by reducing the volatility of the underlying portfolio. It also protects the investor's investment from falling markets. Despite their popularity, there is a very little literature on target volatility strategies. There is a need of the proper method for pricing and hedging of financial derivatives linked to target volatility portfolios.

The basic idea of re-balancing portfolio was introduced by Markowitz (1952) in their work on single period Mean-Variance Optimization (MVO). The underlying idea of MVO was to find an optimal way to allocate investment between different assets by maximizing total expected returns subject to a selected level of risk. More sophisticated models of dynamic re-balancing has been introduced in several articles. One of this kind was introduced by Perold and Sharpe (1988) where they presented a dynamic strategy (CPPI) of asset allocation in a portfolio. Black and Perold (1992) suggested a simple rule of portfolio balancing by investing an amount equal to the difference between the total wealth and a specified floor in a risky asset. Ever since a lot of work can be found on rule-based dynamic asset strategies in the literature (i.e. see Bertrand and Prigent (2001), Herold et al. (2008), Ho et al. (2010) and the references therein).

The basic concept of the target volatility strategy is presented in various documents and articles (see, Giese (2011), Albeverio et al. (2013), Standard and Poor's (2009)). Giese (2011) derives the model for the target volatility strategies and shows that using re-balancing function based on the realized volatility of the risky asset improves the performance of the portfolio. Albeverio et al. (2013) show that the target volatility strategy improves the risk-return characteristics of the portfolio. They derived the risk-neutral pricing framework for financial derivatives linked to TVP. They also demonstrated that financial derivatives linked to target volatility portfolio are less expensive as compared to financial derivative linked to the risky asset. They valuated financial derivatives under the Black-Scholes-Merton model.

The aim of this paper is to investigate the impact of different parameter risks and model assumptions on target volatility portfolios and on the value of financial derivatives linked to TVP. Generally, the unit-linked products have complex payoff structure. However, for illustration, we have considered the simple European option linked to target volatility portfolios. To best of our knowledge, there exists no literature on the impact of the stochastic volatility and re-balancing frequencies on the target volatility portfolios and financial derivatives linked to TVP. In this paper, we analyze the different distributional properties of TVP and various risks associated with financial derivatives linked to TVP. We demonstrate the performance of target volatility portfolios under different financial models. In particular, we assess the impact of stochastic volatility on the pricing and hedging of the European option linked to target volatility portfolio.

The structure of this paper is as follows. In the 2nd Section we present the general framework for the target volatility portfolios and the European option linked to TVP. Section 3 and 4 are devoted to the description of financial models which are used for pricing and hedging of the European option linked to TVP. We also describe the various hedging strategies for the European option linked to TVP. For comparison, we use the Black-Scholes-Merton model as a reference and the Heston model that assumes the volatility of equity asset is not constant. Section 5 presents the numerical approximation of the target volatility portfolios and the European option linked to TVP (TC).

The numerical results are provided in the Section 6, starting with the distributional properties of the target volatility portfolio under the Black-Scholes-Merton model and also under

the Heston model. We also examine the impact of re-balancing frequencies on the distributional properties of the target volatility portfolio. We precede our analysis on the impact of stochastic volatility on the value of European call option linked to TVP in Section 6.2.2. We also check the impact of different re-balancing frequencies on the value of the European option linked to TVP. We analysis the so-called Greeks and implied volatilities of the European call option linked to TVP in the Section 6.3 and Section 6.4, respectively. Finally, we test the impact of lambda (EWMA method) on the value of TC in the Section 6.5.
In Section 7, we first present the discretization of the hedge portfolio. And we analyze and compare the hedging performance of the different hedging strategies under both mentioned financial models.

2 The Financial Model

Let us consider probability space $(\Omega, \mathbb{F}, \mathbb{P})$ carrying $d-$dimensional Wiener process. We suppose that there is a market that consist of $(G+1)$ assets (risky assets and one risk-free asset). The assets $S_0(t), S_1(t), \cdots, S_G(t)$ is assumed measurable with respect to filtration $\mathcal{F}_t$. All assets are traded continuously. Let one of the asset is the risk-free bond $S_0(t)$. The dynamics of the risk-free bond $S_0(t)$ is defined as:

$$\frac{dS_0(t)}{S_0(t)} = r(t) \cdot dt, \quad S_0(0) = 1, \tag{1}$$

where $r(t) > 0$ and $t \in [0, T]$. Let $S_c(t)$ represents the cth risky asset price at time $t \in [0, T]$. The price process $S_c(t)$ evolves according to the following stochastic differential equation:

$$\frac{dS_c(t)}{S_c(t)} = \zeta_c \cdot dt + \sum_{k=1}^{d} \sigma_{c,k}(t) \cdot dW_k(t), \quad S_c(t) > 0, \tag{2}$$

where ζ_c, $c \in \{1, 2, \cdots, G\}$, is a constant drift of the cth risky asset. Moreover $W_k(t)$, $k \in \{1, 2, \cdots, d\}$, denotes the $\mathbb{P}-$Wiener process with the covariance process $d\langle W_i, W_j \rangle = \rho_{i.j} \cdot dt$, and $\rho_{i.j} \in [-1, 1]$ is a correlation between W_i and W_j.
Assume that we can invest in one or more of the $G+1$ assets as defined above. A trading strategy is defined as a stochastic process $\alpha = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \cdots, \alpha_G(t))$ for $t = 0, 1, 2, \cdots T$. Moreover, $\alpha_c(t)$ represents the fraction of wealth that is invested in risky asset $S_c(t)$ at time t. And $\alpha_0(t) = 1 - \sum_{c=1}^{G} \alpha_j(t)$ denotes the fraction of wealth invest in the risk-free asset at time t. Furthermore, the sequence α is assumed to be predictable. The value process of a trading strategy $\alpha = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \cdots, \alpha_G(t))$ is the stochastic process Z defined as:

$$Z(t) = \alpha_0(t) \cdot S_0(t) + \sum_{c=1}^{G} \alpha_c(t) \cdot S_c(t),$$

and the dynamics of a self-financing relative portfolio Z without consumption is expressed as:

$$\frac{dZ(t)}{Z(t)} = \alpha_0(t) \cdot \frac{dS_0(t)}{S_0(t)} + \sum_{c=1}^{G} \alpha_c(t) \cdot \frac{dS_c(t)}{S_c(t)}, \tag{3}$$

by using equation (1) and (2):

$$\frac{dZ(t)}{Z(t)} = \left(\sum_{c=1}^{G} \alpha_c(t) \left(\zeta_c - r(t)\right) + r(t) \right) \cdot dt + \sum_{c=1}^{G} \sum_{k=1}^{d} \alpha_c(t) \cdot \sigma_{c,k}(t) \cdot dW_k(t). \tag{4}$$

In the present article, we consider two assets, an equity index $S(t)$ with constant drift ζ and the risk-free asset $S_0(t)$ with constant interest rate r. We call the portfolio as target volatility portfolio (TVP) and its dynamics is defined as:

$$\frac{dZ(t)}{Z(t)} = \alpha_0(t) \cdot r \cdot dt + \alpha_1(t) \cdot \left(\zeta \cdot dt + \sigma(t) \cdot dW^{\mathbb{P}}(t)\right), \tag{5}$$

where $W(t)$ is the $\mathbb{P}-$Wiener process.

2.1 The Stochastic Weights

We define a stochastic process U which will depend upon a past price trajectory of $\{S(s); t \geq s\}$. Moreover, $U(u_k)$ is estimated annualized historical volatility of the risky asset $S(t)$ at the re-balancing time $u_k \in \mathbb{Z}^+$. Consider that $\alpha_1(t)$ and $\alpha_0(t)$ are stochastic processes and their values depend upon the realized historical volatility of a risky asset S. We have placed the restriction on the sign of $\alpha_0(t)$ and we will not borrowed the amount $\alpha_0(t)$ in the risk-free asset ($\alpha_0(t) \geq 0$). The stochastic weights are defined as a step function:

$$\begin{aligned}
\alpha_1(t) &= \begin{cases} \min\left(\frac{VT}{U(u_k)}, 1\right) & \text{if } u_k \leq t < u_{k+1}; \\ \min\left(\frac{VT}{U(u_M)}, 1\right) & \text{otherwise.} \end{cases} \\
&= \alpha_1(0)\mathbf{1}_{\{0 \leq t < u_1\}} + \sum_{k=1}^{M-1} \min\left(\frac{VT}{U(u_k)}, 1\right) \mathbf{1}_{\{u_k \leq t < u_{k+1}\}} \\
&\quad + \min\left(\frac{VT}{U(u_M)}, 1\right) \mathbf{1}_{\{t \leq u_M < T\}},
\end{aligned} \tag{6}$$

$$\alpha_0(t) = 1 - \alpha_1(t),$$

where VT is a volatility target and its level is set according to the historical realized volatility of the risky asset. The stochastic weights will be kept constant until the next re-balancing time u_k. At time $u_k \leq t < u_{k+1}$, the weights will change its value according to an estimated historical volatility of the risky asset. If at the re-balance time u_k, the estimated historical volatility of the risky asset is higher than the chosen volatility target VT, $U(u_k) > VT$, then $\alpha_1(t)$ takes a value between 0 and 1. Otherwise, $U(u_k) < VT$, $\alpha_1(t)$ is equal to 1.

2.2 Target Volatility Derivative

Consider the filtration $\mathcal{F}_t$ that contains all information about the market up to time t. Albeverio et al. (2013) showed that the discounted target volatility portfolio $(\frac{Z(t)}{S_0(t)}, t \in [0, T])$ is a martingale with respect to the risk-neutral measure $\mathbb{Q}$. The arbitrage-free dynamics of target volatility portfolio is expressed as:

$$\frac{dZ(t)}{Z(t)} = \alpha_1(t) \cdot \left(r \cdot dt + \sigma(t) \cdot dW^{\mathbb{Q}}(t)\right) + \alpha_0(t) \cdot r \cdot dt,$$

$$\frac{dZ(t)}{Z(t)} = r \cdot dt + \alpha_1(t) \cdot \sigma(t) \cdot dW^{\mathbb{Q}}(t), \tag{7}$$

where $\alpha_1(t) + \alpha_0(t) = 1$.

Let us define a price process of a target volatility derivative F that matures at time T. Moreover, an arbitrage-free price of the claim $\Phi(Z(T))$ is described as:

$$F(t) = e^{-r(T-t)} \cdot E_t^{\mathbb{Q}}[\Phi(Z(T))],$$

where the $\mathbb{Q}-$dynamics of $Z(t)$ is given by equation (7).

We consider the European call as the payoff function of $\Phi(Z(T))$. Consequently, the European call option linked to target volatility portfolio (TC) is defined as:

$$V(t) = e^{-r(T-t)} \cdot E_t^{\mathbb{Q}}[(Z(T) - K)^+], \tag{8}$$

Also, K is the strike price and $Z(T)$ is the price of the target volatility portfolio at maturity time T.

2.3 The Annualized Realized Historical Volatility

The value of the weights depends on an annualized historical volatility of the risky asset S and we assume that there are 252 business days. We use the Exponentially Weighted Moving Average (EWMA) method to estimate an annualized daily volatility of the risky asset S.

The main reason for choosing the EWMA method is that the S&P 500 risk control index (see Standard and Poor's (2009)) uses the EWMA method to estimate realized volatilities and it is computationally less expensive. Better models such as GARCH, NGARCH, etc. exist for the estimation of realized volatilities. An important question is the impact of advanced realized volatility estimating models on stochastic weights and on the value of financial derivatives linked to TVP requires more detailed analysis. We leave this subject for future research. Let $R(t_j)$ be defined as the compounded return during day t_j:

$$R(t_j) = \ln\left(\frac{S(t_j)}{S(t_{j-1})}\right),$$

where $\Delta t = t_j - t_{j-1}$, $t_0 = 0 < t_1 < t_2 \cdots < t_N = T$, $j = 1, 2 \cdots, N$,

$$\widetilde{\sigma}^2_{t_j} = \lambda \cdot \widetilde{\sigma}^2_{t_{j-1}} + (1-\lambda) \cdot \frac{1}{\Delta t} \cdot R^2(t_{j-1}), \tag{9}$$

$$\widetilde{\sigma}^2_{t_0} = \ln\left(\frac{S(t_1)}{S(t_0)}\right)^2,$$

where λ is a smoothing or decaying constant and its value lies between 0 and 1.
The weights are re-balanced on $u_k \in (0, T]$ date and $\Delta u = u_k - u_{k-1}$, $u_0 = 0 < u_1 < u_2 \cdots < u_M = T$. Moreover, Δu can be chosen daily, weekly, or monthly.

$$U(u_k) = \widetilde{\sigma}_{t_k}, \tag{10}$$

$$\begin{aligned} \alpha_1(t_j) &= \min\left(\frac{VT}{U(u_k)}, 1\right), \text{ if } u_k \leq t_j < u_{k+1}, \\ \alpha_2(t_j) &= 1 - \alpha_1(t_j). \end{aligned}$$

Example 1 *Suppose that the policyholder invests* 100€ *in a risky asset* $S(t_0) = 100$€ *at time* t_0 *and after one day* $t_1 = 1$, *the risky asset has a value* $S(t_1) = 90$€. *Moreover, assume that the estimated realized volatility of the risky asset is* $\widetilde{\sigma}_{t_1} = 0.12$. *Also assume that, we choose the volatility target* $VT = 0.10$ *and that the target volatility portfolio is re-balanced daily* $u_1 = 1$. *Then by using equation (10), the new allocation of the weights of the portfolio at time* t_1 *will be*

$$\alpha_1(t_1) = \min(\frac{0.10}{0.12}, 1) = 0.833 \ \& \ 1 - \alpha_2(t_1) = 0.166,$$

thus, at time t_1 *the target portfolio will invest* $0.833 * 90 = 74.97$ *in the risky asset and*

$0.166 * 90 = 14.94$ *in the risk-free asset.*

3 Financial Market Models

The focus of our analysis to investigate the impact of stochastic equity volatility on target volatility portfolios and financial derivatives linked to TVP. Our aim is to assess the risk when we ignored the assumption of the stochastic equity volatility. The Black-Scholes-Merton model is the most famous model to explain risky asset $S(t)$ dynamics. However, the Black-Scholes-Merton model has many limitations. The Black-Scholes-Merton model assumes the constant risky asset volatility and only allows normal distribution of the asset returns. These features are inconsistent with the observed financial markets (Bakshi et al. (1997)). Therefore, we have considered the Heston model that assumes the instantaneous variance follows the stochastic process. We use the Black-Scholes-Merton model as a reference model for comparing our numerical results. In addition, we assume that the interest rate $r > 0$ is constant.

3.1 Black-Scholes-Merton Model

Under the Black-Scholes-Merton model, the non-dividend paying risky asset follows the geometric Brownian motion whose dynamics under the physical measure $\mathbb{P}$ is given by:

$$\frac{dS(t)}{S(t)} = \zeta \cdot dt + \sigma_{BSM} \cdot dW^{\mathbb{P}}(t),\ S(0) \geq 0, \tag{11}$$

where σ_{BSM} is the constant risky asset volatility, ζ is the constant drift of the risky asset, and $W^{\mathbb{P}}(t)$ denotes a $\mathbb{P}-$Wiener process. Moreover, the dynamics of the risky asset with respect to the risk neutral measure $\mathbb{Q}$ is given by:

$$\frac{dS(t)}{S(t)} = r \cdot dt + \sigma_{BSM} \cdot dW^{\mathbb{Q}}(t),\ S(0) \geq 0, \tag{12}$$

where r is the constant interest rate and $W^{\mathbb{Q}}$ is the $\mathbb{Q}-$Wiener process.

3.2 Heston Model

The Heston model assumes that the underlying risky asset $S(t)$ follows the geometric Brownian motion, but the instantaneous variance $v(t)$ follows the CIR process(Cox et al. (1985)). The dynamics of the Heston model under the real-world measure $\mathbb{P}$ is represented by the following stochastic differential equations,

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \zeta \cdot dt + \sqrt{v(t)} \cdot dW_1(t), \quad S(0) \geq 0, \\ dv(t) &= \kappa(\theta - v(t)) \cdot dt + \sigma_v \cdot \sqrt{v(t)} \cdot dW_2(t), \quad v(0) \geq 0,\end{aligned} \tag{13}$$

where

- β_H is the market price of risk.
- θ is the long term variance.
- $\kappa > 0$ is the mean reversion speed of the variance process $v(t)$.
- $\sigma_v > 0$ is the volatility of the variance process.
- $W_1(t)$ and $W_2(t)$ are two $\mathbb{P}$-Wiener processes with $\langle dW_1(t), dW_2(t)\rangle$ $= \rho \cdot dt$.
- $\rho \in [-1, 1]$ is a correlation between the $W_1(t)$ and $W_2(t)$.

Further, the dynamics of $S(t)$ and $v(t)$ under the risk-neutral measure $\mathbb{Q}$ are, respectively, given as:

$$\begin{aligned}\frac{dS(t)}{S(t)} &= r \cdot dt + \sqrt{v(t)} \cdot dW_1(t), \quad S(0) \geq 0, \\ dv(t) &= \kappa^*(\theta^* - v(t)) \cdot dt + \sigma_v \cdot \sqrt{v(t)} \cdot dW_2(t), \quad v(0) \geq 0,\end{aligned} \tag{14}$$

where $W_1(t)$ and $W_2(t)$ are two $\mathbb{Q}-$Wiener processes and

$$\kappa^* = \kappa + \beta_H \cdot \sigma_v, \quad \theta^* = \frac{\kappa \cdot \theta}{\kappa + \beta_H \cdot \sigma_v}.$$

The properties of the variance process $v(t)$ are described in Cox et al. (1985) and it is well known that for any time $t > 0$, the variance process follows the non-centrally chi-square distribution. If the condition $2\kappa\theta > \sigma_v^2$, then it guarantees that the variance process will be strictly positive. This condition is called the Feller condition.

4 Hedging Strategies

In this section, we describe the different types of hedging strategies. The hedging strategies are used to reduce the financial risk of the derivatives. We describe the hedging strategies for the Heston model and the Black-Scholes-Merton model. Consider a hedge against the short position in the TC. Also, the position is hedged by taking a long position in the replicating portfolio. We will only invest in the underlying target volatility portfolio Z and the risk-free asset S_0, the dynamics of replicating portfolio is described as:

$$B(t) + \delta(t) \cdot Z(t), \ 0 \leq t \leq T, \tag{15}$$

where $\delta(t) \in \mathbb{R}$ and $B(t) \in \mathbb{R}$ denote the positions taken in the target volatility portfolio $Z(t)$ and the risk-free asset at time t, respectively. In addition, the dynamics of the hedge portfolio $H(t)$ at time t is expressed as:

$$H(t) = B(t) + \delta(t) \cdot Z(t) - V(t), \tag{16}$$

$$B(t) = V(t) - \delta(t) \cdot Z(t), \tag{17}$$

where $V(t)$ is a value of TC at time t.
Moreover, the instantaneous change in the hedge portfolio value is given as:

$$dH(t) = B(t) \cdot r \cdot dt + \delta(t) \cdot dZ(t) - dV(t) \tag{18}$$

Some hedging strategies such as vega hedging and gamma hedging involve derivatives of the underlying. But there are no options on the target volatility portfolio available in the financial market; these strategies will not be applicable to target volatility derivatives. Thus, we are only limited to the underlying target volatility portfolio and the risk-free asset as a hedging instruments.

4.1 No Hedging

In no hedging strategy, we hedge the TC by only taking a long position in risk-free asset S_0 such that

$$\delta(t) = 0, \quad \forall \ t. \tag{19}$$

This strategy is same for the Black-Scholes-Merton model and the Heston model.

4.2 Delta Hedging

The main risk associated with taking a short position in a any financial derivative lies in the movements of the underlying price. The delta hedging strategy uses a position in the underlying in order to protect the portfolio against small changes in the underlying price. In the Black-Scholes-Merton model, $\delta(t)$ is chosen as the delta of a financial derivative. Assume that the target volatility portfolio follows the SDE,

$$\frac{dZ(t)}{Z(t)} = \alpha_0(t) \cdot r \cdot dt + \alpha_1(t) \cdot \left(\zeta \cdot dt + \sigma_{BSM} \cdot dW^{\mathbb{P}}(t)\right), \quad Z(0) \geq 0 \tag{20}$$

where σ_{BSM} is the constant risky asset volatility, ζ is the constant drift of the risky asset, and $W^{\mathbb{P}}(t)$ denotes a $\mathbb{P}-$Wiener process. Under the assumption, $V(t) = V(t, Z(t))$, the price of the TC is a function of the time t and the underlying target volatility portfolio Z. Applying Itô's formula, $dV(t)$ is expressed as:

$$dV(t) = \frac{\partial V(t)}{\partial t} \cdot dt + \frac{\partial V(t)}{\partial Z(t)} \cdot dZ(t) + \frac{1}{2} \cdot \frac{\partial^2 V(t)}{\partial Z^2(t)} \cdot dZ^2(t), \tag{21}$$

$$dV(t) = \left(\frac{\partial V(t)}{\partial t} + \frac{1}{2} \cdot \alpha_1^2(t) \cdot Z^2(t) \cdot \sigma_{BSM}^2 \cdot \frac{\partial^2 V(t)}{\partial Z^2(t)}\right) \cdot dt + \frac{\partial V(t)}{\partial Z(t)} \cdot dZ(t). \tag{22}$$

Thus hedge portfolio (18) changes by:

$$dH(t) = \left(B(t) \cdot r - \frac{1}{2} \cdot \alpha_1(t)^2 \cdot Z(t)^2 \cdot \sigma_{BSM}^2 \cdot \frac{\partial^2 V(t)}{\partial Z^2(t)} - \frac{\partial V(t)}{\partial t}\right) \cdot dt + \tag{23}$$

$$\left(\delta(t) - \frac{\partial V(t)}{\partial Z(t)}\right) \cdot dZ(t). \tag{24}$$

To protect overall position against changes in the underlying, the random terms in equation (23) must be equal to zero. Thus under this setup, the random parts in equation (23) are:

$$\left(\delta(t) - \frac{\partial V(t)}{\partial Z(t)}\right) \cdot dZ(t),$$

and if we choose,

$$\delta(t) = \frac{\partial V(t)}{\partial Z(t)},$$

i.e. the change in the value of the hedge portfolio depends only on time and has no random parts. Delta hedging eliminates the risk of the underlying asset price change associated with the value of an option. In the Black-Scholes-Merton model with continuous time trading and with no transaction cost, delta hedging provides a theoretically perfect hedge. But, in reality, we can not re-balance continuously the hedge portfolio. We have to re-balance the hedge portfolio at discrete intervals which cause imperfection.

4.3 Minimum Variance Hedging

The delta hedging under the Black-Scholes-Merton model provides theoretically a perfect hedge. However, the delta hedging under the Heston model does not provide a theoretically perfect hedge because the market is incomplete. Therefore it is not possible to obtain a theoretical perfect hedge and eliminate all sources of risk. One way is to use the risk-minimizing strategies to reduce the risk. We use the minimum variance strategy which aims at minimizing the variance of the instantaneous change of the hedge portfolio[1]. We follow the same procedure as Bakshi et al. (1997). Consider the dynamics of the target volatility portfolio under the Heston model:

$$\frac{dZ(t)}{Z(t)} = \alpha_0(t) \cdot r \cdot dt + \alpha_1(t) \cdot \left(\zeta \cdot dt + \sqrt{v(t)} \cdot dW_1(t)\right), \quad Z(0) \geq 0,$$

$$dv(t) = \kappa(\theta - v(t)) \cdot dt + \sigma_v \cdot \sqrt{v(t)} \cdot dW_2(t), \quad v(0) \geq 0,$$

where $W_1(t)$ and $W_2(t)$ are two $\mathbb{P}$-Wiener processes with $\langle dW_1(t), dW_2(t)\rangle = \rho \cdot dt$. Moreover, we assume that $V(t) = V(t, Z(t), v(t))$, the price of the TC is a function of the variance $v(t)$, the underlying $Z(t)$, and time t. Applying Itô's formula, $dV(t)$ is expressed as:

$$\begin{aligned} dV(t) &= \frac{\partial V(t)}{\partial t} \cdot dt + \left(\frac{\partial V(t)}{\partial Z(t)} + \frac{\partial^2 V(t)}{\partial Z(t)\partial v(t)} \cdot dv(t)\right) \cdot dZ(t) + \frac{\partial V(t)}{\partial v(t)} \cdot dv(t) \\ &\quad + \frac{1}{2} \cdot \left(\frac{\partial^2 V(t)}{\partial v^2(t)} \cdot dv^2(t) + \frac{\partial^2 V(t)}{\partial Z^2(t)} \cdot dZ^2(t)\right) \end{aligned} \tag{25}$$

$$\begin{aligned} dV(t) &= \left(\frac{1}{2} Z(t) \cdot \alpha_1(t) \cdot \frac{\partial^2 V(t)}{\partial Z^2(t)} + \rho \cdot \frac{\partial^2 V(t)}{\partial Z(t)\partial v(t)}\right) \cdot Z(t) \cdot \alpha_1(t) \cdot v(t) \cdot dt + \\ &\quad \left(\frac{1}{2}\sigma_v^2 \cdot v(t) \frac{\partial^2 V(t)}{\partial v^2(t)} + \frac{\partial V(t)}{\partial t}\right) \cdot dt + \frac{\partial V(t)}{\partial v(t)} \cdot dv(t) + \frac{\partial V(t)}{\partial Z(t)} \cdot dZ(t) \end{aligned} \tag{26}$$

The instantaneous change in the hedge portfolio equation (18) becomes:

[1] This is also known as the local risk minimization strategies. Bakshi et al. (1997), Frey and Stremme (1997) and others have applied minimum variance hedging method to the stochastic volatility model.

$$dH(t) = \cdots dt + \delta(t) \cdot dZ(t) - \frac{\partial V(t)}{\partial v(t)} \cdot dv(t) - \frac{\partial V(t)}{\partial Z(t)} \cdot dZ(t),$$

$$dH(t) = \cdots dt + (\delta(t) - V_Z) \cdot dZ(t) - V_v \cdot dv(t),$$

where $V_Z = \frac{\partial V(t)}{\partial Z(t)}$ and $V_v = \frac{\partial V(t)}{\partial v(t)}$. For the conditional variance the term dt does not matter. Thus, minimum variance problem becomes:

$$\min_{\delta(t)} \langle dH(t) \rangle = \min_{\delta(t)} \left(\begin{array}{c} (\delta(t) - V_Z)^2 \cdot \langle dZ(t) \rangle - V_v^2 \cdot \langle dv(t) \rangle \\ -2\left(\delta(t) - V_Z\right) \cdot V_v \cdot \langle dZ(t), dv(t) \rangle \end{array} \right), \tag{27}$$

The resulting first order conditions are found to be:

$$\delta(t) = V_Z + V_v \cdot \frac{\langle dZ(t), dv(t) \rangle}{\langle dZ(t) \rangle}, \tag{28}$$

Where using the standard multiplication rules for stochastic processes:

$$\langle dZ(t), dv(t) \rangle = Z(t) \cdot \alpha_1(t) \cdot v(t) \cdot \sigma_v \cdot \rho \cdot dt,$$

$$\langle dZ(t) \rangle = Z^2(t) \cdot \alpha_1^2(t) \cdot v(t) \cdot dt,$$

So that,

$$\delta(t) = V_Z + V_v \cdot \frac{\sigma_v \cdot \rho}{Z(t) \cdot \alpha_1(t)}. \tag{29}$$

The above equation is called the MV delta. Under the Black-Scholes-Merton model, the MV delta is equal to the standard delta. When the correlation ρ is equal to zero, the MV delta is same as the standard delta.

5 Numerical Approximation

In this section, we present the discretization of the financial model which in defined section 2. We use the Monte Carlo simulations to approximate the value of the European call option linked to target volatility portfolio. Also, there are no analytical solutions for the Greeks of the European call option linked to TVP. We use the finite difference to approximate the partial derivatives and the Monte Carlo method to compute the respective Greeks.

Recalling equation (5), the process $Z(t)$ is discretized by using the Euler-Maruyama method,

$$\frac{Z(t_j)}{Z(t_{j-1})} = \frac{S(t_j)}{S(t_{j-1})} \cdot \alpha_1(t_{j-1}) + \frac{S_0(t_j)}{S_0(t_{j-1})} \cdot \alpha_0(t_{j-1}), \tag{30}$$

where $\alpha_1(t_0) = 1$, $Z(t_0) = S(0)$, $S(t_j)$ is one path of the risky asset at time t_j, and $\Delta t = t_j - t_{j-1}$, $t_0 = 0 < t_1 < t_2 \cdots < t_N = T$, $j = 1, 2 \cdots, N$.

5.1 Discretization of the Black-Scholes-Merton Model

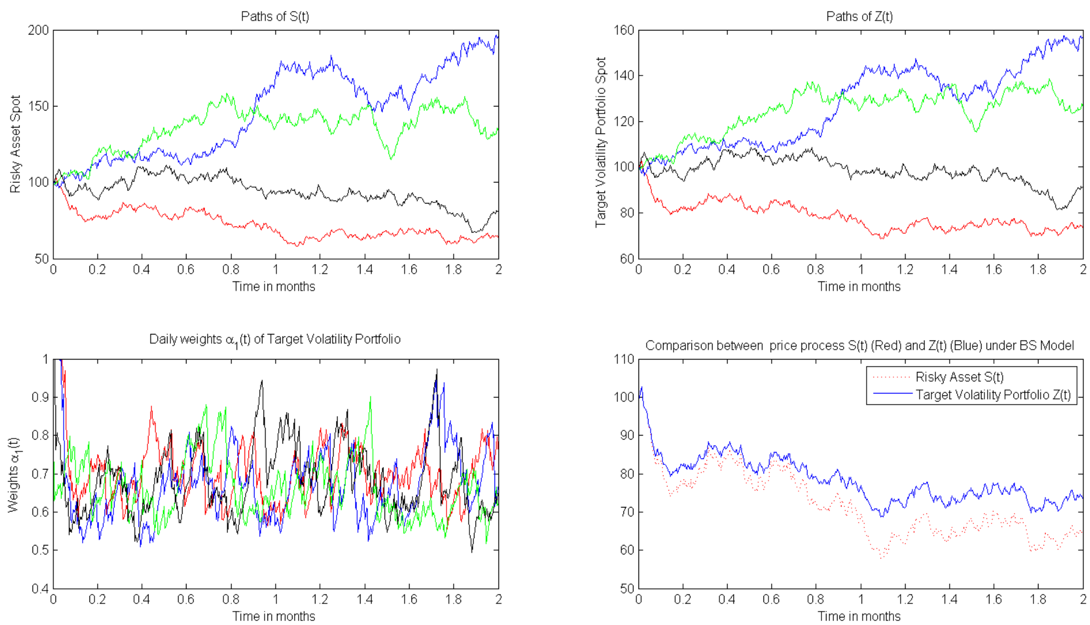

Figure 5.1: Example of paths for Risky Asset $S(t)$ and
Target Volatility Portfolio $Z(t)$ under the Black-Scholes-Merton model.

The risky asset $S(t)$ dynamics under the Black-Scholes-Merton model is given by equation (11). Using the Euler discretization, the path for the risky asset $S(t_j)$ under the Black-Scholes-Merton model is described as:

$$\frac{S(t_j)}{S(t_{j-1})} = e^{(\zeta - \frac{1}{2}\sigma_{BSM}^2)\cdot \Delta t + \sigma_{BSM} \cdot \sqrt{\Delta t} w_{j-1}}, \tag{31}$$

with $j = 1, 2 \cdots, N$, $\Delta t = t_j - t_{j-1}$, $S(t_0) = S(0)$, and $w_{j-1} \in N(0, 1)$. Figure 5.1 shows the example of the risky asset $S(t)$ and the target volatility portfolio $Z(t)$ paths, a comparison between the two price processes, and an example of the risky asset weights $\alpha_1(t)$. We use the following parameters in the case of the Black-Scholes-Merton model: $S(0) = Z(0) = 100$, $\zeta = 0.08$, $\sigma_{BSM} = 0.20$, $T = 2$ years, $VT = 0.15$, and $\Delta u = 1$ (Daily re-balancing). Moreover, we set the smoothing constant $\lambda = 0.94$ (Hull (2006)).

5.2 Discretization of the Heston Model

Under the Heston model, the risky asset $S(t)$ process is discretized in the same way as for the Black-Scholes-Merton model,

$$\frac{S(t_j)}{S(t_{j-1})} = e^{(\zeta - \frac{1}{2}v(t_{j-1}))\cdot\Delta t + \sqrt{v(t_{j-1})\cdot\Delta t}\cdot(\rho\cdot\widetilde{w}_{j-1} + \sqrt{1-\rho^2}\cdot w_{j-1})}, \tag{32}$$

here $\Delta t = t_j - t_{j-1}$, $1 \leq j \leq N$, $S(t_0) = S(0)$, and $v(t_{j-1})$ is a variance process at time t_{j-1}. Additionally, $\widetilde{w}_{j-1}$ and w_{j-1} are two standard-normally distributed random variables. Moreover, the Euler discretization of the variance process $v(t)$ follows as:

$$v(t_j) = v(t_{j-1}) + \kappa(\theta - v(t_{j-1}))\cdot\Delta t + \sigma_v \cdot \sqrt{v(t_{j-1})\Delta t}\cdot\widetilde{w}_{j-1},\ 1 \leq j \leq N, \tag{33}$$

where $v(t_0) \geq 0$, $\Delta t = t_j - t_{j-1}$, and $\widetilde{w}_{j-1}$ is the independent standard-normal random variable. However, the problem with the Euler scheme in equation (33) is that it may simulate negative values for the variance process. It is well known that for any given $v(t_{j-1})$, the probability for $v(t_j)$ to be negative is:

$$\Pr(v(t_j) < 0) = \Phi\left(\frac{-(1-\kappa\Delta t)\cdot v(t_{j-1}) - \kappa\cdot\theta\cdot\Delta t}{\sigma_v\cdot\sqrt{v(t_{j-1})\Delta t}}\right),$$

where $\Phi(X)$ denotes the standard normal cumulative distribution function. Thus there is a probability of ending up with a negative value although our variance process $v(t)$ should be strictly positive. To address this problem, we use the Milstein scheme for $v(t)$ which is described by Kahl and Jäckel (2005) for stochastic volatility models and Kloeden and Platen (1992) for general stochastic processes. The Milstein scheme for the variance process $v(t)$ is given as,

$$v(t_j) = v(t_{j-1}) + \kappa(\theta - v(t_{j-1}))\cdot\Delta t + \sigma_v\cdot\sqrt{v(t_{j-1})\Delta t}\cdot\widetilde{w}_{j-1} + \frac{1}{4}\sigma_v^2\cdot\Delta t\cdot(\widetilde{w}_{j-1}^2 - 1),\ 1 \leq j \leq N, \tag{34}$$

and the above equation is re-written as:

$$v(t_j) = \left(\sqrt{v(t_{j-1})} + \frac{1}{2}\sigma_v\cdot\sqrt{\Delta t}\cdot\widetilde{w}_{j-1}\right)^2 + \left(\kappa(\theta - v(t_{j-1})) - \frac{1}{4}\sigma_v^2\right)\cdot\Delta t,\ 1 \leq j \leq N, \tag{35}$$

According to Gatheral (2011), the Milstein discretization of the variance process reduces the frequency of negative values in the variance process as compared to the Euler scheme. But negative values can still occur. To fix this problem, we use the absorption condition $v(t_j) = \max(0, v(t_j))$, along with the Milstein scheme.

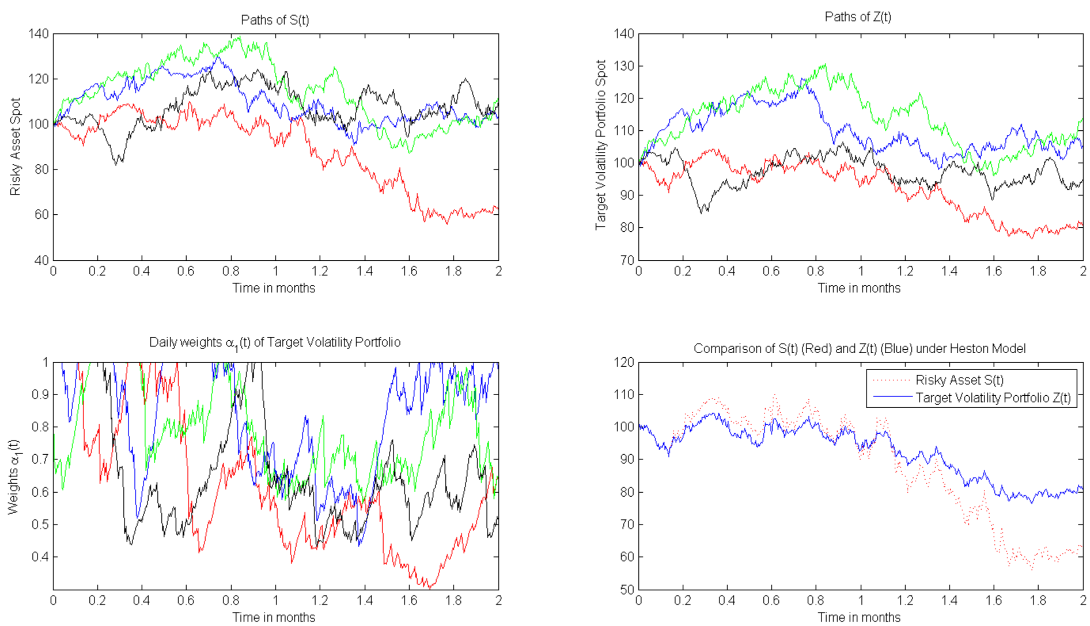

Figure 5.2: Example of paths for Risky Asset $S(t)$ and Target Volatility Portfolio $Z(t)$ under the Heston model.

We set $S(0) = Z(0) = 100$, $\zeta = 0.08$, $T = 2$ years, $VT = 0.15$, $\Delta u = 1$ (Daily re-balancing) and $\lambda = 0.94$. Figure 5.2 shows an example of the paths of the target volatility portfolio $Z(t)$ and the risky asset $S(t)$ under the Heston model. We summarize the Heston model parameters for the base case in Table 1. Moreover, we use the Heston parameters which are stated in Poulsen et al. (2009) and Kling et al. (2011).

κ^*	θ^*	$v(0)$	σ_v	ρ	β_H
4.75	0.22^2	θ^*	0.55	−0.569	0

Table 1: Benchmark parameters for the Heston Model.

6 Numerical Results

6.1 Distributional Properties of the Target Volatility Portfolio

In this subsection, we investigate the performance of the target volatility portfolio (TVP). Moreover, we choose two different investment strategies. We investigate the performance of the portfolio for a variety of investment strategies involving (1) risky portfolio and (2) target volatility portfolio under different financial models. More precisely, the first investment strategy corresponds to the case where the underlying asset is pure risky/equity asset, i.e. $\alpha_1(t)$ is equal to 1 and $\alpha_0(t)$ is equal to zero in the equation (5). The second strategy corresponds to the case where $\alpha_1(t)$ and $\alpha_0(t)$ are defined according to sub-section 2.1. In addition, we consider the dynamics of both portfolios under the $\mathbb{P}$ measure. The distributional properties are characterized by a number of moments which are mean, median, standard deviation, skewness, and kurtosis. The main purpose of analyzing the moments of the TVP is that it will tell us how the target volatility strategy behaves under different models. Moreover, it will also help us to understand the characteristic of financial derivative linked to target volatility portfolio. The moments of the risky portfolio depend on the distributional properties of the dynamics of S. However, in the case of the target volatility portfolio, its distributional properties also depend upon the stochastic weights $\alpha_0(t)$ and $\alpha_1(t)$, VT level, and the dynamics of the risky asset S. We also study the impact of stochastic volatility and re-balancing frequencies on the distributional properties of the target volatility portfolio. We assume the drift ζ of the risky asset S to be 8%, volatility target level $VT = 10\%$, $r = 2\%$, and $S(0) = Z(0) = 100$. Also, we set the number of simulations equal to 100000 for all numerical results.

6.1.1 Target Volatility Portfolio Under the Black-Scholes-Merton Model

In the first example (Table 2 & Figure 6.1.1), we set a $\sigma_{BSM} = 22\%$, $T = 1$, and compare the performance of target volatility portfolio with risky portfolio under the Black-Scholes-Merton model. The second column in table 2 corresponds to the case for risky portfolio. Moreover, the next columns correspond to the case when the underlying portfolio is the target volatility portfolio with various re-balancing frequencies. The first observation from the test is that the target volatility portfolio delivers a lower standard deviation compared to the risky portfolio. The standard deviation tells us that the target volatility strategy really restricts the realized volatility of the portfolio. As we recall, the main purpose of using dynamic re-balancing is to keep the realized volatility of the TVP under a certain level. Hence, the standard deviation of the target volatility portfolio remains below the so-called volatility target level. We find that the mean return of the TVP decreases as compared to the risky portfolio because it invests more money in the risk-free asset in order to maintain the volatility target of the portfolio. We also find an improvement in the TVP Sharpe ratio. Again, this is due to the fact that the TVP has a lower standard deviation compared to the risky portfolio.

Statistic	Risky Portfolio	TVP ($\Delta u = 1$)	TVP ($\Delta u = 5$)	TVP ($\Delta u = 20$)
Mean	0.054485	0.040623	0.042116	0.042828
Median	0.053977	0.040429	0.041855	0.042649
Standard Deviation	**0.22012**	**0.099279**	**0.10698**	**0.11635**
Sharpe Ratio	0.2475	0.4092	0.3937	0.3681
Maximum	1.0541	0.49586	0.50524	0.56154
Minimum	−0.80339	−0.33849	−0.35974	−0.41414
95% Quantile	0.41878	0.20473	0.21895	0.23602
Skewness	−0.0066541	−0.0064373	−0.0086617	−0.0074853
Kurtosis	3.0103	3.0005	2.9866	2.9795

Table 2: Statistic of log returns of the Risky Portfolio vs Target Volatility Portfolio under the BSM model.

Moreover, we have calculated the skewness and kurtosis of the log returns of TVP and risky portfolio. Skewness is a statistical parameter which measures the asymmetry of the distribution of the random variable about its mean. Skewness can be negative or positive. Kurtosis is the measure of the peakedness of the distribution. In case of high numbers of small and large price movements the kurtosis will be higher than 3. If there are only a few small and large price movements, the kurtosis will be less than 3. The kurtosis and skewness of a normally distributed asset are 3 and 0, respectively.

From the observations of table 2 and figure 6.1.1, we see that the log returns of the TVP and risky portfolio are normally distributed under the Black-Scholes-Merton model. This result indicates that the target volatility portfolio can be treated as a normally distributed asset under the Black-Scholes-Merton model. And financial derivatives linked to TVP can be calculated by using exact solutions under Black-Scholes-Merton.

We shall now discuss the impact of different re-balancing frequencies on the target volatility portfolio. To test the importance of the frequency, we examine three different re-balancing frequencies, daily, weekly and monthly. Table 2 and figure 6.1.1 show the results of different re-balancing frequencies and compared with the risky portfolio under the Black-Scholes-Merton model. When the re-balancing frequency is reduced from daily to monthly, the standard deviation tends to increase. We observe that the target volatility portfolio with monthly re-balancing frequency has a standard deviation of 11%. Whereas the daily re-balanced TVP has a standard deviation of 9%, which shows that it achieves the volatility target perfectly. Moreover, the higher standard deviation of the monthly re-balanced TVP suggests that it is exposed to equity risk and there might be large price fluctuations between the re-balancing dates.

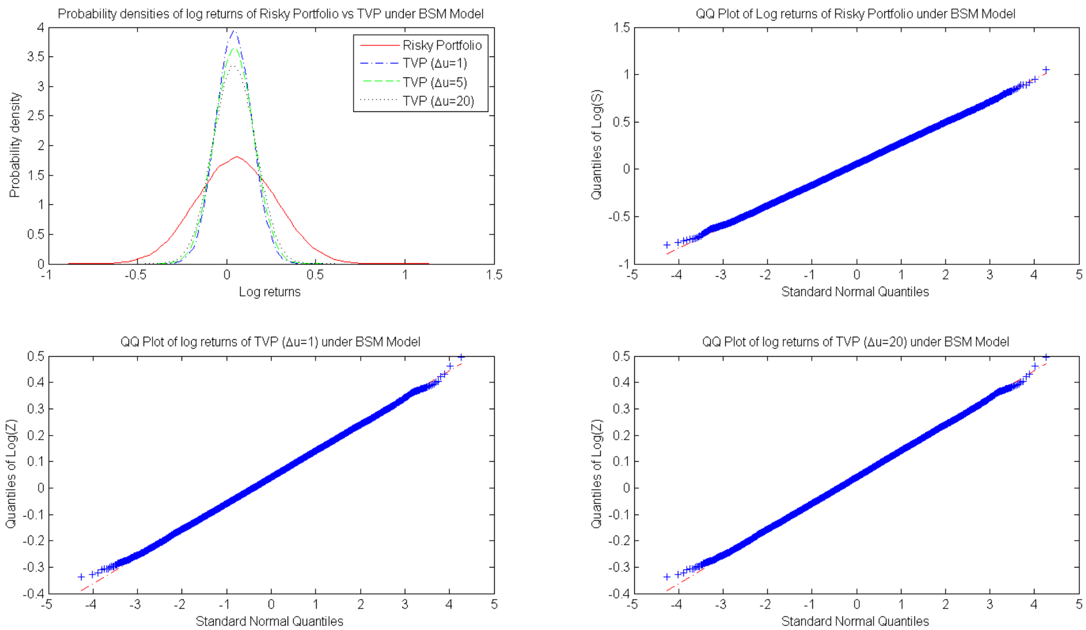

Figure 6.1.1: Estimated probability densities and QQ - plots of the Risky Portfolio vs Target Volatility Portfolio with different re-balancing frequencies under the BSM model.

However, table 2 shows that the weekly and daily re-balanced target volatility portfolios produce similar results in terms of mean returns, standard deviation, and Sharpe ratio. This is an important result. We can deduce that it would be more transaction cost effective to use the weekly re-balancing frequency instead of a daily re-balancing frequency under the Black-Scholes-Merton model.

6.1.2 Target Volatility Portfolio Under the Heston Model

In the second example (table 3 & figure 6.1.2), we analyze the logarithm returns of the risky portfolio and TVP under the Heston model. We consider the Heston parameters which are defined in table 1. Under the Heston model, we find that the mean return and the Sharpe ratio of the target volatility portfolio and risky portfolio have been increased compared to the Black-Scholes-Merton case. The Sharpe ratio of the target volatility portfolio increases by 0.05 in the case of stochastic volatility. We also notice that the Sharpe ratio of the risky portfolio is slightly higher due to the leverage effect. We see from table 3 and figure 6.1.2 that the distribution of the risky portfolio has a sharper peak and long fat tails. Thus, under the Heston model, the logarithmic returns of a risky portfolio have a high peak and returns are skewed to the left compared to the Black-Scholes-Merton model. This feature resembles the real financial market returns.

Statistic	Risky Portfolio	TVP ($\Delta u = 1$)	TVP ($\Delta u = 5$)	TVP ($\Delta u = 20$)
Mean	0.057042	0.045889	0.04732	0.047583
Median	0.079934	0.046773	0.048643	0.049594
Standard Deviation	**0.22409**	**0.10056**	**0.10957**	**0.12189**
Sharpe Ratio	0.2545	0.4563	0.4319	0.3904
Maximum	0.76714	0.44294	0.4677	0.52196
Minimum	−1.3614	−0.36687	−0.4147	−0.51244
95% Quantile	0.37843	0.20964	0.22571	0.24517
Skewness	**−0.71891**	**−0.041125**	**−0.048124**	**−0.10569**
Kurtosis	4.1788	2.9712	2.9559	2.9877

Table 3: Statistic of log returns of the Risky Portfolio vs Target Volatility Portfolio under the Heston model.

We can observe from the table 3 and figure 6.1.2 that the distributions of the daily, weekly and monthly re-balanced target volatility portfolios become more normalized. This is due to the fact that the target volatility strategy efficiently restricts the realized volatility of the risky asset. The relative weights also restrain the influence of risky asset when realized volatility is high. Thus, the returns of the target volatility portfolio are more symmetrically spread around the mean and the distribution is less skewed.
We also compare the effect of the daily, weekly and monthly re-balance frequencies. One can observe from table 3 that skewness is nearly zero for daily and weekly re-balanced target volatility portfolios. Hence, there is minimal impact of stochastic volatility on the TVP with daily and weekly re-balancing frequencies. On the contrary, monthly re-balancing frequency still exhibit some negative skewness. Moreover this result indicates that the daily and weekly re-balanced portfolios are normally distributed under the Heston model.

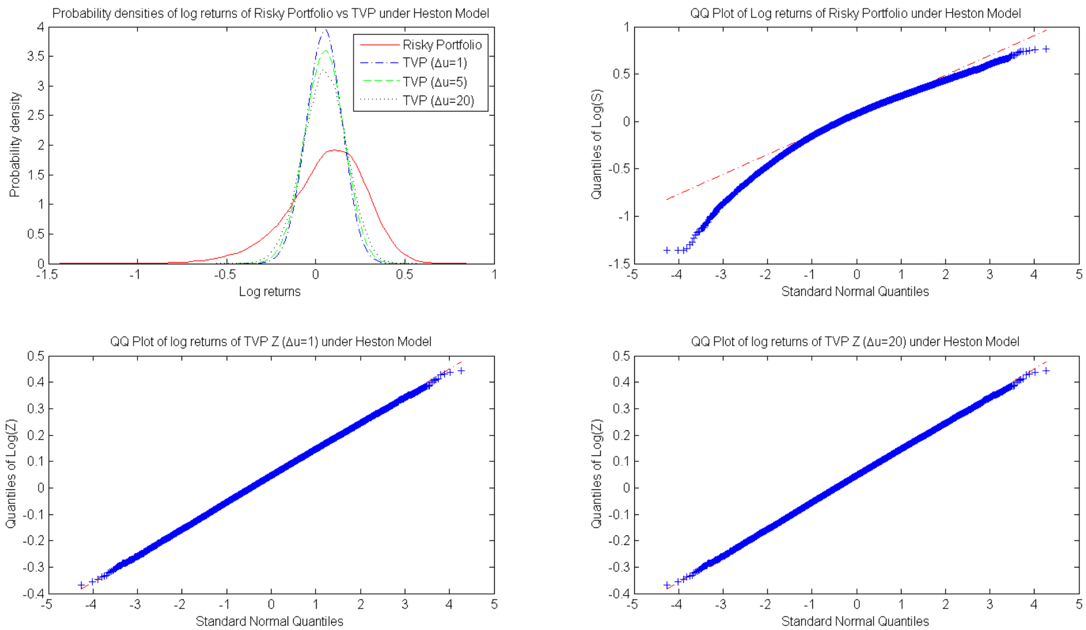

Figure 6.1.2: Estimated probability densities and QQ - plots of the Risky Portfolio vs Target Volatility Portfolio with different re-balancing frequencies under the Heston model.

The study of the distributional properties of the target volatility portfolio viewed from two different perspectives. The study helps the investors to manage their wealth and it explains the key links between investment decision and portfolio choices. Investors can protect their investment from falling markets by investing in target volatility portfolios. On the other hand, insurance companies/investment banks can offer attractive products to investors and reduce financial risk associated with target volatility portfolios. In addition, target volatility portfolios with daily and weekly re-balancing frequencies produce similar results. It would be more transaction cost effective for insurance companies to offer only weekly re-balanced TVP instead of daily re-balanced TVP. We have tested the performance of the target volatility portfolio under different financial models. The target volatility strategy perfectly restricts the influence of the realize volatility of the risky asset. However, under the stochastic volatility, the target volatility portfolio still exhibits slight negative skewness. It shows that there is slighted impact of stochastic volatility on the target volatility portfolio under the measure $\mathbb{P}$. Therefore, insurance companies can face a financial risk if they consider a model with constant equity volatility for risk management purposes (e.g. portfolio management, ruin probabilities, hedging, etc.).

6.2 The European Call Option Linked to TVP

In this section we assess the price of the European options linked to target volatility portfolio. Moreover, we analyze the impact of stochastic volatility on the value of European option linked to target volatility portfolios. In addition to this, we check the impact of different re-balancing frequencies on the price of European options linked to TVP. The European option linked to TVP regards as an Asian option because the value of a target volatility portfolio depends upon the past paths of the equity asset. We consider two different types of portfolios (1) risky portfolio and (2) target volatility portfolio. We also compare the price of European call option linked to risky portfolio (RC) with European call option linked to TVP (TC). Moreover, we consider the strike 100 and spot 100 for all examples.

6.2.1 TC Price under the Black-Scholes-Merton Model

Firstly, we consider the valuation of the European call option linked to TVP (TC) under the Black-Scholes-Merton model. We set $VT = 10\%$ and $r = 2\%$. Table 4 provides the prices of the ATM call option with the maturity of 1 year and various risky asset volatilities. The second column corresponds to the case when the European call option linked to risky portfolio. And the next columns correspond to the case when the European call option linked to TVP with different re-balancing frequencies of the target volatility portfolio.

Volatility σ_{BSM}	RC	TC ($\Delta u = 1$)	TC ($\Delta u = 5$)	TC ($\Delta u = 20$)
15%	6.9618	4.9209	5.1738	5.4198
22%	**9.6981**	**4.9625**	**5.3001**	**5.6844**
25%	10.8706	4.9648	5.3233	5.8392

Table 4: European call option linked to Risky Portfolio (RC) vs European call option linked to TVP (TC) under the BSM model.

The results of table 4 demonstrate some of the important effects related to the European call option linked to TVP. The first effect is that the TC values are significantly smaller compared to the European option linked to risky portfolio. However, the value of the TC increases with the reduce re-balancing frequency. Analyzing data of table 4 (column 3, 4 and 5), one can observe that when we reduce the re-balancing frequency from daily to monthly, the prices are increased by $12 - 14\%$. We recall the results from the previous section and this suggests that an annualized standard deviation tends to increase when we reduce the re-balancing frequency. Hence, the TC prices increase with lower re-balancing frequencies. The second effect reflects that increasing risky asset volatility does not have any impact on the TC prices. This means that the target volatility portfolio efficiently controls the realized volatility of the portfolio by putting more weights on the non-risky asset.

The previous section showed that the target volatility portfolio is a normally distributed asset under the Black-Scholes-Merton model. This implies that we can directly use the Black-Scholes-Merton formula to price the target volatility options instead of the Monte Carlo approximation. In our second test example we wish to establish whether the Black-Scholes-Merton formula can be used to price the TC. In table 5, we price the TC under the Black-Scholes-Merton model by using the Monte Carlo method, and use the following parameters $r = 2\%$, $\sigma_{BSM} = 22\%$, $\lambda = 0.94$, $S(0) = Z(0) = 100$, $K = 100$ and $T = 1$. Table 5 provides the ATM TC prices with different volatility targets. In the second column, we calculate the ATM call option price by using the Black-Scholes-Merton formula. But we set the risky asset volatility equal to the volatility target ($\sigma_{BSM} = VT$). Finally, the next columns correspond to the TC prices with different re-balancing frequencies.

Volatility Target	BSM call price	TC ($\Delta u = 1$)	TC ($\Delta u = 5$)	TC ($\Delta u = 20$)
5%	**3.1206**	**3.1031**	3.4643	4.2419
10%	**5.0169**	**4.9766**	5.3139	5.7196
15%	**6.9618**	**6.8669**	7.2373	7.45912

Table 5: Comparison of ATM call prices under the BSM Model ($\sigma_{BSM} = VT$) and European call option linked to TVP under the BSM model.

Our numerical results confirm that the increasing target volatility level increases the value of European call option linked to TVP. Moreover, if we compare columns 2 and 3 in table 5, we can observe that the analytical Black-Scholes-Merton prices are nearly equal to the TC for daily re-balancing frequency. This is again due to the fact that the target volatility portfolio is a normally distributed asset. However, the Black-Scholes-Merton formula prices are not equal to the TC value for weekly and monthly re-balancing frequencies and there is a difference of 5% and 14% in the prices, respectively. From a financial institution's point of view, using the Black-Scholes-Merton formula to calculate the TC value and Greeks for daily re-balance frequency can save computational time. But using the Black-Scholes-Merton formula for weekly and monthly re-balanced TC options can lead to under-pricing.

6.2.2 TC Price under the Heston Model

In this section we show the impact of the stochastic volatility on the price of European call option linked to TVP. Moreover, we test the impact of various long-term variances and speed of mean reversions on the price of European call option linked to TVP. We use the risk-neutral parameters for the Heston model which are stated in Table 1. We also use the values of long-run variance and the speed of mean reversion for different values of β_H which are described by Kling et al. (2011).

Market price of risk	Speed of mean reversion κ^*	Long term variance θ^*
$\beta_H = 2$	5.85	$(0.198)^2$
$\beta_H = 0$	4.75	$(0.220)^2$
$\beta_H = -2$	3.65	$(0.251)^2$

Table 6: $\mathbb{Q}-$ parameters for different market price of risk.

Kling et al. (2011) describe that the highest values of β_H correspond to a high speed of mean reversion and a low long-term variance, and vice visa. Moreover, $\beta_H = -2$ implies a long-term volatility of 25.1%. In addition, $\beta_H = 0$ and $\beta_H = 2$ imply a long term volatility of 22.0% and 19.8%, respectively.

Market price of risk	RC	TC ($\Delta u = 1$)	TC ($\Delta u = 5$)	TC ($\Delta u = 20$)
$\beta_H = 2$	8.6568	4.8842	5.2410	5.7023
$\boldsymbol{\beta_H = 0}$	**9.5221**	**4.9009**	**5.2891**	**5.7740**
$\beta_H = -2$	10.5752	4.9147	5.3250	5.9199

Table 7: European call option linked to Risky Portfolio (RC) vs European call option linked to TVP (TC) under the Heston model.

For our numerical results in table 7, we have calculated the ATM call option prices under the Heston model with a maturity of 1 year with various risky asset long term volatilities. The second column corresponds to the case when the European option linked to risky portfolio. And, the next columns correspond to the case when the European option linked to TVP. Under the Heston model, prices of TC are again smaller compared to the European call options linked to risky portfolio. We observe that when we increase the long-term equity volatility, it does not affect the TC prices. Moreover, for $\beta_H = 0$ and $\beta_H = -2$, the prices of TC for daily and weekly re-balancing frequencies under the Heston model appear to be same as under the Black-Scholes-Merton model. However, the European call option linked to TVP with monthly re-balancing frequency prices appear to be 1.3% higher compared to the Black-Scholes-Merton model. As we know from section 4.1, monthly re-balance TVP portfolio under the Heston model exhibits some skewness. The increase in the monthly TC prices is explained by this effect.

Volatility Target	TC ($\Delta u = 1$)	TC ($\Delta u = 5$)	TC ($\Delta u = 20$)
5%	**3.0727**	3.4604	4.2336
10%	**4.9046**	5.2946	5.7794
15%	**6.6132**	6.9954	7.3122

Table 8: European call option linked to TVP under the Heston model.

Table 8 presents computation results for European call options linked to TVP for different levels of volatility target. The value of TC increases when we raise the volatility target level. For instant, when we increase volatility target from 10% to 15%, the value of daily re-balanced TC increases from 4.97 to 6.86. This is due to fact that the target volatility strategy allocates more weight in the risky asset and it increases the annual realized volatility of the target volatility portfolio. The analysis provides two important results. First, we have shown that the price of European option linked to TVP is low compared to the European option linked to risky portfolio. The target volatility strategy controls the realize volatility of the underlying portfolio. Therefore, financial derivatives based on the target volatility portfolio (TVP) are less expensive compared to derivatives based on risky portfolio. We also study the impact of different model parameters and impact of stochastic volatility on the value of the European option linked to TVP. Table 7 suggests that the long-term equity volatility has no impact on the value of the European option linked to TVP. In addition, we find that the stochastic equity volatility has no impact on the price of the European option linked to TVP. Therefore, there is no equity volatility risk for financial derivatives linked to TVP.

6.3 Greeks of the European Call Option Linked to TVP

We use Monte Carlo simulation to calculate the different sensitivities of the European call option linked to TVP. Greeks will help us to understand the main risk factors which affect the European call option linked to TVP. We have analyzed the three most commonly known sensitivity measures: delta, gamma, and vega.

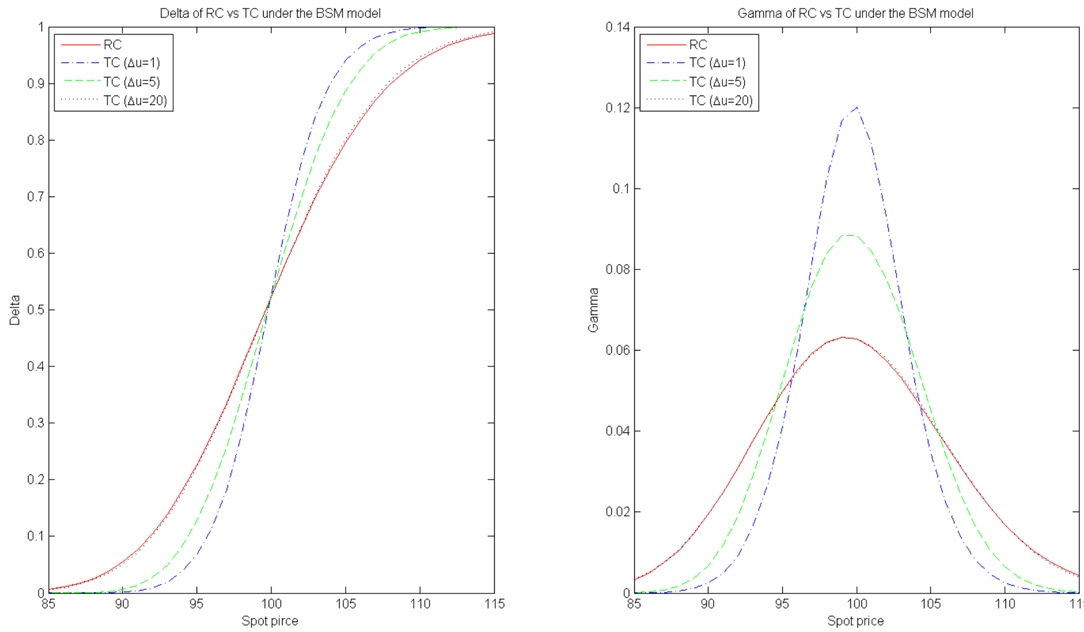

Figure 6.3 (a): Left-hand side: Delta of RC vs TC. Right-hand side: Gamma of RC vs TC under the BSM model as a function of spot price.

In figure 6.3 (a) (left-hand side), we compare the delta of the European call option linked to risky portfolio (RC) with European call option linked to TVP (TC) under the Black-Scholes-Merton model as a function of the spot price. Similarly, the right-hand side of the figure shows the comparison of the gamma of the RC vs TV. We use parameters $T = \frac{1}{12}$, and $VT = 10\%$. Moreover, we present the impact of different re-balancing frequencies on the delta and gamma of the European call option linked to TVP.

First of all, it is clear that the delta of the ITM TC is higher compared to the standard European option. And delta is lower for the OTM TC. Hence, the delta of the TC increases rapidly with rising stock markets and vice versa. Furthermore, the ITM TC with daily re-balance frequency has a larger delta compared to other re-balancing frequencies and its delta is very sensitive to the change in the underlying price. Moreover, the gamma of the TC is higher than the European option linked to risky portfolio. And when gamma is large, the delta is highly sensitive to change in the underlying asset price. TC have higher gamma because they are highly path dependent. When we compare the gamma of different re-balancing frequencies, we see that daily re-balancing frequency has the highest gamma. Thus, the re-balancing frequencies have a significant impact on the delta and gamma of the European call option linked to TVP (TC). In addition, the value of TC with the daily re-balance frequency is more sensitive to changes in an asset price compared to other re-balancing frequencies.

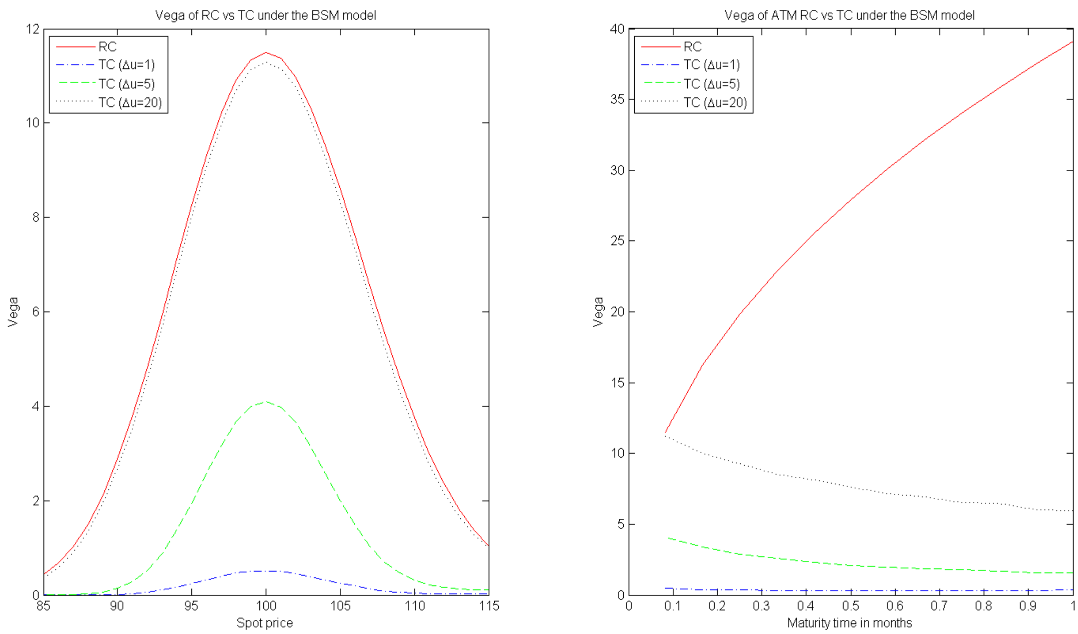

Figure 6.3 (b): Left-hand side: Vega of the TC vs RC.
Right-hand side: Vega of the ATM TC vs RC
with varying maturity.

In the figure 6.3 (b), (left-hand side), we compare the vega of the European call option linked to risky portfolio (RC) with the European call option linked to TVP (TC). The right figure shows the comparison of the vega of the ATM European call option linked to TVP with the European call option linked to risky portfolio as a function of a maturity time.

We can see from the figure at the left-hand side that the vega of the TC is very low compared to the RC. When we check the impact of re-balancing frequency on the vega of the TC, the daily re-balanced target volatility portfolio produces the lowest vega. If we look at the right figure, the vega of the RC is an increasing function with respect to the maturity time. But, the behavior of the vega with regard to the TC is the opposite. Again, when we check the impact of re-balance frequency on the vega of the TC, daily re-balancing frequency produces the lowest vega and its value is constantly near to zero. The vega is low for the European call option linked to TVP because the target volatility portfolio allocate the stochastic weights depending upon the realized historical volatility of the risky asset. When the realized volatility increases, it decreases the weights of the risky asset. Hence we will have less influence of the risky asset. Therefore the European call options linked to TVP are less sensitive to the risky asset volatility and have a lower vega. The daily re-balancing frequency further reduces the influence of the risky asset and produces very low vega.

6.4 Implied Volatilities of the European Call Option Linked to TVP

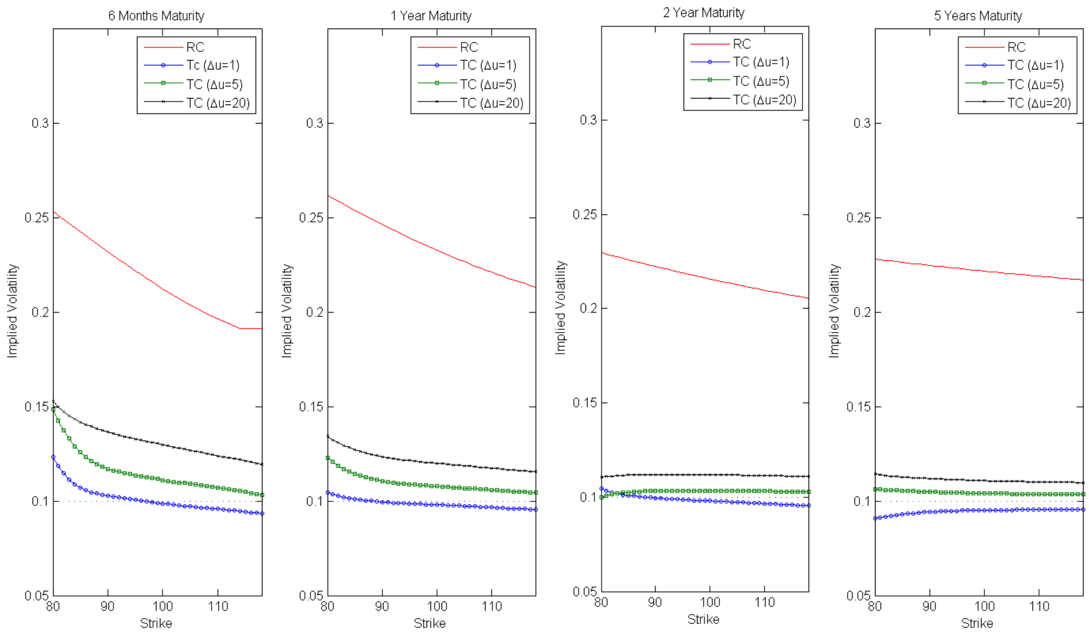

Figure 6.4: Implied volatility of the RC
vs TC with different re-balancing frequencies.

In this section, we check the impact of the stochastic volatility on the implied volatility of the European call option linked to TVP. We also test the impact of a re-balancing frequency on the implied volatility of the TC. To compute the implied volatility, we first calculate the TC prices under the Heston model for different maturities and strikes. We use the parameters which are stated in table 1. To obtain implied volatilities, we need to invert the Black-Scholes-Merton formula and use the Newton-Raphson formula. But we use the pre-defined Matlab function '*blsimpv*' to obtain the implied volatility of the TC. Figure 6.4 shows the comparison of the implied volatility of the European call option linked to risky portfolio with the European call option linked to TVP. It also exhibits the impact of different re-balancing frequencies on the implied volatility of the European call option linked to TVP.

We can see from the figure 6.4 that the implied volatilities of the European call option linked to TVP (TC) for the longer maturities (2 & 5 years) are approximately equal to the volatility target (10%). We observe slight skewness for the 6 months and 1 year maturities. The TC with weekly and monthly re-balancing frequencies show a slight skew and their implied volatilities are higher than the volatility target. On the other hand, the TC with daily re-balancing frequency display less skewness and its implied volatility is equal to the volatility target.

Moreover, the skewness appears to flatten off for the 2 and 5 year maturities. We observe that the implied volatilities are relatively insensitive to the re-balancing frequency for longer maturity options. The daily and weekly re-balancing frequencies produce similar results and their implied volatilities are approximately equal to the volatility target (10%). But the implied volatilities of the monthly re-balanced TC are slightly higher than the volatility target. This is a very important result. The benefits of the target volatility portfolio for longer maturities can be achieved with weekly re-balancing frequency. Hence, it can save transaction costs and computation time for the product designer.

6.5 Impact of Lambda on the European Call Option Linked to TVP

The main objective of the EWMA method (equation 9) is to estimate the historical realized volatility and keep track of the changes in the realized volatility. The parameter lambda λ is known as a smoothing/decay parameter. We had chosen $\lambda = 0.94$ for all the previous numerical results. The Riskmatric suggests that the optimal value of lambda should be equal to 0.94 for daily estimation. But some people use different values of lambda depending upon their past experience and attitude toward risk. In general, a small value of lambda leads to quicker detection of small changes in the underlying and affects the realized volatility estimation. Moreover, the realized volatility changes slowly to recent changes in the underlying when lambda is close to 1. In the current section, we check the impact of lambda on the price of European option linked to TVP under the Black-Scholes-Merton model. Figure 6.5 shows the ATM TC prices for different values of λ. We also check the impact of λ on the daily, weekly and monthly re-balancing frequencies.

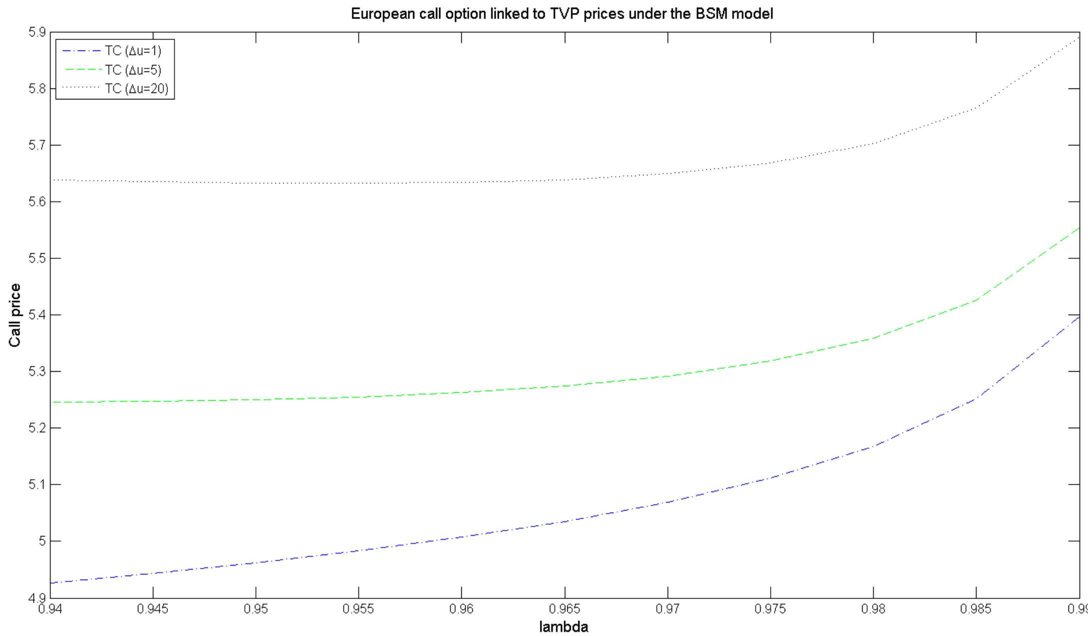

Figure 6.5: The impact of the lambda on the price of TC.
$(T = 1, K = 100, r = 2\%, VT = 10\%, \sigma_{BS} = 22\%)$

Figure 6.5 shows that the European call option linked to TVP with weekly and monthly re-balancing frequencies are relatively insensitive to different values of lambda. The price of the TC with weekly and monthly re-balancing frequencies do not change in the interval of $(0.94, 0.97)$. However, for lambda > 0.97, we observe a small rise in the TC prices and the prices increase by approximately 4%.

The daily re-balanced TC are more sensitive to the parameter lambda. An increase in lambda leads to an increase in the price of the daily re-balanced TC. When $\lambda = 0.99$, the price of TC rises by 9% compared to $\lambda = 0.94$. This is due to the fact that when lambda is close to 1, the estimated realized volatility is slowly changing with the change in the underlying risky asset. Therefore the target volatility strategy allocates more weights in to the risky asset and thus it increases the price of the TC. As a conclusion, insurance companies should correctly specify the value of EWMA's lambda for the valuation of financial derivatives linked to TVP. The miss-specification of lambda's value can lead to under-pricing of financial derivatives linked to TVP.

7 Profit and Loss Analysis

In order to understand the impact of model assumptions on target volatility portfolios and on the value of financial derivatives linked to TVP, we simulate the profit and loss distribution of different hedging strategies. Moreover, we also compare different hedging strategies presented in the section 3.

The dynamic hedging strategies presented in the section 3 require continuous rebalancing of the hedge portfolio. This is not feasible in practice and only discrete time rebalancing is possible. Suppose that the hedge portfolio is rebalanced at discrete time $l_k \in [0, T]$ and $\Delta l = l_h - l_{h-1}$, $l_0 = 0 < l_1 < l_2 \cdots < l_g = T$, and $g = \frac{T}{\Delta l}$. The initial position of the hedge portfolio is defined as:

$$H(0) = B(0) + \delta(0) \cdot Z(0) - V(0) = 0,$$

$$B(0) = V(0) - \delta(0) \cdot Z(0).$$

The hedge portfolio is re-balanced at time step $h = 1, \cdots, g-1$. The hedge portfolio update at time l_h

$$H(l_h) = B(l_{h-1}) \cdot e^{r\Delta l} + \delta(l_{h-1}) \cdot Z(l_h) - V(l_h),$$

$$B(l_{h-1}) = V(l_{h-1}) - \delta(l_{h-1}) \cdot Z(l_{h-1}),$$

where $\delta(l_{h-1})$ is defined according to the prescribe model and the hedging strategy. We consider two different measures to examine the P&L of each hedging strategy. They are defined by the total discounted hedging error (HE) and the total absolute hedging error (AHE):

$$HL = \sum_{h=1}^{g} e^{-r \cdot l_h} \cdot H(l_h), \tag{36}$$

$$AHL = \sum_{h=1}^{g} e^{-r \cdot l_h} \cdot |H(l_h)|, \tag{37}$$

where $H(l_g)$ is the liquidated value of the hedge portfolio at time T. The distribution of the total discounted hedging error will be used to assess the performance of various hedging strategies. We assume that the dynamics of target volatility portfolio under the real-world measure $\mathbb{P}$. We use the risk-neutral $\mathbb{Q}$ dynamics of the Black-Scholes-Merton model and the Heston model to price and compute the delta and vega of the European call option linked to TVP. We assume the following parameters: $\zeta = 8\%$, $r = 2\%$, $S(0) = Z(0) = 100$, $\sigma_{BSM} = 22\%$, $T = 1$ year, and base parameters for the Heston model are defined in Table

1. Moreover, we suppose that the ATM TC and the target volatility portfolio is re-balanced daily ($\Delta u = 1$) and a volatility target level is equal to 10%. The hedge portfolio is re-balanced weekly ($\Delta l = 5$) for all hedging strategies.

Table 9 gives the results for different hedging strategies. The hedging strategies are abbreviated as following: (NBSM) stands for no hedging under the Black-Scholes-Merton model and (NH) represents for no hedging under the Heston model. (DBSM) and (DH) represents for the delta hedging under the Black-Scholes-Merton and the Heston model, respectively. (MV) finally stands for the minimum variance hedging under the Heston model.

We observe from Table 9 that when there is no hedging strategy (NBSM & NH) in place, the HE distribution has a large variance and a long left tail. For instance, mean is -1.7742 and -2.2086 for NBSM and NH, respectively. The negative value of mean represents a loss. Under the delta hedging strategy based on the Black-Scholes-Merton model, the standard deviation is significantly reduced and the mean of the total discounted hedging error (HE) is near to zero.

We analyze the impact of the stochastic volatility on the delta hedging strategy. We simulate the target volatility portfolio under the Heston model and also calculate the respective Greeks under the Heston model. By the introduction of the stochastic volatility, the standard deviation slightly increases. We know from the previous section that there is no impact of stochastic volatility on the price of the daily TC. However, according to the section 5.1.2, the daily re-balanced target volatility portfolio under measure $\mathbb{P}$ still exhibits slight skewness. The slight increase in the standard deviation and the skewness of the total hedging error (HE) is explained due to this effect.

We have also analyzed the minimum variance hedging strategy for the Heston model. Theoretically, the minimum variance should produce better results as compared to the delta hedging strategy. But, in the case of the daily re-balanced target volatility portfolio and TC, the minimum variance produces very similar results as compared to delta hedging under the Heston model. The reason is that the daily re-balanced TC produces a very low vega and the last term of the equation (29) becomes nearly zero. Hence, the minimum variance strategy and the delta hedging under the Heston model produce similar results.

Statistic	NBSM	NH	DBSM	DH	MV
Mean	−1.7742	−2.2086	−0.0111	−0.2033	−0.2093
Median	0.9206	0.7551	−0.0180	−0.1920	−0.2051
Standard Deviation	**7.9343**	**8.3813**	**0.4648**	**0.5055**	**0.4994**
Maximum	5.2551	5.1291	1.7051	1.577	1.5147
Minimum	−46.843	−48.626	−2.0804	−2.3	−2.3099
$\text{VaR}_{0.995}$	−13.345	−14.514	−0.5923	−0.8369	−0.8296
Skewness	**−1.2439**	**−1.2499**	**0.0189**	**−0.1925**	**−0.1952**
Kurtosis	4.3144	4.3498	3.5771	3.7144	3.6519

Table 9: The total discounted hedging error of the no hedging, delta hedging and minimum variance hedging. The hedge portfolio is updated weekly. We consider the ATM TC ($\Delta u = 1$).

We also analyze the impact of the stochastic volatility on the monthly re-balanced target volatility portfolio. Table 10 provides the results of the total absolute error (AHE) of the different hedging strategies. We calculate the absolute hedging error (AHE) because the pattern of profit and loss is clearer. The second and third columns correspond to the case when the target volatility portfolio is re-balanced daily. The next columns correspond to the case when the target volatility portfolio is re-balanced monthly. Moreover, the hedge portfolio is up-dated weekly. And, we assume the ATM European call options linked to TVP. We observe two important results from table 10. Firstly, the hedging errors of DBSM and DH strategy are nearly similar for the daily re-balanced target volatility portfolio. Secondly, by the introduction of stochastic volatility, the standard deviation for monthly re-balanced TVP is increased by 60% as compared to the DBSM. As we know from section 5.1, monthly re-balanced TVP under the Heston model exhibits skewness. Due to this effect, standard deviation of hedging error (AHE) is increased. Moreover, the minimum variance strategy improves the hedging error in the case of the monthly re-balanced TVP. This is due to the effect that monthly re-balanced TC's vega is not zero. We conclude that the stochastic volatility has a significant impact on the hedging strategies and the minimum variance hedging produces best results for monthly re-balanced TC. Therefore, insurance companies should not use a model with constant equity volatility for hedging financial derivatives linked to TVP.

	TC ($\Delta u = 1$)		TC ($\Delta u = 20$)	
Strategy	Mean	SD	Mean	SD
NBSM	38.092	15.905	39.402	17.9
NH	37.327	15.403	38.529	16.97
DBSM	2.8423	0.60191	4.8263	1.1652
DH	2.9033	0.68004	6.097	1.9517
MV	2.8962	0.677	5.7006	1.7476

Table 10: The total absolute hedging error of different hedging strategies. We consider the ATM TC option. The hedge portfolio is up-dated weekly.

8 Conclusion

The target volatility portfolio can be classified as advance risk management strategy. The idea behind the strategy is to shift the allocation of the risky asset to non-risky asset during the high realized volatility of risky asset and vice versa. We find that the mean return and the Sharpe ratio of the target volatility portfolio increased under the Heston model. We also test the impact of re-balancing frequencies on the target volatility portfolios. We find that the distributions of the daily, weekly and monthly re-balanced target volatility portfolios become more normalized under the Heston model. Moreover, the weekly and daily re-balanced target volatility portfolios produce similar results in terms of mean returns, standard deviation, and Sharpe ratio under the Black-Scholes-Merton model and the Heston model. The benefits of the target volatility portfolio can be achieved with weekly re-balancing frequency and it can save transaction costs and computation time for the product designer.

We also assess the price of the European options linked to TVP and test the impact of stochastic volatility on the value of the TC. Our numerical results confirm that the prices of European options linked to TVP for daily and weekly re-balancing frequencies under the Heston model appear to be same as under the Black-Scholes-Merton model. And there is no significant impact of the stochastic volatility on the value of daily and weekly re-balanced TC. We also observe that the analytical Black-Scholes-Merton formula can be used to price the TC. Moreover, the development of the Greeks is also significantly different depending on the selected re-balanced frequency of the target volatility portfolio. Thus the hedging error differs significantly for the different re-balanced frequency of the target volatility portfolio. We analyzed different hedging strategies and analyzed the distribution of the total hedging error. We found that the risk can be reduced significantly by suitable hedging strategies.

In the present paper, we consider the stochastic volatility for the dynamics of the risky asset.

In the future, we plan to consider the stochastic interest models and analyze the insurance products such as variable annuities based on the target volatility portfolios.

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9 Appendix

9.1 A1 European Call option and Greeks under the Black-Scholes-Merton Model

Under the Black-Scholes-Merton model, the exact solution for the European call option at time t on risky asset $S(t)$ is given as:

$$\begin{aligned} C_{BSM}(S(t), K, \sigma_{BSM}, t, T) &= S(t) \cdot N(d_1) - e^{-r\cdot(T-t)} \cdot K \cdot N(d_2), \\ d_1 &= \frac{1}{\sigma_{BSM} \cdot \sqrt{T-t}} \left[\ln(\frac{S(t)}{K}) + \left(r + \frac{1}{2}\sigma_{BSM}^2 \right) \cdot (T-t) \right], \\ d_2 &= d_1 - \sigma_{BSM} \cdot \sqrt{T-t}, \end{aligned} \tag{38}$$

where $N(\cdot)$ denotes the cumulative distribution function for Normal distribution, the strike price is K, and T is the time of maturity. The main reason for defining all the exact formulas for the European call option is that we intend to compare their values with the European call option linked to TVP.
The sensitivities of the European call option with respect to different parameters are called Greeks. Moreover, delta measures the rate of change of the option value with respect to changes in the price of the underlying asset. Also, gamma is the rate of change of delta with respect to changes in the asset price. The Greeks of European call options are given as:

$$\begin{aligned} Delta &= \frac{\partial C_{BSM}}{\partial S(t)} = N(d_1), \\ Vega &= \frac{\partial C_{BSM}}{\partial \sigma_{BSM}} = S(t) \cdot N(d_1) \cdot \sqrt{T-t}, \\ Gamma &= \frac{\partial^2 C_{BSM}}{\partial S^2(t)} = \frac{N(d_1)}{S(t) \cdot \sigma_{BSM} \cdot \sqrt{T-t}}, \end{aligned} \tag{39}$$

9.2 A1 European Call option under the Heston Model

Heston (1993) derives the semi-analytical solutions for the European call and put options by using Fourier inversion techniques. The time t value of a European call option on the risky asset $S(t)$ with a strike K and a maturity at time T is given as:

$$C_{Heston}(S(t), K, v(t), t, T) = e^{x(t)} \cdot P_1 + K \cdot e^{-r\tau} \cdot P_2, \tag{40}$$

$$x(t) = \ln(S(t)),$$

$$\tau = T - t,$$

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\cdot\phi \ln K} f_j(v(t), x(t), \phi)}{i \cdot \phi}\right] \cdot d\phi, \quad j = 1, 2, \tag{41}$$

$$f_j(v(t), x(t), \phi) = \exp(C_j(\phi, \tau) + D_j(\phi, \tau) \cdot v(t) + i \cdot \phi \cdot x(t)),$$

$$C_j(\phi, \tau) = r \cdot i \cdot \phi \cdot \tau + \frac{a}{\sigma_v^2}\left[(b_j - \rho \cdot \sigma_v \cdot i \cdot \phi + d_j) \cdot \tau - 2 \ln\left(\frac{1 - g_j e^{d_j \cdot \tau}}{1 - g_j}\right)\right], \tag{42}$$

$$D_j(\phi, \tau) = \frac{b_j - \rho \cdot \sigma_v \cdot i \cdot \phi + d_j}{\sigma_v^2}\left(\frac{1 - e^{d_j \cdot \tau}}{1 - g_j e^{d_j \cdot \tau}}\right), \tag{43}$$

$$g_j = \frac{b_j - \rho \cdot \sigma_v \cdot i \cdot \phi + d_j}{b_j - \rho \cdot \sigma_v \cdot i \cdot \phi - d_j},$$

$$d_j = \sqrt{(\rho \cdot \sigma_v \cdot i \cdot \phi - d_j)^2 - \sigma_v^2(2 \cdot u_j \cdot i \cdot \phi - \phi^2)},$$

$$a = \kappa \cdot \theta, \; u_1 = \frac{1}{2}, \; u_2 = -\frac{1}{2}, \; b_1 = \kappa - \rho \cdot \sigma_v, \; b_2 = \kappa,$$

where $\mathrm{Re}(\cdot)$ denotes the real part of a complex number and i is the imaginary unit.
The second form for the characteristic equation (42) and (43) are explained in the academic literature (Bakshi et al. (1997), Gatheral (2011)). Gatheral (2011) define these forms as:

$$c_j = \frac{1}{g_j}, \tag{44}$$

$$D_j(\phi, \tau) = \frac{b_j - \rho \cdot \sigma_v \cdot i \cdot \phi - d_j}{\sigma_v^2}\left(\frac{1 - e^{-d_j \cdot \tau}}{1 - c_j e^{-d_j \cdot \tau}}\right),$$

$$C_j(\phi, \tau) = r \cdot i \cdot \phi \cdot \tau + \frac{a}{\sigma_v^2}\left[(b_j - \rho \cdot \sigma_v \cdot i \cdot \phi - d_j) \cdot \tau - 2 \ln\left(\frac{1 - c_j e^{-d_j \cdot \tau}}{1 - c_j}\right)\right]. \tag{45}$$

Albrecher et al. (2006) shows that these forms are equivalent, but the second form (equation 44 and 45) behaves better for longer maturities and unrestricted parameters and well suited for numerical integration.
The improper integral in equation 41 has to be numerically integrated and we use the Trapezoidal method for this purpose.

Working Paper Three

An Analysis of Guaranteed Lifetime Withdrawal Benefits Linked to Target Volatility Portfolio

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Abstract

GLWB is currently the most popular type of a variable annuity. GWLB promises the policyholder to annually withdraw a fixed amount from his investment account for the rest of his life, even if the value of the investment account drops to zero. However, from the insurer's point of view, GLWB contains several types of risk such as mortality risk, interest rate risk and financial risk. Target volatility strategy is used to create a dynamic re-balancing portfolio such that the overall volatility of portfolio maintains a stable level at all time. The strategy shifts the allocation of equity asset to non-risky asset in order to protect the portfolio from equity market crashes. Many insurance companies offer or consider GLWB linked to target volatility portfolios. This paper provides an extensive analysis of the GLWB linked to target volatility portfolio. In particular, we investigate the impact of stochastic volatility and mortality intensity on the pricing of the GLWB linked to target volatility portfolio. Moreover, we examine the lifetime probability of ruin for the target volatility portfolios.

Keywords: Target volatility portfolio, stochastic volatility, variable annuity, stochastic mortality intensity.

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1 Introduction

Variable annuities (VA) are one of the most important types of equity-linked pension products. Variable annuities were introduced in the USA in the early 1970's. Variable annuities have become very popular in the USA and Japan. VA are equity-linked pension contracts where the policyholder invests his retirement savings in the mutual funds, but the benefits received by the policyholder are based on the performance of the underlying funds. Moreover, the policyholder can elect different types of guarantee riders in order to protect his investment by paying a certain amount of insurance fees. There are different types of guarantee riders available in the market. These guarantee riders can be mainly categorized in four subcategories: guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB), guaranteed minimum withdrawal benefits (GMWB), and guaranteed lifetime withdrawal benefits (GLWB). GMIB provides a guaranteed annuity benefit for a certain time period. GMAB provides the policyholder a specific guaranteed value at the maturity of the contract. Under the GMWB, the policyholder can withdraw money from his account even if the value of the account falls to zero. These withdrawals are guaranteed up to the maturity of the contract.

The currently most popular type of variable annuity is GLWB. With this guarantee type, the policyholder can annually withdraw a fixed amount from his investment account for the rest of his life. The insurer provides the guaranteed withdrawals for the rest of the policyholder's life even after the value of the investment account reaches zero. On the other hand, if the value of the account is positive at time of death of the policyholder, any remaining value will be returned to the policyholder's beneficiary. In return for this guarantee, the insurer charges an insurance fee which is deducted as a fixed annual percentage from the policyholder's investment account. Therefore, from an insurer's point of view, GLWB contains several types of risk, namely systematic mortality risk, interest rate risk, and in particular financial risk. Systematic mortality risk refers to unexpected changes in survival probabilities and expected lifetimes of policyholders. Moreover, the insurer can face additional risks, including basis risk and behavioral risk which are not the focus of this paper.

It is well known that insurers suffered unexpectedly large losses during the global financial crises in 2007-2008 due to a rise in the value of the guarantees. The realized and implied equity volatility significantly increased during the financial crisis. As a consequence, it increased the value of most unit-linked equity products and also the value of VA guarantees. The insurers who used poor risk management strategies suffered large losses due to the increase in the guarantee's value and unexpectedly high hedging costs. Several insurance companies have therefore sought new risk management methods to reduce their cost and offer attractive guarantees to policyholders. One of these new popular risk management methods is called the target volatility strategy. There are several new market indices based on the target volatility strategy, e.g. S&P 500 Risk Control Index, EuroStoxx 50 Risk Control Index, Dow Jones Volatility Control Index, etc. The target volatility strategy is used in order to create a dynamically re-balancing portfolio. This portfolio consists of an equity asset and a non-risky asset. The strategy shifts the allocation of the equity asset to the non-risky asset during high realized volatility of equity asset and vice versa. The dynamic allocation is determined by the estimated realized volatility of the equity asset and a pre-defined target level. Hence the overall portfolio volatility level is kept under control. Thereby, the financial derivatives and unit-linked products based on the target volatility portfolio (TVP) are less expensive compared to derivatives based on pure equities or equity indices. Many insurance companies offer variable annuity guarantees based on target volatility portfolios. The target volatility strategy significantly reduces the value of guarantees by controlling the volatility of the underlying funds. On the other hand, it also protects the policyholder's retirement savings from the bearish markets.

Dynamic re-balancing portfolio was first introduced by Markowitz (1952) in his work on Mean-Variance Optimization (MVO). The idea of MVO was to find the optimal way of allocating investment between different asset classes by considering return maximization and risk minimization simultaneously. The basic idea of the target volatility strategy has been described in different articles (see, Giese (2011), Albeverio et al. (2013), Standard and Poor's (2009)). Albeverio et al. (2013) introduce the risk-neutral pricing framework for the financial derivatives and unit-linked insurance products linked to target volatility portfolios. However, they use the deterministic equity volatility for the dynamics of the underlying equity asset.

There have been several articles devoted to the pricing and hedging of the guaranteed lifetime withdrawal benefits (GLWB). Holz et al. (2012) introduce a pricing framework that focuses on GLWB. They analyze the value of a GLWB guarantee for different product designs under the deterministic and optimal policyholder behavior. Piscopo and Haberman (2011) decompose the value of GLWB into living benefits and death benefits. They also analyze the impact of mortality risk on the value of GLWB. Kling et al. (2011) investigate the impact of stochastic equity volatility on the pricing and hedging of GLWB. They also consider different dynamic hedging strategies and compare their performance. Funga et al. (2013) analyze the impact of systemic mortality risk on the valuation and hedging of GLWB. They consider the deterministic equity volatility for underlying funds and a stochastic model for describing mortality intensity.

Existing literature does not, however, provide any detailed numerical analyses of variable annuities linked to the target volatility portfolio. To the best of our knowledge, this paper is the first work to provide such analysis. In particular, we analyze the impact of stochastic volatility on the value of GLWB linked to TVP. Moreover, we analyze the lifetime ruin probability under the stochastic volatility and stochastic mortality intensity.

The paper is organized as follows. In the next section, we propose the general financial model for the target volatility portfolio and we describe the dynamics of the policyholder's investment account. In Section 3, we describe the stochastic modeling of mortality risk and combine the financial and mortality models. In Section 4, we define the no-arbitrage value of GLWB. The definition of the lifetime probability of ruin is presented in Section 5. We use the lifetime probability of ruin to analyze the performance of the target volatility portfolio. Moreover, the numerical results are provided in Section 6, starting with the numerical study of lifetime ruin probabilities. We compare the performance of the target volatility portfolio with a pure equity portfolio. In addition, we perform a sensitivity analysis and show the impact of different parameter risks on the value of GLWB linked to TVP. We also analyze the impact of parameter risk on the profit and loss distributions of GLWB linked to TVP.

2 Financial Modeling

In this section we define a financial market in continuous time where equity asset prices follow a diffusion process on the fixed time horizon $[0, T]$, $0 < T < \infty$.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and $\mathbb{P}$ denotes the real world measure on $(\Omega, \mathcal{F})$. Also, $\mathcal{G}_t = \sigma\left(S(s) : 0 \leq s \leq t\right)$ is a subfiltration of $\mathcal{F}_t$ containing all information about financial market. We define $x + t$ as the age of the policyholder at time $t \in [0, T]$ and ω as the maximum attainable age i.e. survival of the policyholder beyond age ω is assumed to be impossible ($T = \omega - x$).

We suppose that available investment opportunities consist of equity assets $S_1(t), \cdots, S_N(t)$ and a risk-free asset $S_0(t)$. The assets $S_0(t), S_1(t), \cdots, S_N(t)$ are assumed to be measurable with respect to the filtration $\mathcal{F}_t$ and they are traded continuously. The dynamics of the risk-free asset $S_0(t)$ is defined as:

$$\frac{dS_0(t)}{S_0(t)} = r(t) \cdot dt, \quad S_0(0) = 1, \tag{1}$$

where $r(t) > 0$ and $t \in [0, T]$. Let $S_j(t)$, $j = 1, 2, \cdots, N$, represents the jth equity asset price at time $t \in [0, T]$. The price process $S_j(t)$ evolves according to the following stochastic differential equation:

$$\frac{dS_j(t)}{S_j(t)} = \zeta_j \cdot dt + \sum_{k=1}^{d} \sigma_{j,k}(t) \cdot dW_k(t),\ S_j(t) \geq 0,\ j = 1, 2, \cdots, N, \tag{2}$$

where $\zeta_j \in \mathbb{R}^+$ is a constant drift. $W_k(t)$, $k \in \{1, 2, \cdots, d\}$, denotes a $\mathbb{P}-$Wiener process with the covariance structure $d\langle W_i, W_j\rangle = \rho_{i.j} \cdot dt$, and $\rho_{i.j} \in [-1, 1]$ is a correlation between W_i and W_j. Moreover, $\sigma_{j,k}(t)$ is the variance process of the jth equity asset.

Assume that we can invest in one or more of the assets. A trading strategy is defined as a stochastic process $\alpha = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \cdots, \alpha_N(t))$ for $t = 0, 1, 2, \cdots T$, where $\alpha_j(t)$ represents the fraction of wealth that is invested in equity asset $S_j(t)$ at time t. In particularly $\alpha_0(t) = 1 - \sum_{j=1}^{N} \alpha_j(t)$ is the fraction of wealth invested in the risk-free asset at time t. Furthermore, the sequence α is assumed to be predictable. The value process of a trading strategy $\alpha = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \cdots, \alpha_N(t))$ is the stochastic process Z given by:

$$Z(t) = \alpha_0(t) \cdot S_0(t) + \sum_{j=1}^{N} \alpha_j(t) \cdot S_j(t),$$

and the dynamics of a self-financing relative portfolio Z without consumption is expressed as:

$$\frac{dZ(t)}{Z(t)} = \alpha_0(t) \cdot \frac{dS_0(t)}{S_0(t)} + \sum_{j=1}^{N} \alpha_j(t) \cdot \frac{dS_j(t)}{S_j(t)}, \tag{3}$$

by using equations (1) and (2):

$$\frac{dZ(t)}{Z(t)} = \left(\sum_{j=1}^{N} \alpha_j(t) \left(\zeta_j - r(t)\right) + r(t) \right) \cdot dt + \sum_{j=1}^{N} \alpha_j(t) \cdot \left(\sum_{k=1}^{d} \sigma_{j,k}(t) \cdot dW_k(t) \right). \tag{4}$$

In our setup, we assume that the market consists of one equity asset $S(t)$ and one risk-free asset $S_0(t)$ and we name the relative portfolio Z as the target volatility portfolio (TVP):

$$\frac{dZ(t)}{Z(t)} = (\alpha_1(t) \cdot (\zeta - r(t)) + r(t)) \cdot dt + \alpha_1(t) \cdot (\sigma(t) \cdot dW(t)). \tag{5}$$

where $W(t)$ is a $\mathbb{P}-$Wiener process.

The policyholder invests his retirement savings in an investment account that has both equity asset and risk-free asset exposure. Let $I(t)$ denotes the policyholder's investment account. The investment account is depleted by continuous withdrawal/consumption at the rate $g(t)$ and the insurance fee α_f. The dynamics of the policyholder's investment account $I(t)$ is described by the following SDE:

$$dI(t) = -\alpha_f \cdot I(t) \cdot dt + I(t) \cdot \frac{dZ(t)}{Z(t)} - \gamma(t) \cdot dt, \; I(t) \geq 0, \tag{6}$$

$$I(0) = P.$$

We define $c \in \mathbb{R}^+$ to be a point in time where the investment account $I(t)$ depletes to zero, that is

$$c = \inf \{t \in [0, T] : I(t) = 0\} .$$

Note that, after time c, the investment account $I(t)$ becomes:

$$I(t) = 0, \; t \geq c.$$

Here $\gamma(t)$ denotes the annual withdrawal amount deducted by the policyholder at time t, and α_f denotes the annual fee charged by the insurance company. Furthermore, $I(0)$ represents the initial amount invested by the policyholder. Equation (6) holds as long as the investment account $I(t)$ is positive and once the investment account hits zero, it will remain at zero. Suppose that $g(t)$ is the annual withdrawal rate allowed by the insurance company, then the annual withdrawal amount $\gamma(t)$ deducted at time t is defined as:

$$\gamma(t) = g(t) \cdot I(0), \quad 0 \leq \gamma(t) \leq I(t), \quad 0 < t \leq T.$$

Also, it is reasonable to consider that the withdrawal amount $\gamma(t)$ deducted by the policyholder ranges between 0 and the maximum value of the investment account $I(t)$. However, we assume that the withdrawal rate remains constant, i.e. $g(t) = g$. In our valuation analysis, we follow the static approach of Milevsky and Salisbury (2006) in which the policyholder deducts the same withdrawal amount $\gamma(t)$ per annum. In practice, the policyholder can increase the withdrawal amount but the insurance companies impose penalty charges on a exceeded amount.

2.1 The Stochastic Weights

In this part, we introduce the so-called target volatility strategy by imposing a particular structure on the stochastic weights $\alpha_1(t)$ and $\alpha_0(t)$. Let $0 < u_1 < u_2 \cdots < u_M$ be some re-balancing times. In addition, we consider a stochastic process U which will depend upon a past price trajectory of $\{S(s); t \geq s\}$. Moreover, $U(u_k)$ is computed as the annualized realized volatility of the equity asset $S(t)$ at the re-balancing time $u_k \in \mathbb{Z}^+$. We assume that there are 252 business days. We use the Exponentially Weighted Moving Average

(EWMA) method to estimate an annualized daily volatility of the risky asset S. The main reason for choosing the EWMA method is that the S&P 500 risk control index (see Standard and Poor's (2009)) uses the EWMA method to estimate realized volatilities and it is also computationally less expensive. Let $R(t_j)$ be defined as the compounded return during day t_j:

$$R(t_j) = \ln\left(\frac{S(t_j)}{S(t_{j-1})}\right),$$

where $\Delta t = t_j - t_{j-1}$, $t_0 = 0 < t_1 < t_2 \cdots < t_N = T$, $j = 1, 2 \cdots, N$,

$$\widetilde{\sigma}^2_{t_j} = \lambda \cdot \widetilde{\sigma}^2_{t_{j-1}} + (1-\lambda) \cdot \frac{1}{\Delta t} \cdot R^2(t_{j-1}), \tag{7}$$

$$\widetilde{\sigma}^2_{t_0} = \ln\left(\frac{S(t_1)}{S(t_0)}\right)^2,$$

where λ is a smoothing or decaying constant with a value between 0 and 1.
Let $\Delta u = u_k - u_{k-1}$, $u_0 = 0 < u_1 < u_2 \cdots < u_M = T$ and Δu can be chosen daily, weekly, or monthly. We define $U(u_k)$ as:

$$U(u_k) = \widetilde{\sigma}_{t_k}. \tag{8}$$

Consider $\alpha_1(t_j)$ and $\alpha_0(t_j)$ to be two stochastic processes whose values depend upon the realized historical volatility of an equity asset S. We have placed a restriction on the sign of $\alpha_0(t_j)$ and we will not borrowed the amount $\alpha_0(t_j)$ in the risk-free asset ($\alpha_0(t_j) \geq 0$). In particular, we consider the following dynamics for $\alpha_1(t_j)$ and $\alpha_0(t_j)$:

$$\alpha_1(t_j) = \min\left(\frac{VT}{U(u_k)}, 1\right), \; if \; u_k \leq t_j < u_{k+1}, \tag{9}$$

$$\alpha_0(t_j) = 1 - \alpha_1(t_j),$$

where VT is a volatility target whose value is set according to the historical realized volatility of the equity asset. The stochastic weights will be kept constant until the next re-balancing time u_k. At time $u_k \leq t_j < u_{k+1}$, the weights will change their value according to an estimated historical volatility of the equity asset. When the estimated historical volatility of the equity asset is higher than the chosen volatility target VT ($U(u_k) > VT$) at the re-balancing time u_k, then $\alpha_1(t_j)$ takes a value between 0 and 1. Otherwise, ($U(u_k) < VT$)), $\alpha_1(t_j)$ is equal to 1.

2.2 Financial Model

The Black-Scholes-Merton model (Black and Scholes (1973)) is the most influential and famous financial model to explain equity price dynamics. The Black-Scholes-Merton model is popular because it is simple and provides the closed form solution for pricing and hedging of financial derivatives. However, it has many limitations. The Black-Scholes-Merton model enforces constant equity volatility and allows only log-normally distributed asset returns. It has been shown that these features are inconsistent with the observed financial markets (Bakshi et al. (1997)). The observations from the financial market show that the distribution of the asset returns exhibits excess skewness and kurtosis. In the past two decades, research has focused on including stochastic volatility as well as jump components into equity price dynamics. Many authors (e.g. Heston (1993), Bakshi et al. (1997)) have explored new models to explain the equity price dynamics. They have introduced equity models which exhibit excess kurtosis and skewness in the asset returns distribution. In this paper, we consider the Heston model for the equity asset dynamics and interest rate ($r > 0$) is constant. We assume that the market is free of arbitrage and an equivalent martingale measure exists. But we do not assume that the market is complete.

2.2.1 Heston Model

The Heston model was introduced by Steven Heston (Heston (1993)). The model assumes that the conditional underlying equity asset $S(t)$ is a geometric Brownian motion, but the instantaneous variance $v(t)$ follows the CIR process (Cox et al. (1985)). The dynamics of the Heston model under the real-world measure $\mathbb{P}$ is given as:

$$\begin{aligned} dS(t) &= \zeta \cdot S(t) \cdot dt + \sqrt{v(t)} \cdot S(t) \cdot dW_1(t) \quad S(0) \geq 0, \\ dv(t) &= \kappa(\theta - v(t)) \cdot dt + \sigma_v \cdot \sqrt{v(t)} \cdot dW_2(t), \quad v(0) \geq 0, \end{aligned} \tag{10}$$

where $W_1(t)$ and $W_2(t)$ are two $\mathbb{P}$-Wiener processes with $\langle dW_1(t), dW_2(t) \rangle = \rho \cdot dt$ and $\rho \in [-1, 1]$ is a correlation between $W_1(t)$ and $W_2(t)$. Also, θ is the long-term variance, $\kappa > 0$ is the mean reversion speed of the variance process $v(t)$, and $\sigma_v > 0$ is the volatility of the variance process.

Further, the dynamics of $S(t)$ and $v(t)$ under the risk-neutral measure $\mathbb{Q}$ are given as:

$$
\begin{aligned}
dS(t) &= r \cdot S(t) \cdot dt + \sqrt{v(t)} \cdot S(t) \cdot dW_1^{\mathbb{Q}}(t), \quad S(0) \geq 0, \\
dv(t) &= \kappa^*(\theta^* - v(t)) \cdot dt + \sigma_v \cdot \sqrt{v(t)} \cdot dW_2^{\mathbb{Q}}(t), \quad v(0) \geq 0,
\end{aligned} \tag{11}
$$

where $W_1^{\mathbb{Q}}(t)$ and $W_2^{\mathbb{Q}}(t)$ are two $\mathbb{Q}-$Wiener processes and

$$
\kappa^* = \kappa + \beta_H \cdot \sigma_v, \quad \theta^* = \frac{\kappa \cdot \theta}{\kappa + \beta_H \cdot \sigma_v},
$$

here β_H is the market price of risk.

3 Mortality Modeling

It is generally recognized that life expectancy has improved over time. Also, different generations have different mortality patterns (Cairns et al. (2008)). On the other hand the issue of mortality risk has been widely studied in recent years when dealing with the pricing of insurance products. Recently, stochastic models have been used to describe the uncertainty linked to mortality. The stochastic modeling of mortality risk is very similar to the credit risk literature. In this paper, we will follow the doubly stochastic (Cox-process) approach developed by Lando (1998). All the stochastic processes are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathcal{F}_t$, $t \in [0, T]$. Let us assume that the time of the death of the policyholder is modeled as stopping time τ_x admitting stochastic mortality intensity (hazard rate) $\mu_{x+t}(t)$ with respect to filtration $\mathcal{F}_t$. Moreover, $\mathcal{F}_t$ contains all the information about market risk and mortality risk. However, we assume that the financial risk and mortality risk are independent (see for instance Dahl (2004), Biffis (2005)). In our setting, $\mathcal{F}_t = \mathcal{H}_t \vee \mathcal{I}_t$ where $\mathcal{H}_t = \sigma\left(\mathbb{I}_{\tau_x \leq s} : 0 \leq s \leq t\right)$ is the subfiltration generated by the stopping time τ_x and contains information about whether the policyholder is alive or not. Also, $\mathcal{I}_t = \mathcal{G}_t \vee \mathcal{M}_t$ where $\mathcal{G}_t = \sigma\left(S(s) : 0 \leq s \leq t\right)$ and $\mathcal{M}_t = \sigma\left(\mu_{x+s}(s) : 0 \leq s \leq t\right)$ contain all information about the financial market and mortality rates. We consider that the process μ_x satisfies $\int_0^t \mu_{x+s}(s) \cdot ds < \infty$ for all $t > 0$. The random policyholder's time of death $\tau_x \in \mathbb{R}^+$ is defined as the first time when the process $\int_0^t \mu_{x+s}(s) \cdot ds$ is above the random variable E_1:

$$
\tau_x = \inf\left\{t \in [0, T] : \int_0^t \mu_{x+s}(s) \cdot ds \geq E_1\right\}, \tag{12}
$$

where E_1 is an independent exponential random variable with unit parameter 1. The survival probability ${}_{T-t}p_{x+t}(t)$ from time t to $T = \omega - x$ for the policyholder aged $x + t$ at time t is given as:

$$_{T-t}p_{x+t}(t) = \mathbb{P}(\tau_x > T \mid \mathcal{F}_t) = E^{\mathbb{P}}[e^{-\int_t^T \mu_{x+s}(s)\cdot ds} \mid \mathcal{F}_t],$$

where $\mathbb{P}$ is the real probability measure.
Moreover, the $\mathcal{F}_t-$conditional density $f_{x+t}(\cdot)$ of a random residual lifetime τ_x of the policy-holder aged $x+t$ on the set $\tau_x > t$ is:

$$f_{x+t}(s) = \frac{\partial}{\partial s}\mathbb{P}(\tau_x \leq s \mid \mathcal{F}_t) = E^{\mathbb{P}}[\mu_{x+s}(s) \cdot e^{-\int_t^s \mu_{x+u}(u)\cdot du} \mid \mathcal{F}_t]. \tag{13}$$

See Biffis (2005) for further details of equation (13).

3.1 Mortality Model

For our selection of an appropriate stochastic model for describing mortality intensity $\mu_{x+t}(t)$, we rely on the criteria listed by Cairns et al. (2008). Cairns et al. (2008) describe that a good model for stochastic mortality should have non-mean reversion, tractability and consistency with historical data. Also Luciano and Vigna (2005) find that time-homogeneous mean reverting affine processes fail to fit with the observed survival probabilities in life tables. In addition, the mortality intensity observed from the life tables does not present mean reversion behavior. In the rest of this sub-section, we introduce a non-mean reverting affine process for $\mu_{x+t}(t)$. We select the non-mean reverting affine process because it has a closed-form solution for survival probabilities and it is computationally convenient to calibrate the model to life tables. In particular, we choose a CIR process for mortality intensity. Moreover, initial $\mu_x(0)$ is provided by the Gompertz-Makeham law of mortality. More precisely,

$$d\mu_{x+t}(t) = a \cdot \mu_{x+t}(t) + \sigma_\mu \sqrt{\mu_{x+t}(t)} \cdot dW_x(t),\ \ \mu_x(0) > 0, \tag{14}$$

$$\mu_x(0) = \alpha_\mu + \beta_\mu \cdot \gamma_\mu^x, \qquad \alpha_\mu,\ \beta_\mu,\ \gamma_\mu > 0, \tag{15}$$

where $W_x(t)$ is a $\mathbb{P}-$Wiener process and $a > 0$. Further, $\sigma_\mu \geq 0$ represents the volatility of mortality intensity. When $\sigma_\mu = 0$, the evolution of the mortality intensity $\mu_{x+t}(t)$ becomes deterministic and the model coincides with the well known Gompertz model. The main advantage of the CIR process is that it ensures non-negativity of the mortality intensity. However, the process can reach 0 and stay there with positive probability. By using the Girsarov Theorem (Björk (2004)), we see that the processes:

$$dW_x^{\mathbb{Q}}(t) = l_\mu \cdot \sqrt{\mu_{x+t}(t)} + dW_x(t),$$

$$d\mu_{x+t}(t) = (a - l_\mu \cdot \sigma_\mu) \cdot \mu_{x+t}(t) + \sigma_\mu \sqrt{\mu_{x+t}(t)} \cdot dW_x^{\mathbb{Q}}(t),\ \mu_x(0) > 0, \tag{16}$$

represent a $\mathbb{Q}-$Brownian motion and the $\mathbb{Q}$ dynamics of $\mu_{x+t}(t)$. Also, l_μ is the market price of mortality risk.

In this paper, we consider a combined model which is specified by a financial model and a mortality model. We have specified our models under real world measure $\mathbb{P}$ and risk-neutral measure $\mathbb{Q}$. We need the dynamics of our models under the real measure $\mathbb{P}$ for risk management purposes (i.e. ruin probabilities). Also, we require the dynamics of our models under the risk neutral measure $\mathbb{Q}$ for the purpose of no-arbitrage valuation. The mortality derivatives market is still in its early development stage and illiquid. Hence, the mortality market is incomplete and the estimation of the market price of mortality risk l_μ is difficult (see Dahl (2004) and Cairns et al. (2008)). We assume that the risk-neutral measure exists, but it is not unique and the mortality market is free of arbitrage. Also, we set the market price of the mortality risk $l_\mu = 0$.

Under this setup, we get[1]:

$$_{T-t}p_{x+t}(t) == E^{\mathbb{Q}}[e^{-\int_t^T \mu_{x+s}(s) \cdot ds} \mid \mathcal{F}_t] = e^{B_\mu(T-t) \cdot \mu_{x+t}(t)}, \tag{17}$$

$$b_\mu = -\sqrt{a^2 + 2 \cdot \sigma_\mu^2},\ c_\mu = \frac{b_\mu + a}{2},\ d_\mu = \frac{b_\mu - a}{2},$$

$$B_\mu(T-t) = \frac{1 - e^{b_\mu \cdot (T-t)}}{c_\mu + d_\mu \cdot e^{b_\mu \cdot (T-t)}},$$

where the coefficients b_μ, c_μ, d_μ are negative. The survival probability is a decreasing function with respect to an increase in time t if and only if:

$$e^{b_\mu \cdot (T-t)} \cdot (a^2 + 2 \cdot \sigma_\mu^2) > \sigma_\mu^2 - 2 \cdot d_\mu \cdot c_\mu.$$

The density function of the remaining lifetime of the policyholder aged $x+t$ is calculated as:

$$f_{x+t}(s) = \frac{\partial B_\mu(s-t)}{\partial s} \cdot_{s-t} p_{x+t}(t) \cdot \mu_{x+t}(t), \tag{18}$$

$$\frac{\partial B_\mu(s-t)}{\partial s} = \frac{e^{b_\mu \cdot (s-t)} \cdot (c_\mu + d_\mu)}{\left(c_\mu + d_\mu \cdot e^{b_\mu \cdot (s-t)}\right)^2}$$

The values for the parameters α_μ, β_μ, γ_μ, a, and σ_μ are obtained by calibrating the model to

[1] The results follows from Luciano and Vigna (2005).

the survival probabilities obtained from the life table documented in the Human Mortality Database[2] (2013). We calibrate the model with the survival probabilities ${}_tp_{65}(0)$ of American male aged $x = 65$ at time $t = 0$. We use a least-square minimization with respect to the difference between probabilities observed in the data and the model probabilities (equation (17)). Table 1 reports the root mean square error and the estimated values of the parameters.

a	σ_μ	α_μ	β_μ	γ_μ	$error$
0.0963	0.0100	0.0030	1.32×10^{-5}	1.1094	0.0075

Table 1: Benchmark parameters for the mortality model.

The resulting parameters indicate that the mortality model is non-mean reverting and strictly positive. We plot the survival probabilities and densities of the policyholder aged 65 with respect to different values of volatility of mortality intensity σ_μ in Figure 3.1. We use the estimated parameters for the mortality model which are summarized in the Table 1. In addition, we analyze the impact of volatility σ_μ on the survival probabilities and densities. The results show that an increase in volatility σ_μ leads to an improvement in the survival probability ${}_tp_{65}(0)$ of the policyholder. Also, the high volatility of mortality intensity σ_μ leads to an increase in the uncertainty about the policyholder's death time.

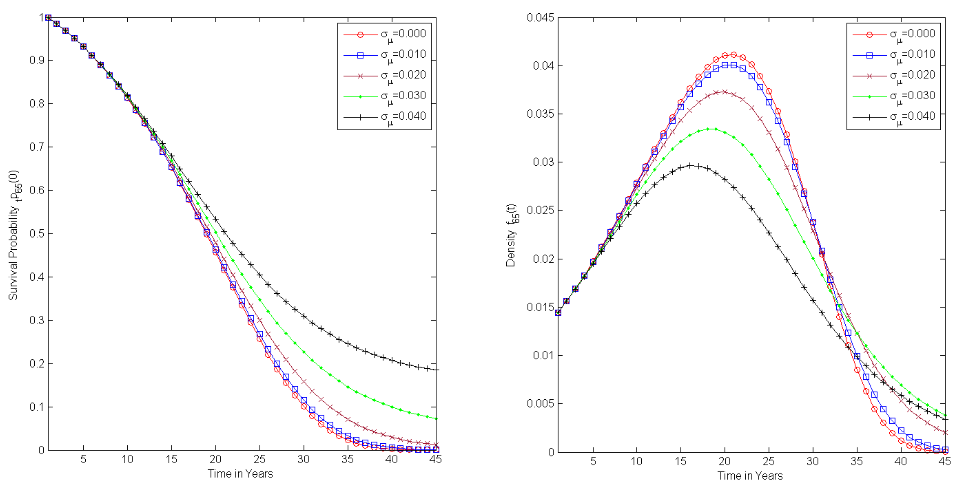

Figure 3.1: Survival probabilities and densities of remaining lifetime of the policyholder aged 65.

[2] http://www.mortality.org/

4 Pricing of GLWB

In this section, we start with the basic insurance valuation results derived by Biffis (2005). We use these results to define the no-arbitrage value of the GLWB. We can view the value of GLWB from two different perspectives (Kolkiewicz and Liu (2012), Hyndman and Wenger (2014)). The policyholder is interested in the total sum of the payments which he receives over the duration of his lifetime. This perspective was suggested by Milevsky and Salisbury (2006). On the other hand, the insurer is concerned about the guaranteed withdrawal amounts that he provides to the policyholder when the investment account has been depleted. The insurer is also concerned about the future insurance fees earned in return for guarantee provision. The second perspective was introduced by Aase and Persson (1994). Moreover, Peng et al. (2012) and Hyndman and Wenger (2014) demonstrate that both of these approaches are equivalent.

4.1 The Policyholder's Perspective

From a policyholder's perspective, the risk-neutral value at time t of GLWB can be seen as the sum of the static withdrawals made by the policyholder and the death benefits. We recall the terminology used above. We consider that the policyholder aged $x+t$ at time t and interest rate r is constant.

Living benefits are defined as the static withdrawals made by the policyholder during the lifetime of the contract. The payments from static withdrawals can be regarded as the immediate life annuity and the risk-neutral value of the immediate life annuity $V_1^{PH}(t)$ at time t is expressed as:

$$V_1^{PH}(t) = \int_t^T E^{\mathcal{Q}}[e^{-\int_t^u r \cdot ds} \cdot 1_{\{\tau_x > t\}} \cdot g \cdot I(0) \mid \mathcal{F}_t] \cdot du, \tag{19}$$

In particular, the financial risk and the mortality risk are independent and the following holds:

$$V_1^{PH}(t) = 1_{\{\tau_x > t\}} \cdot \int_t^T E^{\mathcal{Q}}[e^{-\int_t^u (r+\mu_{x+t}(s)) \cdot ds} \cdot g \cdot I(0) \mid \mathcal{I}_t] \cdot du, \tag{20}$$

$$V_1^{PH}(t) = 1_{\{\tau_x > t\}} \cdot \int_t^T E^{\mathcal{Q}}[e^{-\int_t^u r \cdot ds} \cdot g \cdot I(0) \mid \mathcal{G}_t] \cdot E^{\mathcal{Q}}[e^{-\int_t^u \mu_{x+t}(s) \cdot ds} \mid \mathcal{M}_t] \cdot du, \tag{21}$$

$$V_1^{PH}(t) = 1_{\{\tau_x > t\}} \cdot g \cdot I(0) \cdot \int_t^T e^{-r(u-t)} \cdot_{u-t} p_{x+t}(t) \cdot du, \tag{22}$$

where $1_{\{\tau_x > t\}}$ denotes the indicator function that takes the value of one if the policyholder is still alive at time t or zero otherwise. Moreover, ${}_{u-t}p_{x+t}(t)$ is the survival probability of the policyholder at time t.

The remaining amount in the investment account $I(\tau_x)$ at the policyholder's random time of death τ_x is returned to the policyholder's beneficiary. The risk-neutral value of the death benefit $V_2^{PH}(t)$ at time t is given by:

$$V_2^{PH}(t) = E^{\mathcal{Q}}[e^{-\int_t^{\tau_x} r \cdot ds} \cdot 1_{\{t < \tau_x \leq T\}} \cdot I(\tau_x) \mid \mathcal{F}_t], \tag{23}$$

In particular, the mortality risk and financial risk are independent and using the law of iterated expectations (for details see Biffis (2005)):

$$V_2^{PH}(t) = 1_{\{\tau_x > t\}} \cdot \int_t^T E^{\mathcal{Q}}[e^{-\int_t^u (r + \mu_{x+t}(s)) \cdot ds} \cdot \mu_{x+t}(s) \cdot I(u) \mid \mathcal{I}_t] \cdot du, \tag{24}$$

$$V_2^{PH}(t) = 1_{\{\tau_x > t\}} \cdot \int_t^T E^{\mathcal{Q}}\left[e^{-r(u-t)} \cdot I(u) \mid \mathcal{G}_t\right] \cdot f_{x+t}(u) \cdot du. \tag{25}$$

From a policyholder's perspective, $V_1^{PH}(t)$ and $V_2^{PH}(t)$ are future cash inflows occurring after time t. Under the first approach the value of the GLWB is defined as:

$$V^{PH}(t, \alpha_f) = V_1^{PH}(t) + V_2^{PH}(t). \tag{26}$$

A fair fee rate is a rate $\alpha_f^* \geq 0$ such that

$$V^{PH}(0, \alpha_f^*) = I(0). \tag{27}$$

Equation (27) is a self financing condition for the GLWB and it states that for the GLWB to be fairly priced, we must have an initial amount invested in the product equal to the value of an immediate life annuity plus the death benefits (Milevsky and Salisbury (2006)). The equation (27) does not have any closed form solution and numerical methods (e.g. the Secant method) must be employed to find α_f^*.

4.2 The Insurer's Perspective

The alternative viewpoint, the insurer's perspective, is to explicitly consider the benefits paid by the insurer after the value of investment account $I(t)$ has been depleted. According to the second approach, the value of the GLWB is defined as the difference between the expected benefits paid by the insurer after the investment account has depleted to zero and the expected future insurance fee (Peng et al. (2012)). Recall that the policyholder's investment account depletion time is denoted by c and the time of death of the policyholder is denoted by τ_x.

The insurer guarantees withdrawal payments to the policyholder for his entire lifetime even when the policyholder's account has been depleted at time c. The risk-neutral t-value of the expected benefit is given as:

$$V_1^I(t) = E^{\mathcal{Q}}[\int_c^T e^{-\int_t^v r \cdot ds} \cdot g \cdot I(0) \cdot dv \cdot 1_{\{t<\tau_x \leq T\}} \mid \mathcal{F}_t] \tag{28}$$

$$V_1^I(t) = E^{\mathcal{Q}}[\int_c^T e^{-r(v-t)} \cdot g \cdot I(0) \cdot dv \cdot 1_{\{t<\tau_x \leq T\}} \mid \mathcal{F}_t]$$

$$V_1^I(t) = E^{\mathcal{Q}}\left[\frac{g \cdot I(0)}{r} \cdot \left(e^{-r\cdot(c-t)} - e^{-r\cdot(T-t)}\right) \cdot 1_{\{t<\tau_x \leq T\}} \mid \mathcal{F}_t\right] \tag{29}$$

and using the law of iterated expectations and the independence assumption, we can write equation (29) as:

$$V_1^I(t) = 1_{\{\tau_x > t\}} \cdot \int_t^T E^{\mathcal{Q}}\left[\frac{g \cdot I(0)}{r} \cdot \left(e^{-r(c-t)} - e^{-r(u-t)}\right)^+ \mid \mathcal{G}_t\right] \cdot f_{x+t}(u) \cdot du \tag{30}$$

The expected insurance fee is defined as the fee revenue received by the insurer before the investment account $I(t)$ depletes to zero provided that the policyholder is still alive at time t and dies before or at time T. The risk neutral value of the expected insurance fee at time t is given as:

$$V_2^I(t) = E^{\mathcal{Q}}[\int_t^c e^{-r(v-t)} \cdot \alpha_f \cdot I(v) \cdot dv \cdot 1_{\{t<\tau_x \leq T\}} \mid \mathcal{F}_t]. \tag{31}$$

Moreover, given the independence between the financial risk and the mortality risk, and using the law of iterated expectation:

$$V_2^I(t) = 1_{\{\tau_x > t\}} \cdot \int_t^T E^{\mathcal{Q}}\left[\int_t^{u \wedge c} e^{-r(v-t)} \cdot \alpha_f \cdot I(v) \cdot dv \mid \mathcal{G}_t\right] \cdot f_{x+t}(u) \cdot du, \tag{32}$$

where $u \wedge c = \min\{u, c\}$. Under the second approach, the GLWB value $V^I(t)$ represents the risk exposure to the insurer and it is defined as:

$$V^I(t, \alpha_f) = V_1^I(t) - V_2^I(t).$$

According to Peng et al. (2012), the total value of the GLWB at time 0 plus the initial investment seen from the insurer's perspective equals to the GLWB value seen from the policyholder's perspective.

$$V^{I}(0, \alpha_f) = V^{PH}(0, \alpha_f) - I(0).$$

5 The Ruin Probability

The probability of ruin can be tracked back to 'The gambler's ruin problem' which appeared more than hundreds years ago. The motivation for the gambler's ruin problem was to find a way to compute the failure or success of a gambler who goes to the casino with a fixed amount of money and wants to leave the casino with money. Many papers in the field of finance and insurance such as Milevsky and Robinson (2000) and Huang et al. (2004) have studied the lifetime ruin probability (LPoR). Lifetime ruin probability is the probability that an individual will exhaust his wealth under a fixed periodic consumption. The ruin probability is an important risk management tool. Insurance companies use ruin probability to set risk limits, e.g. they can set certain withdrawal/consumption rates for which the ruin probability is under a certain limit. But, it is also used to analyze the performance of a policyholder's investment account and to determine how much the policyholder can consume from his investment account. In this paper, we calculate the lifetime ruin probability under the stochastic volatility and stochastic mortality. Lifetime ruin probability (LPoR) is defined as the probability that the policyholder's investment account, defined in equation (6), will hit zero while the policyholder is still alive:

$$LPoR = \mathbb{P}\left[\inf_{0 \le t \le \tau_x} I(t) = 0 \mid I(0) = P\right] \tag{33}$$

6 Numerical Results

In this section we study the effect of different financial and mortality intensity parameters on the ruin probabilities, GLWB value, and fair fee rate. We start by implementing the financial model defined in Section 1. The processes $S(t)$, $Z(t)$ and $I(t)$ are discretized by using the Euler-Maruyama method. In addition, we use the Milstein scheme for the variance process $v(t)$. The annualized historical volatility $U(\cdot)$ is computed using the EWMA method and we set the smoothing constant $\lambda = 0.94$ (Hull (2006)). The target volatility portfolio is re-balanced every 20th trading day and we set the volatility target (VT) equal to 10% and interest rate $r = 4\%$. We employ the Monte-Carlo method to approximate the ruin probabilities, the GLWB value, death benefits and living benefits. We set the number of Monte-Carlo simulations equal to 100000 for all results. The amount invested in the policyholder's investment account at time $t = 0$ is $I(0) = 100$, x is the age of the policyholder at time $t = 0$, and the maximum attainable age ω is set to 110. We summarize the Heston model parameters for the base case in Table 2 and we use the Heston parameters which are stated in Poulsen et al. (2009) and Kling et al. (2011).

κ^*	θ^*	$v(0)$	σ_v	ρ	β_H
4.75	0.22^2	θ^*	0.55	-0.569	0

Table 2: Benchmark parameters for the Heston Model.

6.1 Example of Ruin Probabilities

In this sub-section, we provide a numerical study of the lifetime ruin probability, equation (33), and we use the Monte-Carlo method to approximate the lifetime ruin probability. We follow the procedure described in Milevsky and Robinson (2000) where the lifetime ruin probability is approximated by generating a policyholder's death times τ_x and then simulating the paths of the investment account $I(t)$. Consistently with Brigo and Mercurio (2007), the time of death is modelled as the "first jump time", defined in equation (12), of the Cox process. The first jump time of the Cox process occurs as soon as the integrated mortality intensity μ_{x+t} reaches a randomly set level. For the reader's convenience, we summarize the main points of the procedure:

1. Simulate a random variable $\widehat{E_1}$ ~Exp(1).

2. Simulate the path of the mortality intensity $\mu_{x+t}(t_i)$ from $t_0 = 0 < t_1 < t_2 \cdots < t_p = \omega - x$, where $\Delta t = t_{i+1} - t_i$ and $i = 1, 2, \cdots, p$.

3. Determine a realization of the integrated mortality intensity $\psi(t) = \int_0^t \mu_{x+s}(s) \cdot ds$ at times $t = t_1, t_2 \cdots, t_p = \omega - x$ as follows:

$$\widehat{\psi}(t_k) = \sum_{i=0}^{k} \mu_{x+t_i}(t_i) \cdot \Delta t, \text{ where } k = 1, 2, \cdots, p.$$

4. Obtain a random death time $\widehat{\tau}_x$ as:

$$\widehat{\tau}_x = \min \left\{ \left\{ t_k : \widehat{\psi}(t_k) \geq E_1 \right\} \cup \{\omega - x\} \right\}.$$

5. Generate a path of the investment account $I(t_i)$ under the real measure $\mathbb{P}$ from $t_0 = 0$ to $t_p = \widehat{\tau}_x$. Set a counter variable ξ_j as ξ_j =1 if $I(\widehat{\tau}_x) = 0$ and $\xi_j = 0$ otherwise.

6. Repeat the steps (1) to (5) N times and generate N realizations $\xi_1, \xi_2, \cdots, \xi_N$ of the counter variable. Formally, we obtain the lifetime probability of ruin:

$$LPoR \approx \frac{\sum_j^N \xi_j}{N}.$$

We assume the drift ζ of the equity asset S to be 8% and we use the financial parameters which are stated in Table 2. The current market insurance fee for the GLWB is between 1−1.25% per annum (see Funga et al. (2013)). Therefore, we set the insurance fee equal to 1% per annum. We choose 2 different dynamics for the policyholder's investment account $I(t)$. We investigate the lifetime ruin probability for a variety of investment strategies involving (1) equity portfolio and (2) target volatility portfolio (TVP). More precisely, the first strategy corresponds to the case where the underlying asset is a pure equity asset S described in equation (10), i.e. $\alpha_1(t)$ is equal to 1 and $\alpha_0(t)$ is equal to zero in the equation (5). The second strategy corresponds to the case where $\alpha_1(t)$ and $\alpha_0(t)$ are defined according to sub-section 1.1.

Retirement age	Expected age at death	Annual Withdrawal Rate per 100$				
		2%	4%	6%	8%	10%
65	**83.48**	**2.8**%	**13.4**%	**28.1**%	**43.2**%	**56.5**%
70	84.21	1.6%	9.2%	21.1%	34.7%	47.3%
75	86.07	0.7%	5.5%	14.2%	25.1%	36.3%
80	88.25	0.3%	2.7%	7.9%	15.4%	24.2%

Table 3: Ruin probability approximation for a portfolio consisting of 100 percent equity at different retirement ages.

Table 3 provides the results of the lifetime probability of ruin for the equity portfolio. It shows the lifetime ruin probabilities for annual withdrawal rates ranging from 2% to 10% per 100$ initial investment. The first column of Table 3 shows the policyholder's retirement age and the second column displays the policyholder's expected age at death. The mean and standard deviation of the log-returns of the equity portfolio are 0.061 and 0.23, respectively. According to Table 3, if the policyholder retires at the age of 65 and invests 100$ in the equity portfolio and withdraws 6$ per annum, the lifetime probability of ruin is 28.1 percent. We observe from Table 3 that the ruin probabilities decrease with higher retirement age. In case the policyholder aged 65 postpones his retirement for 5 years and retires at age 70, then ruin probabilities decrease by about $20 - 30\%$. The ruin probabilities decrease because the policyholder will live shorter and there is less chance that he will consume his entire wealth in his lifetime. Table 3 indicates that lowering the desired withdrawal rates for the policyholder aged 65 from 6% to 4% can reduce the probability of ruin to 13.4%. Moreover, when the withdrawal rate is reduced to 2%, we see that the probability of ruin decreases to 2.8%. Conversely, the ruin probabilities increase when withdrawal rates are increased. For instance, if the policyholder aged 65 increases the withdrawal rate to 10%, the probability of ruin would be 56.5%, which is the unacceptably high number. This is due to the fact that the policyholder will consume more wealth and the investment account will be depleted much rapidly.

Retirement age	Expected age at death	Annual Withdrawal Rate per 100$ 2%	4%	6%	8%	10%
65	**83.48**	**0.01%**	**3.13%**	**21.24%**	**46.33%**	**64.58%**
70	84.21	0.00%	1.48%	13.29%	34.39%	52.69%
75	86.07	0.00%	0.48%	6.82%	21.82%	38.34%
80	88.25	0.00%	0.14%	2.52%	10.74%	22.87%

Table 4: Ruin probability approximation for the target volatility portfolio at different retirement ages.

For next investment choice, we consider the target volatility portfolio. Table 4 shows the ruin probabilities for various retirement ages and withdrawal rates per 100$ for the target volatility portfolio. The standard deviation and mean of the log-returns of the target volatility portfolio are 0.10 and 0.05 per annum, respectively. The main purpose of using the target volatility strategy is to keep the realized volatility of the portfolio under some control. The standard deviation of the target volatility portfolio remains around the so-called volatility target ($VT = 10\%$). When we look at the means and the standard deviations of these portfolios, the equity portfolio proves to be a high-risk and high-return portfolio whereas the target volatility portfolio is a low-risk and low-return portfolio. We observe from the Table 3 and Table 4 that the ruin probability decreases with lower standard deviation and vice versa.

Consider the base case of the policyholder aged 65 who invests in the target volatility portfolio and withdraws 6$ per year. In this case, Table 4 shows that the ruin probability is 24% lower compared to the equity portfolio. We also see that when we reduce the withdrawal rate to 4%, the risk of ruin is 3.13%. In addition, if the policyholder waits until age 70 to retire and withdraws 6$ per annum, the ruin probability drops to 13.29%. We note a general tendency that the ruin probabilities decrease for the target volatility portfolio for the instances shown at the lower left sides of tables 3 and 4. In contrast, as regards the cases that are shown toward the upper right of the tables, the ruin probabilities are higher for the target volatility portfolio as compared to the equity portfolio. The values of the ruin probability are higher due to the policyholder's consumption pattern. If the policyholder wants a high income from a portfolio, the risk of ruin can only be reduced by investing in a high risk and high return portfolio. When the policyholder invests in the equity portfolio, the ruin probability values are lower compared to the target volatility portfolio (upper right Table 3). But these numbers are still unacceptably high. If the policyholder withdraws a reasonable amount ($4-6$$ per annum), the target volatility strategy significantly reduces the ruin probabilities.

Retirement age	Tolerated Probability of Ruin 1%	5%	10%	20%
65	3.35%	4.33%	4.89%	5.82%
70	3.76%	4.78%	5.59%	6.63%
75	4.30%	5.64%	6.49%	7.83%
80	5.23%	6.74%	7.85%	9.57%

Table 5: Withdrawal amount that results in a given probability of ruin for different retirement age.

The study of the ruin probabilities can be viewed from two different perspectives. The study helps the policyholder to manage his wealth and it also explains the key links between investment decision, withdrawal rates, and retirement planning. On the other side, the ruin probabilities can be used to assess the financial risk face by insurance companies. The GLWB is an insurance product where the policyholder invests his retirement saving in an investment account. Moreover, we know that the withdrawal payments paid by the insurer after the policyholder's account value hits zero is known as expected benefit. By calculating the ruin probability, the insurance company can assess the odds of paying expected benefit. Most insurance companies impose restrictions on the asset allocation of the investment account in order to limit the effect of volatility. Hence, they can reduce the probability of ruin. The target volatility strategy effectively restricts the realized volatility of the portfolio and reduces the ruin probability for certain withdrawal rates. For instance, if the insurer offers a GLWB with target volatility portfolio and an annual withdrawal rate of 4%, there is only

3.13% chance that the insurer has to pay expected benefit.

We can invert the equation (33) and calculate the maximum withdrawal rates for a given tolerated lifetime probability of ruin. Table 5 displays the withdrawal rates for the policyholder who invests in the target volatility portfolio. We observe from Table 5 that the withdrawal rates increase when the policyholder delays his retirement age. Naturally, the withdrawal rates increase when we increase the tolerance level of ruin. For example, the policyholder aged 65 who desires a ruin probability of only 5 percent can afford to consume the maximum of 4.33$ per 100$ of his initial investment every year. And the withdrawal rate increases from 4.33$ to 4.78$ if the policyholder delays his retirement for 5 years. Also, if he is willing to tolerate a ruin probability of 10%, the maximum withdrawal rate increases from 4.33$ to 4.89$ per 100$. Similarly, the insurer can assess the risk of paying expected benefit for given withdrawal rates and can therefore offer specific withdrawal rates to the policyholder that will keep the insurer's risk within the acceptable span.

6.2 Sensitivity Analysis with Respect to Financial Risk

In this sub-section we show the impact of the equity asset stochastic volatility on the fair fee rate α_f^*. We use equation (27) to calculate the fair fee rate α_f^* and use the Secant method to find α_f^*. Again, we consider 2 different dynamics for the policyholder's investment account $I(t)$. We investigate the impact of stochastic volatility on the fair fee rate for strategies involving (1) an equity portfolio and (2) a target volatility portfolio (TVP). The policyholder invests $I(0) = 100$ in the investment account and the policyholder's age is $x = 65$ at time $t = 0$. Moreover, we test the impact of various long-term variances and speeds of mean reversions on the fair fee rate. We use the values of long-run variance and the speed of mean reversion for different values of β_H which are described by Kling et al. (2011).

Market price of risk	Speed of mean reversion κ^*	Long-term variance θ^*
$\beta_H = 2$	5.85	0.198^2
$\beta_H = 0$	4.75	0.220^2
$\beta_H = -2$	3.65	0.251^2

Table 6: $\mathbb{Q}$ – parameters for various market prices of risk.

Kling et al. (2011) describe that high values of β_H correspond to a high speed of mean reversion and a low long-term variance, and vice visa. Moreover, $\beta_H = 0$ and $\beta_H = 2$ imply a long term volatility of 22.0% and 19.8%, respectively. Also, $\beta_H = -2$ implies a long-term volatility of 25.1%.

Market price of volatility risk	Withdrawal Rate 5%	5.5%	6%
$\beta_H = 2$, $r = 4\%$	0.63%	0.96%	1.46%
$\beta_H = 0$, $r = 4\%$	**0.73%**	**1.10%**	**1.62%**
$\beta_H = -2$, $r = 4\%$	0.87%	1.30%	1.84%
$\beta_H = 0$, $r = 3\%$	1.23%	1.89%	2.92%
$\beta_H = 0$, $r = 5\%$	0.43%	0.65%	0.95%

Table 7: Fair fee rate $\alpha_f^*(\%)$ per annum for a portfolio of 100% equity under different market prices of volatility risk and interest rates.

Table 7 provides the results of the fair fee rates for the equity portfolio. Table 7 shows the fair fee rates α_f^* for annual withdrawal rates ranging from 5% to 6%. The first column of Table 7 corresponds to the market prices of volatility risk and different interest rate levels. We observe that the fair fee rates appear to be very sensitive with respect to the long-term equity asset volatility. For example, if the policyholder withdraws 5.5$ per annum then α_f^* is equal to 1.10% for $\beta_H = 0$ and $r = 4\%$. Also, α_f^* increases from 1.10% to 1.30% per annum when we increase the long-term volatility to 25%. The high long-term volatility levels dramatically increase the fair fee rate of the GLWB linked to the equity portfolio. The relationship between the fair fee rates and the long-term volatilities is positive and it is consistent with the financial theory. Financial theory suggests that options are more expensive when the volatility of the underlying asset is high. However, the fair fee rates drop by around $33 - 35\%$ when the withdrawal rates are reduced from 5.5% to 5%. This can be explained by considering equation (27) where α_f^* is defined as the fee rate that makes the initial amount equals to the living benefits plus the death benefits. Since the reduction in the withdrawal rates decrease the GLWB value (equation (26)) and α_f^* takes a lower value to balance the equation (27).

Table 8 shows the results of the α_f^* for the GLWB linked to the target volatility portfolio. We observe from Table 8 that the fair fee rates are very small for the GLWB linked to the TVP compared to the GLWB linked to the equity portfolio. For instance, for $\beta_H = 0$ which corresponds to a long-term volatility of 22%, α_f^* is 0.15% per annum for the GLWB linked to the target volatility portfolio when the policyholder withdraws 5.5$ each year. In contrast, α_f^* is equal to 0.73% for the GLWB linked to the equity portfolio. We also observe that the equity asset volatility has no influence on the fair fee rates for GLWB linked to TVP. But, the fair fee rates for the GLWB linked to equity portfolio are sensitive to long-term equity volatility levels. As we know that the target volatility portfolio consists of an equity asset S and the risk-free asset S_0. We estimate the historical volatility of the equity asset over the most recent time period. The asset weights are re-balanced according to this estimated realized volatility of the equity asset and a pre-defined volatility target level. When the equity asset volatility is high, the target volatility strategy puts more weight on the risk-free asset and vice versa, such that the overall realized volatility of the target volatility portfolio is kept under control. The volatility of the target volatility portfolio is always lower compared to the volatility of the equity portfolio. Hence, any financial derivative linked to the TVP will have a lower value. Moreover, high equity asset volatility levels will not affect the price of financial derivatives linked to TVP. Since the GLWB value is lower for the case of the target volatility portfolio, we therefore need a lower α_f^* to balance equation (27). Also, the increasing levels of equity volatility will not affect fair fee rates for the GLWB linked to TVP.

Market price of volatility risk	Withdrawal Rate 5%	5.5%	6%
$\beta_H = 2$, $r = 4\%$	0.14%	0.31%	0.68%
$\beta_H = 0$, $r = 4\%$	**0.15%**	**0.32%**	**0.69%**
$\beta_H = -2$, $r = 4\%$	0.15%	0.33%	0.69%
$\beta_H = 0$, $r = 3\%$	0.49%	0.92%	1.88%
$\beta_H = 0$, $r = 5\%$	0.07%	0.15%	0.31%

Table 8: Fair fee rate $\alpha_f^*(\%)$ per annum for the target volatility portfolio under different market prices of volatility risk and interest rates.

Contrary to the case of long-term equity volatility, the α_f^* for the GLWB linked to TVP is very sensitive to the various interest rate levels. For example, the α_f^* increases from 0.32% to 0.92% when the interest rate level drops from 4% to 3% for an annual 5.5% withdrawal rate. The present value of living benefits and death benefits increases with falling interest rate levels. As a result, the fair fee rate takes a higher value to balance the equation (27). In general, the target volatility portfolios are very sensitive to low interest rate levels. The performance of the target volatility portfolio decreases in the low interest rate environment because the target volatility strategy invests a substantial amount in the risk-free asset. And the return of the risk-free asset decreases with falling interest rates. Therefore, the low interest rate level dynamically increases the price of the financial derivatives linked to the target volatility portfolio. We observe from Table 7 and Table 8 that the low interest rate level ($r = 3\%$) increases the fair fee rate by around 200% for the GLWB linked to the target volatility portfolio. Conversely, fair fee rates for the GLWB linked to the equity portfolio have increased by around 70%.

6.3 Sensitivity Analysis with Respect to Mortality Risk

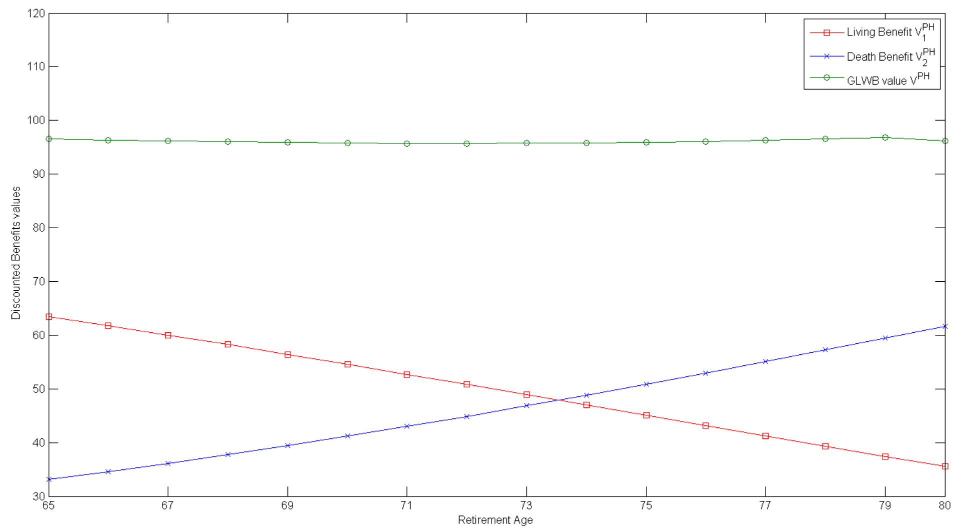

Figure 6.3 (a): The values of the GLWB, the living benefit and the death benefit for different retirement ages.

In this sub-section we analyze the relationship between the GLWB value V^{PH} and the policyholder's age x at the start of the contract. Moreover, we show the impact of different withdrawal rates g and different volatilities of the mortality intensity σ_μ on the fair fee rates α_f^* for the GLWB linked to the target volatility portfolio. We set the interest rate equals to 4% and base parameters are defined in Table 2. Figure 6.3 (a) shows the value of the GLWB for different policyholder's ages. In order to study the changes in the GLWB value with respect to age, we decompose the GLWB value into the living benefit V_1^{PH} and the death benefit V_2^{PH}. The value of the living benefit decreases as the age of the policyholder increases at the start of the contract. The living benefit value decreases because the numbers of possible withdrawals decreases. The policyholder aged 70 has a lower lifetime ruin probability as compared to the policyholder aged 65. The expected remaining lifetime decreases when we increase the retirement age. The remaining life time of the policyholder aged 70 is statistically lower and he is likely to make fewer withdrawals from his investment account compared to the policyholder aged 65. On the other hand, the value of the death benefit increases when we increase the retirement age because the death benefit is discounted for a shorter time period. The death benefit value is relatively small compared to the value of the living benefits for the policyholder aged 65. This is due to the fact that the withdrawal amount deducted from the investment account is rather high and the death benefit is discounted for a longer period. When the age of the policyholder increases at the start of the contract, the death benefit becomes more significant and the living benefit becomes less important.

Figure 6.3 (b) shows the impact of σ_μ on the fair fee rate α_f^* for a policyholder aged 65 and 70, respectively. We observe that the fair fee rate α_f^* is an exponential function with respect to the volatility of the mortality σ_μ. The fair fee rate α_f^* increases from 0.7% to 1.6% per annum when we increase the volatility of mortality σ_μ from 0 to 0.035 for the case the policyholder aged 65. We observe from the Figure 3.1 that the survival probability improves with respect to a rise in σ_μ.

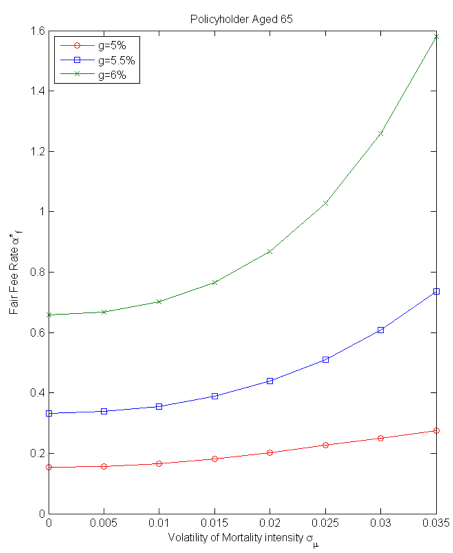

Figure 6.3 (b): Sensitivity of the fair fee rate α_f^* with respect to the volatility of mortality intensity σ_μ.

Moreover, the policyholder's life expectancy also improves with respect to an increase in σ_μ. The value of the living benefit increases exponentially as the policyholder lives longer and the number of withdrawals increases. Consequently, the amount available in the investment account I at the time of death decreases. Overall, the GLWB value V^{PH} increases as the policyholder's life expectancy increases. Therefore, the fair fee rate takes a higher value to balance the equation (27). Furthermore, the fair fee rates are significantly lower for the policyholder aged 70 compared to the policyholder aged 65. This can be explained by the fact that the policyholder aged 70 withdraws amounts from his investment account for a shorter time period compared to the policyholder aged 65. And the overall GLWB value V^{PH} for the policyholder aged 70 is smaller. As a consequence, the fair fee rate α_f^* takes a lower value to balance equation (27).

The sensitivity analysis provides two important results. First of all, we have shown that the fair fee rates for the GLWB linked to TVP are very low compared to the GLWB linked to the equity portfolio. We also study the impact of different model parameters on the fair fee rates. If the insurer failed to specify the parameters correctly at the beginning of the contract, it would lead to a significant impact on the pricing of GLWB and the insurer's profit and loss distribution. The sensitivity analysis provides the impact of different risks inherent in parameter specification. We study the impact of different model parameters on the fair fee rate and find that the fair fee rate is extremely sensitive to low interest rates. Setting correct interest rate levels should therefore be the insurer's top priority when pricing the GLWB. Table 8 suggests that the long-term equity volatility has no impact on the fair fee rates for the GLWB linked to the target volatility portfolio. However, the underestimation of the volatility of mortality intensity σ_μ can lead to incorrect pricing of the GLWB.

6.4 Profit and Loss Analysis

Case	α_f^*	Mean	Median	Std	Skewness	$\mathrm{VaR}_{0.995}$
r						
3%	1.88%	1.622	1.895	1.124	−0.416	−0.835
4%	**0.69%**	**0.507**	**0.586**	**0.447**	**−0.540**	**−0.512**
5%	0.31%	0.199	0.231	0.181	−0.550	−0.210
β_H						
2	0.69%	0.512	0.600	0.447	−0.536	−0.508
0	**0.69%**	**0.507**	**0.586**	**0.447**	**−0.540**	**−0.512**
−2	0.69%	0.502	0.581	0.448	−0.546	−0.520
σ_μ						
0.00	0.65%	0.458	0.534	0.443	−0.557	−0.524
0.010	**0.69%**	**0.507**	**0.586**	**0.447**	**−0.540**	**−0.512**
0.020	0.79%	0.589	0.690	0.449	−0.526	−0.501

Table 9: The distribution statistics for the Profit and Loss.

In order to better understand the parameter risk faced by the insurer, we simulate the profit and loss distribution for different scenarios. We perform a no-hedging strategy and all liabilities are solely financed by the fair fee rate α_f^* charged by the insurer. The insurer guarantees the periodic withdrawal for the lifetime of the policyholder even after the policyholder's investment account reaches zero. The withdrawal payments paid by the insurer after the policyholder's account has reached zero are called the expected benefits, but to the insurer the expected benefits are in fact a liability. We assume that the insurer has sold a pool of GLWB contracts. We assume that the pool consists of 1000 policyholders all aged 65. Moreover, each policyholder has invested 100$ in the target volatility portfolio and cannot change the underlying fund. We charge a α_f^* according to the result shown in the previous sub-sections. We compute 1000 random death times by simulating the paths of the mortality intensity process. Note that we use the same random death times for all computations. In addition, we assume that if a policyholder dies after his investment account value has depleted to zero, the expected insurance fee (equation (28)) charged by the insurer will be used to finance the liability. But, if a policyholder dies before the account value drops to zero, the expected insurance fee is considered as profit that grows with an interest rate r. We perform 100000 simulations to obtain the statistics for the discounted P&L and we discount the P&L at time $t = 0$. The base parameters are defined in Table 2 and we set the withdrawal rate equal to 6.5%. The results are shown in Table 8 including the fair fee rate, the mean, median, standard deviation (Std), and Value-at-Risk of P&L. The results from the base parameters are indicated in bold letters.

The results indicate that the target volatility strategy effectively restricts the realized volatility of the portfolio. Hence the long-term equity volatility has no significant impact on the P&L distribution for the GLWB linked to the target volatility portfolio. Further, the mean and standard deviation of the P&L increases when we increase the fair fee rates and vice versa. This is due to the fact that the expected insurance fee directly depends on the fair fee rate. On the other hand, the interest rate levels have significant effect on the P&L distribution. The value of the expected benefit increases with falling interest rates. Hence, the standard deviation and the VaR of the P&L distribution sharply increase with lower interest rate levels. The insurer should assess the interest rate level carefully when pricing the GLWB linked to the target volatility portfolio. We also analyze the model risk in the situation where an insurer assumes a deterministic mortality model. We impose the condition $\sigma_\mu = 0$ in equation (17). One can observe that deterministic mortality leads to incorrect pricing of the GLWB. Therefore, the mean of the P&L drops and the VaR deteriorates. Consequently, the top priority of the insurer when pricing the GLWB linked to target volatility portfolio should be the interest rate risk immediately followed by the mortality risk.

7 Conclusion

In the this paper, we present the detailed numerical analysis of the guaranteed lifetime income benefits (GLWB) linked to the target volatility portfolio. The target volatility strategy is a new risk management method that reduces the realized volatility of the portfolio and keeps the realized volatility under control. Moreover, the GLWB is the most popular type of variable annuity. It allows the policyholder to annually withdraw fixed amount from his investment account for the rest of his life, even if the value of the investment account falls to zero. The paper provides a detailed description of the target volatility strategy, GLWB, and lifetime probability of ruin for the target volatility portfolio.

The results indicate that the target volatility strategy significantly reduces the lifetime ruin probabilities. In particular, if the withdrawal rate ranges from $4-6\%$ per annum, the target volatility strategy significantly reduces the ruin probabilities. The study also helps to explain the key links between investment decision, withdrawal rates, and retirement planning.

In particular, the valuation of the GLWB linked to target volatility portfolio should be of interest to insurers and regulators. The fair fee rates for GLWB linked to TVP is around $0.14-0.68\%$ per annum and fair fee rates for GLWB linked to equity portfolio is around $0.68-1.84\%$ per annum. We found that the fair fee rates are much lower for GLWB linked to TVP compared to GLWB linked to equity portfolio. The results show that the fair fee rates for GLWB linked to TVP are insensitive to long-term equity volatility but the fair fee rates increase exponentially with decreasing interest rate levels. Also, the fair fee rates for GLWB linked to TVP increase with increasing volatility of mortality. In addition, the P&L distributions show that Value-at-Risk increases with low interest rate levels. Therefore, correct interest rate levels should be the insurer's top priority when pricing the GLWB linked to target volatility portfolio.

In the future, research could aim at extending our results to consider stochastic interest models and also analyze the impact of policyholder behavior on the fair fee rate.

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