Report RP6 - Deliverable 3.3

Report on validation and implementation of developed modelling approaches

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For more details on the papers described in the report, please contact me at the following e-mail address cpisani@econ.au.dk

Camilla Pisani
Aarhus, October 2015

Part 1

Simulation of stochastic volatility models: The Heston model and the Wishart model

The first part of this report deals with simulation schemes for the square root process and its multidimensional generalization given by the Wishart. Those two processes have many possible applications in finance such as for example as instantaneous variance in the Heston model for the one-dimensional case or as variance-covariance matrix of a group of assets for the multidimensional case. Also the Wishart can be used to model more sources of randomness in the instantaneous variance of a single asset.

In the paper *"Simple simulation schemes for CIR and Wishart processes"* [2] we propose weak second order simulation schemes for the square root process and for the Wishart process. The main idea behind them is to split the infinitesimal generator thus reducing the problem to the simulation of the square of a matrix valued Ornstein-Uhlenbeck process to be added to a deterministic process. The resulting schemes are really easy to implement and prove to work quite well, especially when compared to the commonly used Euler-Maruyama scheme. A drawback of our simulation schemes is that convergence is granted only under one condition on the parameters. In the case of the Heston model however, this constraint is less restrictive than the Feller condition.

Attached to this report you can find two working papers focussing on some applications of the simulation schemes mentioned above to the case of the Heston model and the Heston model augmented with jumps. The first working paper *"Second order discretization schemes for square root processes and applications to volatility derivatives"* deals with pricing of realized variance options and VIX options using a Monte Carlo simulation method where the instantaneous variance is simulated according to the simulation scheme in Baldi and Pisani (2013). Comparisons with inversion of Laplace transform thecniques and the formula in Lian and Zhu (2011) are provided.

In the second small paper *"The Heston model in the FX market - a calibration exercise"* we perform a "calibration exercise" of the Heston model to the FX market, again using the second order simulation schemes for the square root process in Baldi and Pisani (2013).

[2] Paolo Baldi and Camilla Pisani, International Journal of Theoretical and Applied Finance 16, 1350045 (2013) DOI: 10.1142/S0219024913500453

Report: Second order discretization schemes for square root processes and applications to volatility derivatives

Camilla Pisani†

***Abstract*—The purpose of this work is to apply the schemes suggested in Baldi and Pisani [5] to the computation of prices of volatility derivatives, in particular of options on the realized variance of an asset and VIX options. The models considered are the standard Heston [10] stochastic volatility model and the same model augmented with compound Poisson jumps in the instantaneous variance dynamics. Numerical illustrations are given in the case without jumps and with Inverse Gamma jump-sizes.**

Keywords: simulation of square root processes, processes with jumps, option pricing, realized variance, VIX options.

I. The simulation scheme

When computing prices under the Heston stochastic volatility model via Monte Carlo, application of a simple Euler Maruyama scheme to the square root process can led to negative values under the square root. This can have important consequences on the following computation of option prices. For this reason, new simulation schemes have been recently proposed (see e.g. Andersen [3], Alfonsi [2] and Baldi and Pisani [5]). Exact simulation methods are also available (for example in Broadie and Kaya [7]). Those schemes however reveal to be quite time consuming and therefore useful in practice in the case one needs to simulate the process at one or few times only.

The purpose of this paper is to investigate how the second order simulation schemes introduced in Baldi and Pisani [5] perform when applied to derivative pricing. In particular pricing of realized variance options and VIX options will be considered. Before going into details we recapitulate the main points behind the simulation technique and the definition of realized variance and VIX (for an overview of volatility derivatives we refer to Kallsen [12]). In section IV we will also see how to extend the simulation scheme when jumps are added to the square root dynamics.

Let us first focus on the Heston stochastic volatility model. We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ where $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration the price process S_t is adapted to and $\mathbb{Q}$ is a pricing measure. We assume that the risk-neutral dynamics of the log-asset price $X_t = \log(S_t)$ and its instantaneous variance v_t are given by

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_t \\ dv_t &= (a - \kappa v_t)dt + \sigma\sqrt{v_t} dB_t \end{aligned} \tag{1}$$

where the processes W and B are standard Brownian motions with correlation parameter ρ. The volatility process can be reformulated equivalently as

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dB_t \tag{2}$$

where κ is the mean reversion parameter, θ the long run and σ is the volatility of variance. Whenever $v_0 \geq 0$ and $a \geq \frac{\sigma^2}{2}$ the process is always positive. Below we summarize the scheme used, working for every set of parameters satisfying $a \geq \frac{\sigma^2}{4}$. For further explanations and details we refer to [5].

Given an interval $[0, T]$ and a regular grid on it with discretization size-step $h = \frac{T}{N}$

$$t_0 = 0 < t_1 = \frac{1}{N} < \cdots < t_N = T$$

1) call $n := \lfloor \frac{4a}{\sigma^2} \rfloor$
2) define p_1 as the transition probability associated to

$$v_{t_{i+1}} = v_{t_i} + \Big(a - n\,\frac{\sigma^2}{4}\Big)h$$

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3) call

$$\omega_{t_{i+1}} = Y_{t_i} e^{-\frac{\kappa}{2}h} + \frac{\sigma}{2}\sqrt{\psi_k(h)}\,W$$

4) and define p_2 as the transition probability associated to

$$v_{t_{i+1}} = \omega_{t_{i+1}}^T \omega_{t_{i+1}}$$

where W is a $n \times d$ matrix whose entries are independent and $N(0,1)$ distributed, Y_{t_i} denotes any $n \times 1$ matrix such that $Y_{t_i}^T Y_{t_i} = v_{t_i}$ and $\psi_k(h) = \frac{1}{k}(1 - e^{-kh})$.

5) Finally, apply

$$q(t) = p^{(2)}(\tfrac{t}{2}) \circ p^{(1)}(t) \circ p^{(2)}(\tfrac{t}{2})$$

that reveals to be a second order discretization scheme for the square root process v_t thanks to the composition rule in Theorem 1.17 in [2].

Alternative simulation schemes based on the same composition rule have been proposed in [5]. The results obtained with them are comparable to those obtained with the scheme q above. For easiness of exposition we focus on the scheme q only.

II. Realized variance options

As a first application we will focus on vanilla options written on the realized variance of an asset. Given a time interval $[0, T]$ and a grid $0 = t_0 < t_1 < \cdots < t_N = T$ on it we define *realized variance* of S_t over $[0, T]$

$$RV_N = \sum_{n=1}^{N} (X_{t_n} - X_{t_{n-1}})^2,$$

where $X_t = \log(S_t)$ denotes the log-asset price. RV_N can therefore be considered as a measure of the variation in the price realized over the time interval $[0, T]$. The realized variance is often approximated by the quadratic variation $[X]_T$ of the log-asset price

$$[X]_T = \lim_{\Pi \to 0} \sum_{n=1}^{N} (X_{t_n} - X_{t_{n-1}})^2.$$

This is justified by the fact that the realized variance converges to $[X]_T$ as the mesh-size of the time-grid, $\Pi = \sup_{n=1,\dots,N}(t_n - t_{n-1})$, goes to zero and it is proved to work quite well in practice when considering daily data (see e.g. [6]). We will adopt this assumption and indicate with the term of realized variance its continuous-time limit given by the quadratic variation. Under the Heston model the realized variance is the same as the integrated variance which is more tractable from a mathematical point of view, and easier to simulate. We shall therefore consider call options written on the annualized integrateded variance

$$V_T = \frac{1}{T}\int_0^T v_t dt \qquad (3)$$

and indicate as

$$C_{V_T}(K, T) = e^{-(r-\delta)T}\mathbb{E}\left(V_T - K\right)^+$$

the price correspondent to a strike K and a maturity T, for an interest rate r and a dividend yield δ.

A. *Some numerical illustrations for realized variance options*

In this section we perform some numerical examples showing how the simulation schemes in [5] perform with respect to a simple Euler-Maruyama scheme when applied to the computation of prices of call options on the realized variance. In order to do that we will make use of Monte Carlo simulations. In particular, we simulate the integrated variance in (3) discretizing the integral and applying the trapezoidal rule on each sub-interval. The correspondent approximation will be given by

$$V_T = \frac{1}{T}\int_0^T v_t dt = \frac{1}{T}\sum_{i=0}^{N}(t_{i+1} - t_i)v_{t_i} \qquad (4)$$

where v_{t_i} is simulated using whether the scheme in [5] or a simple Euler Maruyama scheme. In order to have a third-comparison method we make use of the formula in Carr et al. [8]. Indicating by

$$\mathcal{L}_C(u) = \mathbb{E}[e^{-uC}]$$

the Laplace transform of $C_{V_T}(K, T)$ (seen as a function of the strike price only) and by

$$\mathcal{L}_V(u) = \mathbb{E}[e^{-uV_T}]$$

the Laplace transform of V_T we have

$$\mathcal{L}_C(u) = \int_0^\infty e^{-uK} C_{V_T}(K, T) dK = \frac{\mathcal{L}_V(u) - 1}{u^2} + \frac{V_0}{u}\,. \qquad (5)$$

From the previous expression we can compute the Laplace transform of the call price and using some algorithm for the inversion of Laplace transforms we can recover the price itself. We refer to Abate et al. [1], Weeks [22] and Iseger [11] for an overview of different inversion methods. Before showing the numerical results we recall the expression of the Laplace transform of V_T under the Heston model (we refer to Lamberton and Lapeyre [14] for a proof of the formula):

$$\mathcal{L}_V(u) = \mathbb{E}[e^{-uV_T}] = e^{\alpha(u,T)+\beta(u,T)\,v_0}\ , u \geq 0$$

with functions $\alpha(u,t)$ and $\beta(u,t)$ given by

$$\begin{aligned} \alpha(u,t) &= -\tfrac{2a}{\sigma^2}\log\tfrac{(\gamma(t)-\kappa)e^{-\gamma(t)t}+\gamma(t)+\kappa}{2\gamma(t)} - \tfrac{2a}{\kappa+\gamma(t)}u \\ \beta(u,t) &= -\tfrac{2u}{t}\tfrac{1-e^{-\gamma(t)T}}{(\gamma(t)-\kappa)e^{-\gamma(t)t}+\gamma(t)+\kappa} \end{aligned}$$

and $\gamma(t) = \sqrt{\kappa^2 + 2\sigma^2 u/t}$.

Next we exhibit some numerical illustrations. For simplicity we will consider zero interest rates and zero dividend yields. Fig 1 shows a possible outcome for options with maturity T of 3-months and parameters from Bakshi et al. [4] that is $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$. We performed 10^5 simulations with discretization step of $T/10$. We also tried parameters from Kokholm and Stisen [13], namely $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$. Again the numerical illustrations confirm the goodness of our simulation scheme. Fig. 2 shows one possible outcome for a maturity of 3-months.

Increasing the maturity still proves good results as shown in Table I reporting some numerical values for different strikes and maturity of 3, 6 and 9 months. Note however that with the two choices of parameters above the condition $a = \kappa\theta \geq \frac{\sigma^2}{4}$ is satisfied. When the condition does not hold convergence of the method is not granted anymore.

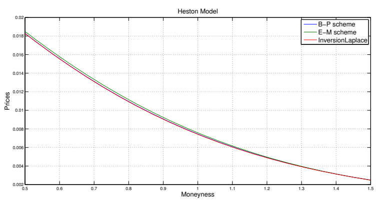

Figure 1: *Comparison of call options prices on V_T under the Heston model with $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$, $T = 3$ months.*

III. VIX OPTIONS

As a second application we study the performance of the scheme in [5] when applied to pricing of VIX options. The Chicago Board Options Exchange Volatility Index (VIX) introduced in 1993, gives a measure of the implied volatility of the S&P 500 stock index options with maturity 30

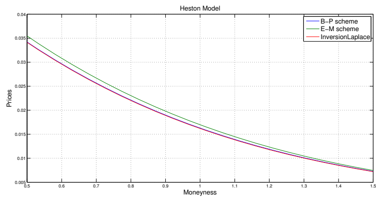

Figure 2: *Comparison of call options prices on V_T under the Heston model with $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $T = 3$ months.*

K	Prices Inversion	Prices Baldi-Pisani	Prices Euler-Maruyama
$T = 0.25$			
0.0261	0.0120	0.0121	0.0123
0.0348	0.0074	0.0074	0.0076
0.0435	0.0044	0.0044	0.0045
$T = 0.5$			
0.0261	0.0134	0.0134	0.0139
0.0348	0.0093	0.0092	0.0097
0.0435	0.0064	0.0063	0.0066
$T = 0.75$			
0.0261	0.0141	0.0140	0.0151
0.0348	0.0102	0.0101	0.0110
0.0435	0.0073	0.0073	0.0080
$T = 0.25$			
K	Prices Inversion	Prices Baldi-Pisani	Prices Euler-Maruyama
0.0476	0.0237	0.0237	0.0249
0.0635	0.0162	0.0161	0.0170
0.0794	0.0109	0.0108	0.0114
$T = 0.5$			
0.0575	0.0288	0.0290	0.0314
0.0766	0.0200	0.0202	0.0218
0.0958	0.0138	0.0139	0.0150
$T = 0.75$			
0.0642	0.0315	0.0318	0.0357
0.0856	0.0217	0.0220	0.0246
0.1070	0.0148	0.0151	0.0169

Table I: Prices of call options on V_T in the Heston model with $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$ (top) and $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$ (bottom).

days. Under few continuity assumptions the square of the VIX can be interpreted as the risk-neutral expectation of the log-contract

$$VIX_t^2 = -\frac{2}{\tau}\mathbb{E}\left[log\frac{S_{t+\tau}}{F_t^{t+\tau}}\Big|\mathcal{F}_t\right] \tag{6}$$

where $F_t^{t+\tau} = S_t e^{r\tau}$ is the 30 day forward price of the S&P500 at time t and $\tau = 30/365$. Under the dynamics (1), the expectation in (6) can be computed explicitly (see Lin [16] for a proof)

$$VIX_t = \sqrt{Av_t + B} \tag{7}$$

where

$$\begin{aligned} A &= \tfrac{1}{k\tau}(1-e^{-k\tau}), \\ B &= \theta(1-A). \end{aligned} \tag{8}$$

We will consider VIX options with price

$$C_{VIX}(K,T) = e^{-(r-\delta)T}\mathbb{E}\left(F_{VIX}(0,T) - K\right)^+$$

in correspondence to a strike K and a maturity T, where r and δ still denote the interest rate and the dividend yield respectively and $F_{VIX}(0,T)$ is the price at time 0 of a VIX future with expiration time T. It should be kept in mind indeed that the underlying in VIX options is the VIX future price rather than the VIX itself. This has important consequences in pricing but also in the application of standard formulae such as the call-put parity in which the underlying price has to be substituted by the future price. However, since the VIX future has the same expiration date as the call option and $F_{VIX}(0,T) = VIX_T$ we can rewrite the expression above as

$$C_{VIX}(K,T) = e^{-(r-\delta)T}\mathbb{E}\left(VIX_T - K\right)^+$$

For a more detailed description of VIX options we refer to Seep ([20] and [21]).

A. *Some numerical illustrations for VIX options*

In an analogous way as in section II-A we now perform some numerical examples showing how the simulation schemes in [5] perform with respect to a simple Euler-Maruyama scheme when applied to the computation of prices of VIX options. We will still apply Monte Carlo and simulate the value of the VIX from equation (7) where v_t is simulated using whether the scheme in [5] or a simple Euler Maruyama scheme. As third-comparison method we will use the formula in Lian and Zhu [15] expressing the price $C_{VIX}(K,T)$ of a VIX call option with maturity T and starting VIX value $VIX_0 = x$ as

$$C_{VIX}(K,T) = \frac{e^{-(r-\delta)T}}{2A\sqrt{\pi}} \quad (9)$$

$$\int_0^\infty Re\left[e^{\omega B/A}\mathcal{C}_v(\omega)\frac{1-erf(K\sqrt{\omega/A})}{(\sqrt{\omega/A})^3}\right]d\omega_I$$

where $\omega = \omega_R + i\omega_I$ is a complex variable, $\mathcal{C}_v$ is the characteristic function of the instantaneous variance with $v_0 = \frac{x^2-B}{A}$ and erf is the complex error function defined as

$$erf(Z) = \frac{2}{\sqrt{\pi}}\int_0^Z e^{-s^2}ds$$

A similar equation for the VIX future is also given

$$F(T, VIX_0 = x) =$$
$$\frac{1}{2A\sqrt{\pi}}\int_0^\infty Re\left[\mathcal{C}_v(\omega)\frac{1}{(\sqrt{\omega/A})^3}\right]d\omega_I$$

Moreover, under the dynamics in equation (1) it is possible to compute $\mathcal{C}_v$ explicitly

$$\mathcal{C}_v(\omega) = \mathbb{E}[e^{-\omega v_t}] = e^{\phi(\omega,t)+v_0\psi(\omega,t)}$$

with $\omega = \omega_R + i\omega_I$, $\omega_R, \omega_I \in \mathbb{R}$, $i = \sqrt{i}$ and functions $\phi(\omega,t)$ and $\psi(\omega,t)$ as below

$$\begin{aligned}\phi(\omega,t) &= \tfrac{2k\theta}{\sigma^2}\log\left(\tfrac{2ke^{tk}}{\sigma^2\omega(e^{kt}-1)+2ke^{kt}}\right)\\ \psi(\omega,t) &= -\tfrac{2k\omega}{\sigma^2 iu(e^{kt}-1)+2ke^{tk}}\end{aligned}$$

(see [14]). The numerical illustrations for VIX options also prove good results, as shown in Figure 3 and Figure 4 reporting the cases correspondent to a maturity of 3 months and parameters $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$ and $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$ respectively. Table II reports some numerical values for the same parameter sets and maturity of 3, 6 and 9 months.

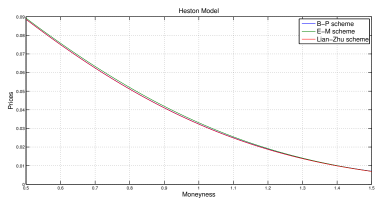

Figure 3: *Comparison of call options prices on the VIX under the Heston model with* $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$, $T = 3$ *months.*

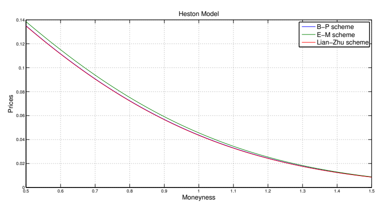

Figure 4: *Comparison of call options prices on the VIX under the Heston model with* $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $T = 3$ *months.*

IV. Extension to the case with jumps

In this section we generalize the Heston model allowing for jumps in the instantaneous variance

K	Prices Lian-Zhu	Prices Baldi-Pisani	Prices Euler-Maruyama
		$T = 0.25$	
0.1268	0.0566	0.0565	0.0573
0.1690	0.0323	0.0322	0.0329
0.2113	0.0161	0.0161	0.01641
		$T = 0.5$	
0.1268	0.0560	0.0562	0.0581
0.1690	0.0349	0.0350	0.0365
0.2113	0.0201	0.0203	0.0212
		$T = 0.75$	
0.1193	0.0596	0.0596	0.0637
0.1590	0.0397	0.0397	0.0425
0.1988	0.0251	0.0251	0.0270

K	Prices Lian-Zhu	Prices Baldi-Pisani	Prices Euler-Maruyama
		$T = 0.25$	
0.2007	0.0810	0.0811	0.0845
0.2676	0.0435	0.0436	0.0458
0.3345	0.0207	0.0207	0.0218
		$T = 0.5$	
0.2181	0.0902	0.0902	0.0973
0.2908	0.0501	0.0502	0.0551
0.3635	0.0251	0.0252	0.0280
		$T = 0.75$	
0.2269	0.0947	0.0948	0.1043
0.3026	0.0531	0.0530	0.0600
0.3783	0.0270	0.0268	0.0313

Table II: Prices of call options on the VIX in the Heston model with $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$ (top) and $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$ (bottom).

and we investigate how the simulation algorithm in [5] performs in this more general case. In particular, given a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{Q})$ with the same assumptions as for the Heston model, the joint dynamics of the log-asset price and the instantaneous volatility are given by

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_t \\ dv_t &= (a - \kappa v_t)dt + \sigma\sqrt{v_t} dB_t + dJ_t \, . \end{aligned} \tag{10}$$

where the processes W and B are standard Brownian motions with correlation parameter ρ and J is a compound Poisson process

$$J_v = \sum_{i=1}^{N(t)} Z_i, \quad Z_i \sim i.i.d.\ Z \tag{11}$$

with $N(t)$ being a Poisson process with intensity l. For more details on Compound Poisson Processes we refer to [19]. For our numerical experiments we will consider the specific case where the jump size Z follows an Inverse Gamma law $I\Gamma(\nu, \mu)$ with density

$$f_{I\Gamma}(x) = \frac{\mu^\nu}{\Gamma(\nu)} x^{-\nu-1} e^{-\mu/x} \mathbf{1}_{x\geq 0}$$

and Laplace transform

$$\mathcal{L}_Z(u) = \frac{2(\mu u)^{\nu/2}}{\Gamma(\nu)} K_\nu(\sqrt{4\mu u})$$

where K_ν is the modified Bessel function of the second kind. Under these dynamics the realized variance of the log-asset price can still be represented as integrated variance, allowing for easy simulation of the integral in equation (3) by means of the trapezoidal rule. Natural extensions allowing for jumps both in the log-asset price and in the instantaneous variance could be considered as well. Here we restrict ourselves to the case where jumps are allowed in the instantaneous variance only, as those are the dynamics the simulation scheme deals with.

In order to perform simulations of the instantaneous variance under the dynamics in (10) we make use of jump-adapted approximation schemes as suggested in Platen [17] (see also Platen and Bruti-Liberati [18]). Results on the strong convergence of these schemes are available. Remind however that our simulation scheme is assured to be only weak convergent of the second order. This does not prevent us from applying jump-adapted schemes and verify the goodness of the approximation a-posteriori based on the numerical illustrations.

Let us describe briefly how to perform the simulations. Starting from a regular time grid we simulate the Poisson process and add the jump times to the initial grid. Once built the final grid we proceed as follows: between discretization points we simulate the diffusion part using our scheme or a simple Euler-Maruyama scheme. We then add the effect of a jump whenever we encounter a jump-time.

An explicit expression for the Laplace transform of the integrated variance is also available in the case where we add jumps to the instantaneous variance and it is given by

$$\mathcal{L}_V(u) = \mathbb{E}[e^{-uV_T}] = e^{\alpha(u,T) + v_0\,\beta(u,T) + l\,\theta(u,T)}$$

for $u \geq 0$ with functions $\alpha(u,t)$, $\beta(u,t)$ and $\theta(u,t)$ defined as follows

$$\begin{aligned} \alpha(u,t) &= -\tfrac{2a}{\sigma^2} \log \tfrac{(\gamma(t)-\kappa)e^{-\gamma(t)t} + \gamma(t) + \kappa}{2\gamma(t)} - \tfrac{2a}{\kappa+\gamma(t)} u \\ \beta(u,t) &= -\tfrac{2u}{t} \tfrac{1-e^{-\gamma(t)t}}{(\gamma(t)-\kappa)e^{-\gamma(t)t} + \gamma(t) + \kappa} \\ \theta(u,t) &= \textstyle\int_0^t \left(\mathcal{L}_Z(-\beta(u,s)) - 1\right) ds \end{aligned}$$

and $\gamma(t) = \sqrt{\kappa^2 + 2\sigma^2 u/t}$ (see e.g. Duffie et al. [9]). We can therefore still apply equation (5) and use it as a third comparison method for pricing of options on V_T. Figure 5 and Figure 6 provide the numerical illustrations in correspondence to a maturity of 3-months and parameters $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$, $l = 1.5$, $\nu = 4$ $\mu = 0.03$ and $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $l = 1.5$, $\nu = 4$ $\mu = 0.03$ respectively. Few numerical values for maturities of 3, 6 and 9 months are shown in Table III.

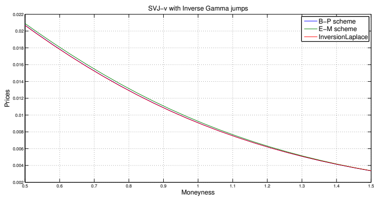

Figure 5: *Comparison of call options prices on V_T under the SVJ-v with Inverse Gamma jumps with $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$, $l = 1.5$, $nu = 3$ $mu = 0.03$, $T = 3$ months.*

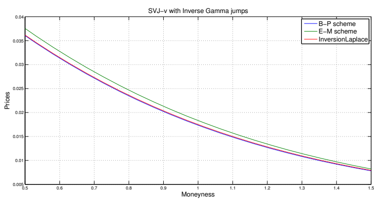

Figure 6: *Comparison of call options prices on V_T under the SVJ-v with Inverse Gamma jumps with $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $l = 1.5$, $\nu = 3$ $\mu = 0.03$, $T = 3$ months.*

At the same time an explicit expression for $\mathcal{C}_v$ is also available for the case with jumps and it is given by

$$\mathcal{C}_v(\omega) = \mathbb{E}[e^{\omega v_t}] = e^{\phi(\omega,T)+v_0\psi(\omega,T)+l\delta(\omega,T)} \tag{12}$$

(see [9] or [15] for a derivation of the formula) with functions $\phi(\omega, t)$, $\psi(\omega, t)$ and $\delta(\omega, t)$ as below

$$\begin{aligned} \phi(\omega,t) &= \frac{2k\theta}{\sigma^2}\log\left(\frac{2ke^{tk}}{\sigma^2\omega(e^{kt}-1)+2ke^{kt}}\right) \\ \psi(\omega,t) &= -\frac{2k\omega}{\sigma^2\omega(e^{kt}-1)+2ke^{tk}} \\ \delta(\omega,t) &= \int_0^t \left(\mathcal{L}_Z(-\psi(\omega,s))-1\right)ds \end{aligned}$$

allowing for an application of formula (9) also in the case with jumps but under the condition

$$0 < \omega_R < min\left(\frac{1}{\mu_V}, \frac{2k}{(1-e^{-kT})\sigma_V^2 + 2\mu_V ke^{-kT}}\right) \tag{13}$$

$T = 0.25$			
K	Prices Inversion	Prices Baldi-Pisani	Prices Euler-Maruyama
0.0261	0.0140	0.0140	0.0144
0.0348	0.0091	0.0090	0.0093
0.0435	0.0056	0.0056	0.0057
$T = 0.5$			
0.0261	0.0170	0.0171	0.0176
0.0348	0.0122	0.0123	0.0127
0.0435	0.0087	0.0087	0.0090
$T = 0.75$			
0.0261	0.0190	0.0192	0.0200
0.0348	0.0142	0.0144	0.0150
0.0435	0.0105	0.0107	0.0112

$T = 0.25$			
K	Prices Inversion	Prices Baldi-Pisani	Prices Euler-Maruyama
0.0476	0.0254	0.0255	0.0265
0.0635	0.0175	0.0176	0.0183
0.0794	0.0119	0.0120	0.0123
$T = 0.5$			
0.0575	0.0316	0.0317	0.0344
0.0766	0.0221	0.0223	0.0243
0.0958	0.0154	0.0155	0.0169
$T = 0.75$			
0.0642	0.0351	0.0350	0.0397
0.0856	0.0244	0.0243	0.0278
0.1070	0.0169	0.0167	0.0193

Table III: Prices of call options on V_T in the SVJv model with $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $l = 1.5$, $\nu = 3$ $\mu = 0.03$ (top) and $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $l = 1.5$, $\nu = 3$ $\mu = 0.03$ (bottom).

As regards the simulation of the VIX starting from the simulation of the instantaneous variance v_t instead, the equivalent of Formula 7 for the case with jumps reads as follows

$$VIX_t = \sqrt{A_J v_t + B_J} \tag{14}$$

where A_J is the same as in the case without jumps, whereas a jump component is added to B_J with respect to B in equation (8):

$$\begin{aligned} A_J &= \frac{1}{k\tau}(1-e^{-k\tau}), \\ B_J &= \left(\frac{l\mu}{k(\nu-1)}+\theta\right)(1-A_j). \end{aligned} \tag{15}$$

Numerical illustrations are given in Figure 7 and Figure 8, whereas Table IV reports few numerical values for $T = 3, 6, 9$.

V. Conclusion

We performed some numerical tests on the simulation scheme in Baldi and Pisani [5] when applied to pricing of realized variance options and VIX options. The simulation procedure considered is a weak second order scheme for CIR processes. As a natural application we therefore considered the Heston model where the instantaneous variance follows CIR dynamics. We also showed how to apply the simulation procedure to the case with jumps. The numerical illustrations provided give

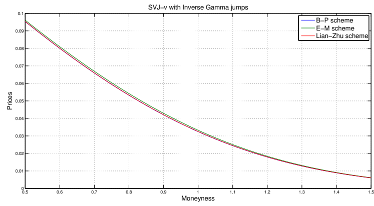

Figure 7: *Comparison of call options prices on the VIX under the SVJ-v with Inverse Gamma jumps with* $v_0 = 0.0348$, $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$, $l = 1.5$, $nu = 3$ $mu = 0.03$, $T = 3$ *months.*

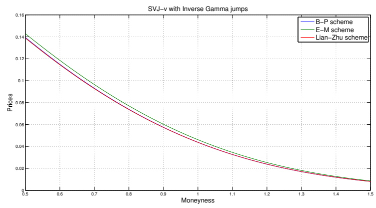

Figure 8: *Comparison of call options prices on the VIX under the SVJ-v with Inverse Gamma jumps with* $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $l = 1.5$, $\nu = 3$ $\mu = 0.03$, $T = 3$ *months.*

good results both in the case with jumps and without jumps showing improvements with respect to a simple Euler Maruyama scheme. Remind however that the simulation procedure in [5] is granted to converge only under some restrictions on the parameters. On the other hand, whenever the restriction is satisfied and applicability of the scheme is possible, this provides with good results.

K	Prices Lian-Zhu	Prices Baldi-Pisani	Prices Euler-Maruyama
		$T = 0.25$	
0.1382	0.0595	0.0598	0.0607
0.1843	0.0325	0.0328	0.0335
0.2304	0.0152	0.0154	0.0157
		$T = 0.5$	
0.1399	0.0645	0.0649	0.0607
0.1865	0.0389	0.0391	0.0407
0.2331	0.0214	0.0214	0.0225
		$T = 0.75$	
0.1424	0.0670	0.0678	0.0709
0.1898	0,0416	0.0421	0.0444
0.2373	0.0239	0.0241	0.0256

K	Prices Lian-Zhu	Prices Baldi-Pisani	Prices Euler-Maruyama
		$T = 0.25$	
0.2069	0.0828	0.0825	0.0865
0.2759	0.0438	0.0435	0.0462
0.3449	0.0202	0.0201	0.0214
		$T = 0.5$	
0.2261	0.0926	0.0928	0.0998
0.3015	0.0506	0.0506	0.0556
0.3769	0.0247	0.0246	0.0275
		$T = 0.75$	
0.2358	0.0974	0.0980	0.1070
0.3144	0.0537	0.0541	0.0606
0.3930	0.0265	0.0268	0.0307

Table IV: Prices of call options on the VIX in the SVJv model with $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $l = 1.5$, $\nu = 3$ $\mu = 0.03$ (top) and $v_0 = 0.044$, $\kappa = 2.55$, $\theta = 0.119$, $\sigma = 0.78$, $l = 1.5$, $\nu = 3$ $\mu = 0.03$ (bottom).

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Report: The Heston model in the FX market - a calibration exercise

Camilla Pisani†

The Heston model is nowadays considered as a benchmark model in derivative pricing. The simulation of the square root dynamics in the instantaneous variance through a simple Euler Maruyama scheme can present few drawbacks. For this reason alternative simulation schemes have been recently proposed. The aim of this report is to document how the simulation scheme in Baldi and Pisani [4] performs when applied to calibration in the FX market.

I. The Heston model

The Heston model ([7]) was together with the models in Hull and White [8] and Stein and Stein [11] one of the first stochastic volatility models to appear in order to represent the dynamics of financial assets. As opposite to the Black and Scholes model here the volatility of the stock price is not a constant but rather a stochastic process itself. This allows to better reproduce some features of asset prices, with respect to the case with constant volatility. Indeed, it has been empirically proved that the implied volatility, that is the volatility found when comparing market prices with Black and Scholes prices, is not constant over time and/or strikes. Moreover, the distribution of log asset prices is non-Gaussian but rather characterised by heavy tails and high peaks. These features calls for alternatives to the Black and Scholes model. We now go back to the Heston model. Under standard assumptions, the risk-neutral dynamics of the asset price S_t and its instantaneous variance v_t are given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dB_t \end{aligned}$$

where the processes W and B are standard Brownian motions with correlation parameter ρ, κ is the mean reversion parameter, θ the long run and σ is the volatility of variance. Whenever $v_0 \geq 0$ and $a = \kappa\theta \geq \frac{\sigma^2}{2}$ the process is always positive.

One of the reasons leading to the popularity of this model is probably the possibility to easily interpret the parameters of the instantaneous variance v_t. Moreover, a semi-analytical formula for pricing of options under the Heston model is available ([7]). Also, simulation schemes have been proposed (for example in [1], [2], [5], [9]). In our numerical experiments we will make use of Monte Carlo methods where the volatility is simulated using the second order discretization scheme in Baldi and Pisani [4].

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II. Simulation

When computing prices under the Heston stochastic volatility model via Monte Carlo, application of a simple Euler Maruyama scheme to the square root process can led to negative values under the square root. This can have important consequences on the following computation of option prices. For this reason, new simulation schemes have been recently proposed (see e.g. Andersen [2], Alfonsi [1] and Baldi and Pisani [4]). Exact simulation methods are also available (for example in Broadie and Kaya [5]). Those schemes however reveal to be quite time consuming and therefore useful in practice in the case one needs to simulate the process at one or few times only.

Here we use the second order simulation schemes introduced in Baldi and Pisani [4] which revealed to outperform simple Euler-Maruyama scehemes. Below we summarize the main points behind the simulation technique. We observe that results on the convergence of the schemes are proved only under the condition $a = \kappa\theta \geq \frac{\sigma^2}{4}$ which is however less restrictive than the Feller condition.

Given an interval $[0, T]$ and a regular grid on it with discretization size-step $h = \frac{T}{N}$

$$t_0 = 0 < t_1 = \frac{1}{N} < \cdots < t_N = T$$

1) call $n := \lfloor \frac{4a}{\sigma^2} \rfloor$
2) define p_1 as the the transition probability associated to
$$v_{t_{i+1}} = v_{t_i} + \Big(a - n\,\frac{\sigma^2}{4}\Big)h$$
3) call
$$\omega_{t_{i+1}} = Y_{t_i} e^{-\frac{\kappa}{2}h} + \frac{\sigma}{2}\,\sqrt{\psi_k(h)}\,W$$
4) and define p_2 as the transition probability associated to
$$v_{t_{i+1}} = \omega^T_{t_{i+1}} \omega_{t_{i+1}}$$
where W is a $n \times d$ matrix whose entries are independent and $N(0,1)$ distributed, Y_{t_i} denotes any $n \times 1$ matrix such that $Y^T_{t_i} Y_{t_i} = v_{t_i}$ and $\psi_k(h) = \frac{1}{k}(1 - e^{-kh})$.
5) Finally, apply
$$q(t) = p^{(2)}(\tfrac{t}{2}) \circ p^{(1)}(t) \circ p^{(2)}(\tfrac{t}{2})$$
that reveals to be a second order discretization scheme for the square root process v_t thanks to the composition rule in Theorem 1.17 in [1].

Alternative simulation schemes based on the same composition rule have been proposed in [4]. The results obtained with them are comparable to those obtained with the scheme q above. For easiness of exposition we focus on the scheme q only.

III. Calibration

Calibration of a model is not as easy as it might seems. Indeed, numerical issues may appear because of lack of reliability and precision in the pricing method used but also because of the calibration procedure itself. Furthermore, identification problems may appear when different parameter sets lead to similar calibration performances. For this reason many papers dealing with calibration problems have recently appeared in the literature (see e.g. Avellaneda et al. [3], Samperi [10], Christoffersen and Jacobs [6] and references therein). In order to identify the parameters of the financial model considered, one possibility is to minimize the average squared differences between market and model prices

$$\sum_{K,T} (C_{Market}(K,T) - C_{Model}(K,T))^2$$

However, it is natural to think that for example an error of 1€ has a minor impact on prices around 50€ with respect to prices around 1€. We could therefore opt for some relative error instead

$$\sum_{K,T} \left(\frac{(C_{Market}(K,T) - C_{Model}(K,T))}{C_{Market}(K,T)} \right)^2$$

Unfortunately, this choice presents few drawbacks as well, as short time to maturity OTM options which have small values with respect to other options are assigned a bigger weight in the optimization process. In order to obtain a better homoskedasticity in the data the following loss function based on implied volatilities can be used instead

$$\sum_{K,T} (ImpVol_{Market}(K,T) - ImpVol_{Model}(K,T))^2$$

For our numerical experiments we will indeed consider the last loss function expression and in order to minimize it, we will simply make use of a non-linear least squares procedure, while referring to the papers mentioned above for possible refinements or alternative calibration procedures. Optimization by non-linear least squares can be easily achieved through the MATLAB function `lsqnonlin`.

Figure 1 shows prices corresponding to a calibration of the euro/yen exchange to 3 months options, whereas Figure 2 exibiths the corresponding implied volatilities. The calibrated parameters are: $\kappa = 8.48$ $\theta = 0.02$ $\sigma = 0.72$ $\rho = -0.39$ and the relative average square error is $5 * 10^{-7}$. Figure 3 and Figure 4 show the results corresponding to a calibration on 6 months options which leads to the following parameters: $\kappa = 3.51$ $\theta = 0.02$ $\sigma = 0.51$ $\rho = -0.38$, and a relative average square error of $3.08 * 10^{-6}$. Finally, Figure 5 and Figure 6 show prices and implied volatilities corresponding to a calibration on 9 months options leading to the following parameters: $\kappa = 1.87$ $\theta = 0.02$ $\sigma = 0.4$ $\rho = -0.42$ and a relative average square error of $1.5 * 10^{-5}$.

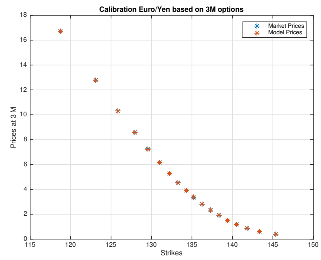

Figure 1: *Option prices on the euro/yen exchange rate under the Heston model with parameters calibrated from* 6 *months options.*

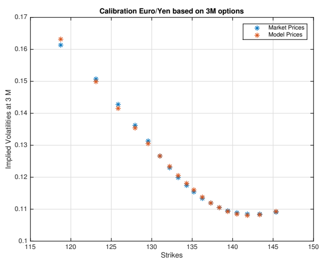

Figure 2: *Implied volatilities on the euro/yen exchange rate under the Heston model with parameters calibrated from* 6 *months options.*

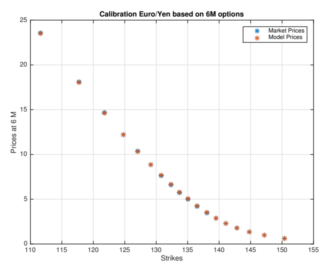

Figure 3: *Option prices on the euro/yen exchange rate under the Heston model with parameters calibrated from* 6 *months options.*

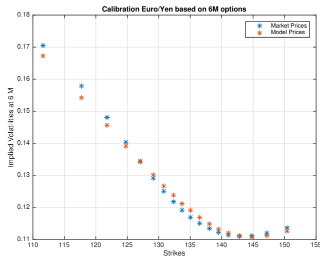

Figure 4: *Implied volatilities on the euro/yen exchange rate under the Heston model with parameters calibrated from* 6 *months options.*

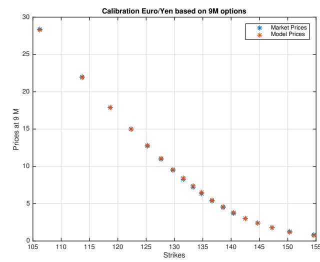

Figure 5: *Option prices on the euro/yen exchange rate under the Heston model with parameters calibrated from* 9 *months options.*

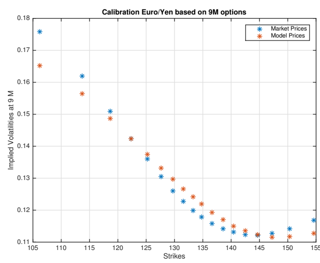

Figure 6: *Implied volatilities on the euro/yen exchange rate under the Heston model with parameters calibrated from* 9 *months options.*

IV. Conclusion

In this work we made few simple calibration exercises of the Heston model in the FX market, using the second order simulation scheme in Baldi and Pisani [4]. The aim was to show how this simulation scheme performs when applied in practical situations. Remind that the discretizations schemes used applies also in the case of the Wishart model which adding factors to the instantaneous volatility of the log-asset prices, possibly leads to improvements of the calibration performances.

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Part 2

The Impact of Jump Distributions on the Implied Volatility from Realized Variance and VIX options

The second part of this report focuses on modelling of the implied volatility curve obtained from volatility derivatives, a class of derivative securities where the payoff explicitly depends on some measure of the volatility of an underlying asset. In particular, we provide with a summary of the working paper *"The Impact of Jump distributions on the Implied Volatility of Variance"* (Nicolato, Pisani and Sloth) where we consider the Heston stochastic volatility model augmented with jumps in the instantaneous variance and we show how the particular distribution of the jumps has an impact on the qualitative shape of the implied volatility from realized variance options and VIX options.

The content of this section is naturally related to the first part of the report, as is evident for example from the paper *"Second order discretization schemes for square root processes and applications to volatility derivatives"*. In the first part we focussed on the Heston model and possible generalizations mainly from a simulation point of view, whereas in the second part we focus more on a modeling perspective as we stress the importance in the choice of the jump distribution.

Report: "The Impact of Jump Distributions on the Implied Volatility of Variance"

E. Nicolato† C. Pisani‡ D. Sloth*

***Abstract*—We consider a tractable affine stochastic volatility model that generalizes the seminal Heston [15] model by augmenting it with jumps in the instantaneous variance process. In this framework, we consider options written on the realized variance of an asset and VIX options, and we examine the impact of the distribution of jumps on the associated implied volatility smile. We provide with sufficient conditions for the asymptotic behavior of the implied volatility of variance for small and large strikes. In particular we show that, by selecting alternative jump distributions, one can obtain fundamentally different shapes of the implied volatility of variance smile – some clearly at odds with the upward-sloping volatility skew observed in variance markets.**

Keywords: Jump distributions, stochastic volatility, option pricing, realized variance, volatility derivatives.

I. Motivation

The empirical literature offers ample evidence indicating the presence of jumps both in stock prices and in their volatilities (see e.g. Eraker et al. [11], Eraker [10], Chernov et al. [8], Broadie et al. [5]). From an equity-derivatives perspective, jumps in the price level might explain the steep and negative skew that the implied volatility surface exhibits at short expiries. From the volatility-derivatives perspective, jumps in the variance level might be helpful in reproducing the upward-sloping volatility skews implied e.g. from market prices of VIX options.

However, whereas for equity derivatives the particular distribution of the jumps does not have a profound impact on the implied volatility, at least from a qualitative point of view, it does when considering options on the realized variance of an asset and VIX options.

The aim of our paper is to give some precise mathematical formulations relating the distributional properties of the jumps in the instantaneous variance with the behavior of the implied volatility of variance.

II. Realized Variance Options in the SVJ-v Model

Given a regular grid $0 = t_0 < t_1 < \cdots < t_N = T$ on the time interval $[0, T]$ we define *realized variance* of the asset S_t over the grid given as

$$V_T = \sum_{n=1}^{N} \left(\log(S_{t_n}/S_{t_{n-1}})\right)^2 ,$$

where $X_t = \log(S_t)$ denotes the log-asset price. The realized variance gives therefore a measure of the variation in the price realized over some period of time.

Common practice is to approximate the realized variance by the quadratic variation $[X]_T$

$$[X]_T = \lim_{\Pi \to 0} \sum_{n=1}^{N} (X_{t_n} - X_{t_{n-1}})^2.$$

where Π denotes the mesh-size of the time-grid, $\Pi = \sup_{n=1,\dots,N}(t_n - t_{n-1})$. This in practice works quite well for daily data as shown in Broadie and Jain [6]. From now on we therefore identify the realized variance with its continuous time limit

$$V_T = [X]_T .$$

which is more tractable from a mathematical point of view. For simplicity, we also consider an economy with zero interest rates and dividend yields.

We assume that under a risk neutral measure $\mathbb{Q}$ the log-price X and its instantaneous variance v evolve according to the following dynamics

$$\begin{aligned} dX_t &= -\tfrac{1}{2} v_t \, dt + \sqrt{v_t} dW_t \\ dv_t &= \lambda(\eta - v_t)dt + \varepsilon \sqrt{v_t} dB + dJ_t \ . \end{aligned} \qquad (1)$$

The processes W and B are possibly correlated Brownian motions while J is an increasing and

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driftless Lévy process which is independent of (W, B). Furthermore, we assume that the Lévy measure of J is absolutely continuous, and we denote by ν_J its density. Thus, the unit-time Laplace transform $\mathcal{L}_J(q) = \mathbb{E}[e^{-qJ_1}]$, takes the form

$$\mathcal{L}_J(q) = e^{\kappa_J(q)}$$

where

$$\kappa_J(q) = \int_0^\infty (e^{-qx} - 1)\nu_J(x)dx, \quad q \geq 0\,,$$

where the Lévy density is such that $\int_0^1 x\nu_J(x)dx < \infty$. Finally, the parameters λ, η and ε are non-negative constants.

The stochastic volatility model (1) generalizes the seminal Heston [15] model by augmenting the square root process describing the instantaneous variance to allow for jumps. It can also be seen as a particular instance of the double-jump stochastic volatility (SVJJ) model introduced by Duffie et al. [9] where both the instantaneous variance v and the log-price process X are affected by jumps. To emphasize the fact that, in (1), jumps are allowed only at the variance level, we refer to this model as the **SVJ-v model**. Notice that if we set $\eta = \varepsilon = 0$ in the full SVJ-v model (1) the instantaneous variance v moves uniquely by jumps

$$dv_t = -\lambda v_t dt + dJ_t. \tag{2}$$

This is the *non-Gaussian Ornstein-Uhlenbeck* (OU for short) model class proposed by Barndorff-Nielsen and Shephard[2]. Specifications of the OU type are extremely tractable from a mathematical viewpoint since the solution of (2) takes the explicit form

$$v_t = e^{-\lambda t}v_0 + \int_0^t e^{-\lambda(t-s)}dJ_s.$$

The main advantage of the general SVJ-v model (1) is that the quadratic variation coincides with the integrated variance and therefore the realized variance is given by

$$V_T = \int_0^T v_t dt.$$

In affine models, such a quantity is easy to handle as its Laplace transform is known in closed form. Duffie et al. [9] show that $\mathcal{L}_V(q,T) = \mathbb{E}[e^{-qV_T}]$ is given by

$$\mathcal{L}_V(q,T) = e^{\kappa(q,T)}$$

where

$$\kappa_V(q,T) = A(q,T) + v_0 B(q,T) + C(q,T)\,, \qquad q \geq 0$$

with the function A, B and C satisfying the ODEs

$$\frac{\partial}{\partial t}A = \eta\lambda B \tag{3}$$

$$\frac{\partial}{\partial t}B = -\lambda B + \frac{1}{2}\varepsilon^2 B^2 - q\,, \tag{4}$$

$$\frac{\partial}{\partial t}C = \kappa_J(-B)\,, \tag{5}$$

with initial conditions $A(q,0) = B(q,0) = C(q,0) = 0$. For the full SVJ-v model with $\eta, \varepsilon > 0$, the explicit solutions read as follows

$$A(q,t) = -\log\left(\frac{\gamma + \lambda + (\gamma - \lambda)e^{-\gamma t}}{2\gamma}\right)^{\frac{2\lambda\eta}{\varepsilon^2}} - \frac{\lambda\eta}{\gamma + \lambda}t$$

$$B(q,t) = -2q\frac{(1 - e^{-\gamma t})}{\gamma + \lambda + (\gamma - \lambda)e^{-\gamma t}}$$

$$C(q,t) = \int_0^t \kappa_J(-B(q,s))ds\,,$$

where

$$\gamma = \gamma(q) = \sqrt{\lambda^2 + 2\varepsilon^2 q}\,.$$

In the OU specification (2) we have that $A = 0$, while B simplifies to $B(q,t) = -q\frac{1-e^{-\lambda t}}{\lambda}$.

III. VIX OPTIONS

Since 2006 options have traded on CBOE's VIX index and constitute today a relatively liquid market of variance derivatives. The VIX index tracks the price of a portfolio of options on the S&P 500 index (SPX index). As shown by Carr and Wu [7], VIX squared approximates the conditional risk-neutral expectation of the realized variance of SPX over the next 30 calendar days. As such, it can be interpreted as the fair swap rate of a variance swap - an OTC contract in which one exchanges payments of realized variance against receiving a fixed variance swap rate.

It is immediate to show that under the general SVJ-v dynamics (1), the VIX squared - the price of future realized variance - is simply given by an affine transformation of the instantaneous variance

$$\text{VIX}_T^2 = \mathbb{E}_T\left[\tfrac{1}{\tau}\int_T^{T+\tau} v_t dt\right] = a v_T + b \tag{6}$$

where

$$\begin{aligned} a &= \tfrac{1}{\lambda\tau}(1 - e^{-\lambda\tau}), \\ b &= (\tfrac{\mathbb{E}[J_1]}{\lambda} + \eta)(1 - a) \end{aligned} \tag{7}$$

and $\tau = 30/365$. The Laplace transform of the instantaneous variance v_T takes the characteristic affine form

$$\mathcal{L}_v(u) = \mathbb{E}[e^{-uv_T}] = e^{\alpha(u,T)+\beta(u,T)v_0} \quad (8)$$

where the functions $\alpha(q,t)$ and $\beta(q,t)$ are given by

$$\beta(q,t) = \frac{2\lambda q}{\varepsilon^2 q - (2\lambda + \varepsilon^2 q)e^{\lambda t}}$$

$$\alpha(q,t) = \frac{-2\lambda\eta}{\varepsilon^2}\log\left(1 - \frac{\varepsilon^2 q}{2\lambda}(e^{-\lambda t} - 1)\right) + \int_0^t \kappa_J(-\beta(q,s))ds,$$

and κ_J is the cumulant exponent of the jump process. For the full derivation of these results, see e.g. Lian and Zhu [17].

IV. Main Tools

Let H_T be a positive random variable with finite first moment which, without loss of generality, we normalize to one, i.e., $\mathbb{E}[H_T] = 1$ and denote the distribution function, the tail function and the Laplace transform of H_T by

$$F_H(x) = \mathbb{Q}(H_T \le x), \qquad \overline{F}_H(x) = 1 - F_H(x),$$
$$\mathcal{L}_H(x) = \mathbb{E}[e^{-xH_T}],$$

$x \ge 0$. The price of a call option on H with strike K and maturity T is given by $C(K) = \mathbb{E}\,(H_T - K)^+$ whereas the corresponding put price $P(K)$ can be obtained by the put-call parity relation.

We define the implied volatility $I(K)$ associated with $C(K)$ as the solution of the equation

$$C(K) = \Phi\left(\frac{\log(1/K)}{I(K)\sqrt{T}} + \frac{I(K)\sqrt{T}}{2}\right) - K\Phi\left(\frac{\log(1/K)}{I(K)\sqrt{T}} - \frac{I(K)\sqrt{T}}{2}\right),$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal law. If the underlying variable H_T satisfies the condition $F_H(x) < 1$ for all $x > 0$, then $I(K)$ is well defined for all $K \ge 1$ and one can study the asymptotic behavior as $K \to \infty$. Similarly, if $F_H(x) > 0$ for all $x > 0$, then $I(K)$ is well defined for all $K < 1$ and it can be analyzed as $K \to 0$.

The starting point of our analysis on the behavior of $I(K)$ at extreme strikes, also called *smile wings*, is given by the two following Theorems, summarizing the results obtained in the context of equity options first by Lee [16] (who relates the smile wings to the number of moments of the underlying distribution H_T) and subsequently by Benaim and Friz ([3], [4]), and Gulisashvili ([13],[12], [14]).

The function ψ appearing in the formulations below is given by

$$\psi(x) = 2 - 4(\sqrt{x^2 + x} - x),$$

and $g(x) \sim h(x)$ means that $g(x)/h(x) \to 1$ as either $x \to 0$ or $x \to \infty$ depending on the context. Also, recall that a positive, measurable function f on $\mathbb{R}_+$ is said to be *regularly varying* at ∞ with index $\alpha \in \mathbb{R}$ if the following holds

$$\lim_{x\to\infty} \frac{f(\xi x)}{f(x)} = \xi^\alpha,$$

for all $\xi > 0$. In this case we write $f \in R_\alpha$. When $f \in R_0$, then we say that f is *slowly varying* at ∞. It can be shown that $f \in R_\alpha$ if and only if it takes the following form

$$f(x) = x^\alpha \ell(x)$$

where $\ell \in R_0$.

Let us start by considering the behavior of $I(K)$ at large strikes.

Theorem IV.1. *Assume $F_H(x) < 1$ for all $x > 0$. Then the following statements hold for the implied volatility $I(K)$ at large strikes.*

(i) *Let $\tilde{p}_H = \sup\left\{p : \mathbb{E}[H_T^{p+1}] < \infty\right\}$, then*

$$\limsup \frac{I^2(K)T}{\log(K)} = \psi(\tilde{p})$$

as $K \to \infty$.

(ii) *If $\tilde{p}_H < \infty$, then we can replace* lim sup *with the limit and write*

$$I^2(K) \sim \psi(\tilde{p}_H)\frac{\log(K)}{T} \quad as,$$

$K \to \infty$ if and only if we can find $f_1, f_2 \in R_{-\alpha}$ with $\alpha = \tilde{p}_H$, such that $f_1(K) \le C(K) \le f_2(K)$ for all $K > K_0$, with K_0 large enough.

(iii) *If $\tilde{p}_H = \infty$, then*

$$I(K) \sim \frac{1}{\sqrt{2T}}\log(K)\left(\log\frac{1}{C(K)}\right)^{-1/2}$$

as $K \to \infty$.

The large strike formula (IV.1) is derived in [16], while statements (IV.1) and (IV.1) can be found in [12].

Let us now consider the behavior of $I(K)$ at small strikes.

Theorem IV.2. *Assume $F_H(x) > 0$ for all $x > 0$. Then the following statements hold for the implied volatility $I(K)$ at small strikes.*

(i) *Let* $\tilde{q}_H = \sup \left\{q : \mathbb{E}[H_T^{-q}] < \infty\right\}$, *then*

$$\limsup \frac{I^2(K)T}{\log(H_0/K)} = \psi(\tilde{q})$$

as $K \to 0$.

(ii) *If* $\tilde{q}_H < \infty$, *then we can replace* $\limsup$ *with the limit and write*

$$I^2(K) \sim \psi(\tilde{q})\frac{\log(K)}{T}$$

as $K \to 0$, *if and only if we can find* $f_1, f_2 \in R_{-\alpha}$ *with* $\alpha = \tilde{q}_H + 1$ *such that* $f_1(1/K) \leq P(K) \leq f_2(1/K)$ *for all* $K < K_0$, *with* K_0 *small enough.*

(iii) *If* $\tilde{q}_H = \infty$, *then*

$$I(K) \sim \frac{1}{\sqrt{2T}}\left(\log\frac{1}{K}\right)\left(\log\frac{K}{P(K)}\right)^{-1/2}$$

as $K \to 0$.

Simple applications of the theorems above together with the use of Karamatas Theory of regular variation and Tauberians Theorems (relating the behavior of a positive random variable with that of its Laplace transform), allow us to carry out a precise mathematical analysis relating the behavior of the jump distribution with that of the implied volatility of variance. In particular, we provide sufficient conditions for the asymptotic behavior of the volatility of variance for small and large strikes.

V. Some numerical illustrations

As an illustrative example, we report here the case of the implied volatility from VIX options under the Heston model, the SVJ-v model with exponential jumps and the SVJ-v model with inverse Gamma jumps. We consider a maturity of 3 months.

In the first case we take parameters from Bakshi et al. ([1]) which are obtained by calibration to out-of-the-money options on the S & P 500: $v_0 = 0.0348$, $\lambda = 1.15$, $\eta = 0.0348$, $\varepsilon = 0.39$. The corresponding implied volatility exhibit a downward sloping skew, against market data.

In the second case we keep the same parameters for the diffusive part, while we augment the dynamics of the instantanous variance with Compund Poisson jumps having an exponential jump-size distribution with $l = 1.5$ and $\beta = 1/0.3429$. Still we observe a downward sloping skew.

Finally we keep the same parameters in the diffusive part, the same intensity of jumps, and the same moment and variance of the jump distribution but we change the exponential density into an Inverse Gamma density. The correspondent parameters are therefore $\nu = 4.5$, $\mu = 1.2$. The implied volatility now exhibit an upward sloping skew according to market data.

VI. Conclusion

We have considered options on realized variance and VIX options in a tractable affine stochastic volatility model that generalizes the Heston model [15] by augmenting it with jumps in the instantaneous variance. The model allowed us to isolate the unique impact of the jump distribution and to show how this has a profound effect on the characteristics and shape of the implied volatility of variance smile. We provided with sufficient conditions for the asymptotic behavior of the implied volatility of variance for small and large strikes. In particular, we showed that by selecting alternative jump distributions, one obtains fundamentally different shapes of the implied volatility smile, some clearly at odds with the upward-sloping volatility skew observed in variance markets.

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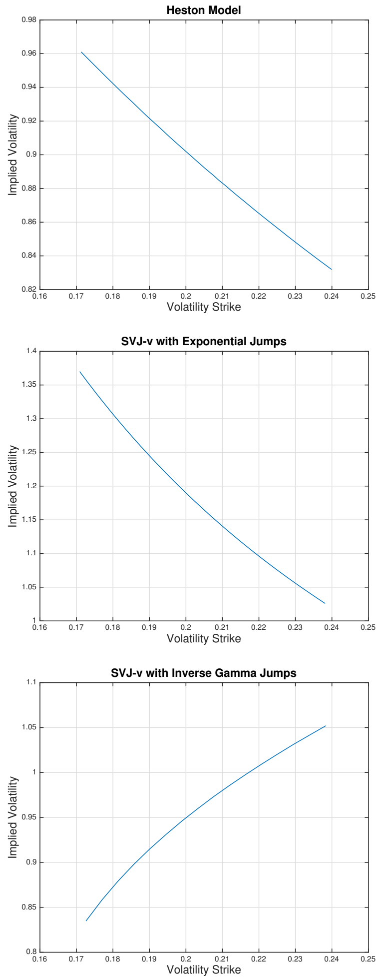

Figure 1: *Implied volatility from VIX options in the Heston model (up), the SVJ-v model with exponential jumps (middle) and the SVJ-v model with Inverse Gamma jumps (bottom)*

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Part 3

The correlation skew in the MVMD model

The last part of this report deals with some applications of the MVMD model introduced in Brigo, Mercurio and Rapisarda (2004). This is a multidimensional local volatility model, that is the volatility is a deterministic function of time and the underlying price, as opposed to the models considered in the first two parts of the report where the volatility was a stochastic process.

We report an extended abstract of the working paper: *"The Multivariate Mixture Dynamics Model: Shifted dynamics and correlation skew"* (Brigo, Pisani, Rapisarda) where we analyze the correlation skew under some generalizations of the MVMD model. As an application we consider triangular relationships in the FX market and study whether the model is able to consistently reproduce the implied volatility of cross rates, once the single components are calibrated to univariate shifted LMD models.

Report: The Multivariate Mixture Dynamics Model - Shifted dynamics and correlation skew

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The Multi Variate Mixture Dynamics model (MVMD) introduced by Brigo, Mercurio and Rapisarda [3] and recently described in a deeper way in Brigo, Rapisarda and Sridi [4] is a tractable, dynamical, arbitrage-free multivariate model characterized by transparency on the dependence structure (closed form formulae for terminal correlations, average correlations and copula funtion are available) and by complete decorrelation between assets and instantaneous variances. Each single asset is modeled according to a density-mixture model while the same property holds for the multivariate process of all assets, whose density is a mixture of multivariate basic densities.This allows for consistency of single asset and index/portfolio smile. Finally, the MVMD model can be seen as a Markovian projection of a multivariate uncertain volatility model, denominated MUVM model. While the latter is interesting from a mathematical point of view and as a tool to understand some features of the MVMD model, the projected model prevails in terms of smoothness and consistency.

In this paper we introduce a shifted MVMD model where shifted dynamics on the single assets are reconnected into a multivariate model using a similar procedure as for the non shifted case and under this model, we investigate the correlation skew. This can be considered as a descriptive tool/metric similar to the volatility smile in the one-dimensional case, with the difference that it describes implied dependence instead of volatility. Indeed, it is observed in practice under normal market conditions that assets are relatively weakly correlated with each other. However during period of market stress higher correlations (in absolute value) are observed. This fact suggests that a single correlation parameter for all options quoted on a basket of assets, or an index, say, may not be sufficient to reproduce all option prices on the basket/index for a given expiry. In fact, this is what is observed empirically when inferring a multidimensional dynamics from a set of single–asset dynamics. Among others, this has been shown in the context of equities in Bakshi et al. [2] for options on the S&P 100 index and in Langnau [5] for options on the Euro Stoxx 50 index and on the DAX index.

In a recent book, Austing [1] provided with a review on some of the most popular multi-assets products which lead to different possible definitions of the implied correlation. The definition we use is based on options having payoff

$$(S_1(T)S_2(T) - K)_+ . \tag{1}$$

This has a straightforward application in the foreign exchange market within the study of triangular relationships. Imagine for example S_1 and S_2 to represent the exchange rates USD/EUR and EUR/JPY respectively. The cross asset $S_3 = S_1 S_2$ would then represent the USD/JPY exchange rate and the corresponding payoff in equation (1) would be the payoff of a call option on the USD/JPY.

We investigate whether the model is able to consistently reproduce the implied volatility of S_3, once the single components S_1, S_2 are calibrated to univariate shifted LMD models. Consistency properties of this kind are important for example for reconstructing the time series of less liquid cross currency pairs (for instance Yen) from more liquid ones (such as Dollar and Euro).

We compare the performance of the shifted MVMD model in terms of implied correlation with those of the shifted SCMD model where

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the dynamics of the single assets are reconnected näively by introducing correlation among their Brownian motions. Finally, we introduce a model with uncertain volatilities and uncertain correlation featuring greater flexibility in reproducing a cross rate smile.

Key words: MVMD model, Mixture of densities, Multivariate local volatility, Correlation Skew, Random Correlation, Calibration, Cross rates

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