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Numerical Methods in the Pricing of Contracts: Numerical Solution for Backward SDEs

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Fourier-Hermite Expansion Algorithm for Backward SDEs

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Abstract

In this paper we are concerned with the numerical approximation of the class of Markovian backward stochastic differential equations (BSDEs) with a Gaussian prescribed terminal conditions. By developing the solution of a backward stochastic differential equation (BSDE) of our class as a Fourier-Hermite expansion in a Hilbert space, we show that the problem of solving the BSDE is equivalent to solve a system of infinite dimensional countable ordinary differential equation (CODE). From this, we derive a numerical algorithm for the BSDE via the standard Euler scheme based on the solution of the countable system of ordinary differential equations. In the last part, we provide numerical experiments to test the performance of the algorithm.

Key words: BSDE, CODE, Ordinary Differential Equation (ODE), Hermite, Martingale.
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1 Introduction

In this paper we are concerned with the numerical approximation of markovian backward stochastic differential equations (BSDEs) on a certain time interval $[0, T]$ with a Gaussian prescribed terminal conditions. Backward stochastic differential equations were first introduced by Bismut [4] in the linear case and later developed by the works of Pardoux and Peng [22]. We can find their applications in mathematical finance, in insurance, in partial differential equations studies, etc.

In the past decade, BSDEs have attracted a lot of attention and have been intensively studied in particular in stochastic optimal control. See for instance a part of the papers mentioned above El Karoui et al. [11], Cheridito et al.[7], Barles et al. [3] and several other papers. In general, many of these equations do not have an explicit or closed solution. Some efforts are made to provide some probabilistic numerical solutions. A four step scheme has been proposed by Ma et al. in [19] to solve numerically forward backward stochastic differential equations. In [2], Bally has proposed a random time discretization scheme. Discrete time approximation schemes have been also proposed by Bouchard and Touzi in [5], Chevance in [8], etc. In the last work of Chevance, the convergence needs strong regularity assumptions of the coefficients of the BSDE. Recently Gobet et al. [13] give an discrete algorithm based on Monte Carlo method to solve BSDEs. A Fourier-cosine method was proposed by Ruijter and Oosterlee in [24] and recently a convolution method by Hyndman et al. [16].

As introduced above, the purpose of this paper is to develop a probabilistic numerical scheme to solve markovian backward stochastic differential equations (BSDEs). We will discuss the particular case where the terminal condition is assumed to be a Gaussian functional. By developing the solution of the markovian BSDE as a Fourier-Hermite expansion in a Hilbert space, we show that the problem of solving the BSDE is equivalent to solve a system of infinite dimensional countable ordinary differential equation (CODE). From this, we derive a numerical algorithm for the BSDE via the standard Euler scheme based on the solution of the countable system of ordinary differential equations.

This paper proceeds as follows. In the first part of our work, we will introduce the backward stochastic differential equations and give some general results on their studies. In the next step, we will introduce the "generalised" Hermite polynomials and develop the solution of the markovian BSDE as a Fourier-Hermite expansion in a Hilbert space. We show their connection to countable system of ordinary differential equations (CODE) and derive a numerical algorithm to
solve the BSDE based on the standard Euler scheme. In the last part we provide numerical experiments to test the performance of the Euler scheme algorithm.

Notations and Assumptions

We recall the notations used in the paper of El Karoui et al. [12]. We consider a filtered probability space \((\Omega, \mathcal{F}, P, \mathbb{F})\) with \(\mathcal{F} = \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) a complete natural filtration of a \(d\)-dimensional Brownian motion \(W\) on this space and \(T\) a fixed finite horizon time. For all \(m \in \mathbb{N}^*, \) and \(x \in \mathbb{R}^m, \) \(|x|\) denotes the Euclidean norm of \(x.\) For the matrix \(A \in \mathbb{R}^{m \times d},\) we define its Shur norm by \(|A| = \sqrt{\text{Trace}(AA^*)}\). The matrix \(A\) can be considered as an element of the space \(\mathbb{R}^{m \times d}.

- \(L^2_m(\mathcal{F}_t) := \{(X_t)_{t \in [0,T]} \in \mathbb{R}^m, \mathcal{F}_t - \text{measurable} \text{ and } ||X||_{L^2} = E[||X||^2_{L^2}] < \infty\}.
- \(S^2(\mathbb{R}^m) := \{(Y_t)_{t \in [0,T]} \in \mathbb{R}^m, \text{continuous and adapted} \text{ s.t.} ||Y||^2_{S^2} = E[ \sup_{t \in [0,T]} |Y_t|^2 ] < \infty).\)
- \(H^2(\mathbb{R}^m) := \{(Z_t)_{t \in [0,T]} \in \mathbb{R}^m, \text{continuous and adapted} \text{ s.t.} ||Z||^2_{H^2} = E[\int_0^T |Z_s|^2 ds] < \infty}\).
- For \(x, y \in \mathbb{R}^k, (x, y)\) denotes the usual inner product on \(\mathbb{R}^k.\)
- \(L^2(\mathbb{N}) := \{(x_i)_{i \in \mathbb{N}}, \text{such that } \sum_i |x_i|^2 < \infty\},\) equipped by its natural inner product, \(L^2(\mathbb{N})\) is a Hilbert space. We can identify each element of the space \(L^2(\mathbb{N})\) as an infinite dimensional vector. We will use this identification and clarify the cases unless explicitly stated otherwise.
- For \(x \in \mathbb{R}^m,\) we define the gradient operator \(\nabla_\times := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}).\)
- The first assumptions are:
  \[
  \text{(H)} \begin{cases}
    \text{there exists a positive constant } K > 0, \text{ such that } dt \otimes dP \text{ a.s.,} \\
    \text{(H1)} |g(t, x_1, y_1, z_1) - g(t, x_2, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|), \forall (x_i, y_i, z_i)_{i=1,2}, \\
    \text{(H2)} \sup_{t \in [0, T]} |g(t, 0, 0, 0)| \leq K, \\
    \text{(H3)} \text{the function } \phi \text{ is Lipschitz.}
  \end{cases}
  \]
- All the equalities and inequalities between random variables are understood in the \(P\)-almost sure sense unless explicitly stated otherwise.
2 Definitions and Estimation

In this section, we introduce the general concept of backward stochastic differential equations with respect to the standard Brownian motion. In the last part we recall a classical estimation of backward stochastic differential equations theory which is very useful for our further work. Considering a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$, backward stochastic differential equations are a new class of stochastic differential equations. The main difference is that these equations are specified with a prescribed terminal value as shown in the following equation

$$
\begin{cases}
-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t; & 0 \leq t < T \\
Y_T = \xi & t = T.
\end{cases}
$$

(2.1)

The previous system can be written equivalently as the following integral equation

$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,
$$

(2.2)

where

- $\xi$ is the terminal condition of the equation (2.1). $\xi$ is also assumed to be an $\mathcal{F}_T$-measurable and square integrable random variable,

- the mapping $(t, y, z) \mapsto f(t, y, z)$ is called in general the generator.

Assuming that the generator $f$ is measurable, a solution of the backward stochastic differential equations (2.1) is a couple of progressively measurable processes $(Y, Z)$ such that:

$$
\begin{cases}
\text{i) } \int_0^T |Z_s|^2 ds < \infty \text{ and } \int_0^T |f(s, Y_s, Z_s)| ds < \infty, \\
\text{ii) } (Y_t, Z_t) \text{ satisfies the equation } (2.1).
\end{cases}
$$

(2.3)

Under the assumptions (H1), (H2) one can shows that

$$(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^{m \times d}).$$

In general, we don not have a unique solution to the equation (2.1). The existence and uniqueness of the solution can be assumed by the conditions of Pardoux and Peng [21].
Remark 2.1. If the generator function $f$ is identically equal to zero, the backward stochastic differential equations (2.2) is reduced to the following stochastic equation. In this case we have

$$Y_t = \xi - \int_t^T Z_s dW_s.$$  

This previous simplification of the backward stochastic differential equation (2.2) can be associated to the martingale representation theorem in the filtration generated by the Brownian motion. In this case, the solution $Y$ is a martingale and we have

$$Y_t = \mathbb{E}(\xi|\mathcal{F}_t).$$

Proposition 2.1. Pardoux and Peng [21].

If the function $f$ satisfies the assumptions (H1) and (H2), then the couple $(Y, Z)$ satisfies the inequality

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_s|^2 ds \right) \leq \mathbb{E} \left( |\xi|^2 + \int_0^T |f(t, 0, 0)|^2 ds \right). \quad (2.4)$$

The previous inequality shows how the solution of the BSDE is governed by the terminal condition and the generator function.

3 Backward Stochastic Differential Equations and Hermite polynomials

The purpose of this paper is to develop a probabilistic numerical scheme to solve the Markovian BSDE (3.5) where the terminal condition is assumed to be a Gaussian functional. More precisely, we assume that the terminal condition $\xi$ has the following representation $\xi := \Phi(W_T)$. The function $\Phi$ is smooth enough and $W_T$ denotes the Brownian motion at the horizon time $T$. In this section, we will introduce Hermite polynomials, list some useful properties of these polynomials and highlight their connection to BSDE via the conditional expectation operator. Let us consider the following BSDE

$$
\begin{cases}
    -dY_t = g(t, W_t, Y_t, Z_t) dt - Z_t dW_t, & 0 \leq t < T, \\
    Y_T = \phi(W_T) & t = T.
\end{cases}
\quad (3.5)
$$

From the work of Pardoux and Peng [22], we can represent the couple of process $(Y, Z)$ by the solution $u$ of the below parabolic partial differential equation ($\ast$)

$$
\forall t \in [0, T], \quad Y_t = u(t, W_t) \quad \text{and} \quad Z_t = (\nabla_x u)(t, W_t),
$$
where the function $u$ solves the parabolic partial differential equation
\[
(*) \quad \begin{cases}
\dfrac{\partial u}{\partial t}(t, x) + \frac{1}{2} \Delta u(t, x) + g(t, x, u, \nabla u) = 0 \\
u(T, x) = \Phi(x), \quad \text{with} \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\end{cases}
\]
The term $\nabla u$ denotes the gradient of $u$ with respect to the variable $x$ and the differential operator $\Delta$ denotes the Laplacian operator,
\[
\Delta u(t, x) =: \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(t, x).
\]
Our work will be focus on the numerical solution of the BSDE (3.5) in one-dimensional setting via a system of infinite dimensional countable ordinary differential equation.

### 3.1 Hermite Polynomials and Martingales

The hermite polynomials appear and are used in many areas such as physics, chemistry, mathematics, etc. These polynomials appear naturally in the study of the propagation of the heat equation, in the study of quantum harmonic oscillator, etc. The most famous application of Hermite polynomials is in the Schrödinger theory of quantum physics. The system of the probabilist’s Hermite polynomials $(H_n(x))_{n \in \mathbb{N}}$ can be easily defined by the Rodrigues’s formula
\[
H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad \text{with} \quad n \geq 1 \text{ and } H_0 = 1.
\]
for every $n$ positive integer. The components of the system $(H_n(x))_{n \in \mathbb{N}}$ are orthogonal polynomials with respect to the Gaussian weight function $\phi(x) = \exp(-\frac{1}{2}x^2)/\sqrt{2\pi}$, $x \in \mathbb{R}$. Hence for $(n, m) \in \mathbb{N}^2$, any pair $(H_n(x), H_m(x))$ satisfies the orthogonality relationship
\[
\int_{-\infty}^{\infty} H_n(x)H_m(x)\phi(x) \, dx = \sqrt{n!m!} \delta_{nm}, \quad \delta_{nm} = \|n=m|, \quad (3.6)
\]
where $\delta_{nm}$ denotes the Kronecker symbols. The first few Hermite polynomials are:

\[
\begin{align*}
H_0(x) &= 1 \\
H_1(x) &= x \\
H_2(x) &= x^2 - 1 \\
H_3(x) &= x^3 - 3x \\
H_4(x) &= x^4 - 6x^2 + 3 \\
H_5(x) &= x^5 - 10x^3 + 15x.
\end{align*}
\]

In general the Hermite polynomials satisfy the following recursion relation

\[
H_{n+1}(x) = xH_n(x) - nH_{n-1}(x). \tag{3.7}
\]

This recursion relationship is very useful for generating values of $H_n(x)$ for a given $x$ in a fast way. If we assume that $H_n$ has the following representation $H_n(x) = \sum_{k=0}^{n} a_{n,k}x^k$ where $a_{n,k} \in \mathbb{R}$, then we can deduce immediately from (3.7) that the individual coefficients satisfy the recursion formula

\[
a_{n+1,k} = a_{n,k-1} - na_{n-1,k} \quad \text{with} \quad a_{0,0} = 1 \quad \text{and} \quad a_{n,-1} = 0 \quad \forall n \geq 0. \tag{3.8}
\]

We can also integrate Hermite polynomials analytically against any Gaussian density. For this reason we need first the non-central moments of a Gaussian random variable $Z \sim \mathcal{N}(\mu, \sigma)$, which are given by

\[
\int_{-\infty}^{\infty} z^n e^{-\frac{1}{2} \left(\frac{z-\mu}{\sigma}\right)^2} \, dz = \mathbb{E}[Z^n] = (i\sigma)^n H_n \left( \frac{\mu}{i\sigma} \right). \tag{3.9}
\]

Although the last expression involves the imaginary unit $i$, the outcome is always a real number since the complex numbers cancel out for each integer value of $n$. We will introduce a new notation for a "generalised" Hermite polynomial:

\[
H_n^{[\theta]}(x) := \theta^{\frac{n}{2}} H_n \left( \frac{x}{\sqrt{\theta}} \right), \tag{3.10}
\]

which gives (as stated above) also a real value for any $\theta > 0$. Taking the limit for $\theta \to 0$ we obtain the result

\[
H_n^{[0]}(x) = x^n. \tag{3.11}
\]
The "generalised" Hermite polynomials satisfy the orthogonality relationship

$$
\int_{-\infty}^{\infty} H_n^{[0]}(x)H_m^{[0]}(x) \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi \theta}} \, dx = \theta^n n! \delta_{nm},
$$

(3.12)

with respect to the Gaussian density with mean 0 and variance \( \theta \). We can use formula (3.9) for the non-central moments to derive the result

$$
\int_{-\infty}^{\infty} H_n(z) \frac{e^{-\frac{1}{2}(\frac{z-\mu}{\sigma})^2}}{\sqrt{2\pi \sigma^2}} \, dz = E \left[ \sum_{k=0}^{n} a_{n,k} Z^k \right] = \sum_{k=0}^{n} a_{n,k} H_k^{[-\sigma^2]}(\mu).
$$

(3.13)

In the last inequality, we have extended and used the definition (3.10) for imaginary number. The last expression can be simplified by using the umbral composition formula for (generalised) Hermite polynomials

$$
\sum_{k=0}^{n} a_{n,k} H_k^{[-\sigma^2]}(\mu) = \left( H_n^{[1]} \circ H^{[-\sigma^2]} \right)(\mu) = H_n^{[1-\sigma^2]}(\mu).
$$

(3.14)

Again we note that the last expression also yields the correct answer for \( \sigma^2 > 1 \). Using the same type of derivation, we can generalise this result to

$$
\int_{-\infty}^{\infty} H_n^{[0]}(z) \frac{e^{-\frac{1}{2}(-\sigma z)^2}}{\sqrt{2\pi \sigma^2}} \, dz = H_n^{[0-\sigma^2]}(\mu).
$$

(3.15)

For the generalised Hermite polynomials, we have the following addition formula

$$
H_n^{[0]}(x + y) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} H_k^{[0]}(x).
$$

(3.16)

The last formula can be proven directly via a Taylor expansion of \( H_n^{[0]}(x + y) \). We also have the following integration formula for the product of three Hermite polynomials:

$$
\int_{-\infty}^{\infty} \frac{e^{-t^2}}{\sqrt{\pi}} H_l(t)H_m(t)H_n(t) \, dt = \frac{2^{\frac{1}{2}(1+m+n)!}m!n!}{(\frac{1}{2}(l+m-n))! (\frac{1}{2}(m+n-l))! (\frac{1}{2}(l+n-m))!}
$$

with \( l + m + n \) even, and \( l + m \geq n \) and \( m + n \geq l \) and \( l + n \geq m \). This is still in raw form for physicists Hermite polynomials.

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1Source: [http://functions.wolfram.com/Polynomials/HermiteH/21/02/01/0004/](http://functions.wolfram.com/Polynomials/HermiteH/21/02/01/0004/)
From the generalised Hermite polynomials expression, it is easy to see that $H_n^{[t]}(W_t)$
define a sequence of martingale. Applying formula (3.15), we obtain immediately that

$$
\mathbb{E} \left[ H_n^{[T]}(W_T) \middle| \mathcal{F}_t \right] = H_n^{[t]}(W_t) \quad \forall \ 0 \leq t \leq T.
$$  \hspace{1cm} (3.18)

We will call these *Hermite martingales*. For each *Hermite martingale*, we obtain the explicit Martingale Representation formula

$$
H_n^{[T]}(W_T) - H_n^{[t]}(W_t) = \int_t^T nH_n^{[s]}(W_s) \, dW_s,
$$  \hspace{1cm} (3.19)

where we have used the fact that $\partial H_n^{[t]}(W_t)/\partial x = nH_n^{[t-1]}(W_t)$. Taking the limit
for $\theta \to 0$ we obtain the result

$$
H_n^{[0]}(x) = x^n.
$$  \hspace{1cm} (3.20)

For a positive $\theta$, we introduce below the normalised system $(\tilde{H}_n^{[\theta]} \theta \geq 0)$. Each element
$\tilde{H}_n^{[0]}$ of the system $(\tilde{H}_n^{[\theta]} \theta \geq 0)$ is defined by the following normalization

$$
\tilde{H}_n^{[\theta]}(x) := \frac{1}{\sqrt{\theta^n n!}} H_n^{[\theta]}(x), \quad n \in \mathbb{N}.
$$

This system of the generalized and normalised Hermite polynomials satisfies

$$
\int_{-\infty}^{\infty} \tilde{H}_n^{[\theta]}(x) \tilde{H}_m^{[\theta]}(x) \frac{e^{-\frac{1}{2} x^2}}{\sqrt{2\pi \theta}} \, dx = \delta_{nm}, \quad m, n \in \mathbb{N}.
$$  \hspace{1cm} (3.21)

and follows the martingale equality

$$
\mathbb{E} \left[ \tilde{H}_n^{[T]}(W_T) \middle| \mathcal{F}_t \right] = \left( \frac{t}{T} \right)^{n/2} \tilde{H}_n^{[t]}(W_t), \quad \forall \ 0 \leq t \leq T.
$$  \hspace{1cm} (3.22)

The partial derivative with respect to the space variable of the normalized polynomials is given by

$$
\partial_x \tilde{H}_n^{[\theta]}(x) = \left( \frac{n}{\theta} \right)^{1/2} \tilde{H}_n^{[\theta]}(x).
$$  \hspace{1cm} (3.23)

We can express any Gaussian functional random variable $Y(T, W_T)$ as

$$
Y(T, W_T) = \sum_{k=0}^{\infty} \alpha_k(T) \tilde{H}_k^{[T]}(W_T) \quad \text{with} \quad \alpha_k(T) := \mathbb{E} \left[ Y(T, W_T) \tilde{H}_k^{[T]}(W_T) \right].
$$  \hspace{1cm} (3.24)
where the sequence $\alpha$ belongs to the space $L^2(\mathbb{N})$. The above series is assumed
to converge in $L^2_*(\mathcal{F}_T)$ sense. If we define the process $Y(t, W_t)$ as the conditional
expectation of $Y(T, W_T)$, we obtain the following expression due to the martingale
equality (3.22).

$$Y(t, W_t) = \mathbb{E} \left[ \sum_{k=0}^{\infty} \alpha_k(T) \mathcal{H}_k^{[T]}(W_T) | F_t \right] = \sum_{k=0}^{\infty} \left( \frac{t}{T} \right)^{K/2} \alpha_k(T) \mathcal{H}_k^{[t]}(W_t). \quad (3.25)$$

On the other hand, if we apply formula (3.24) to the random variable $Y(t, W_t)$, we obtain (again using the orthogonality of $\mathcal{H}_k^{[t]}(W_t)$)

$$Y(t, W_t) = \sum_{k=0}^{\infty} \alpha_k(t) \mathcal{H}_k^{[t]}(W_t) \quad \text{with} \quad \alpha_k(t) := \mathbb{E} \left[ Y(t, W_t) \mathcal{H}_k^{[t]}(W_t) \right]. \quad (3.26)$$

By identification, we obtain additionally the linear relation

$$\alpha_k(t) = \left( \frac{t}{T} \right)^{K/2} \alpha_k(T), \quad \forall t \leq T. \quad (3.27)$$

Furthermore, in the system of the generalized and orthogonal Hermite polynomials, we can also express the Gaussian functional random variable $Y(T, W_T)$ as

$$Y(T, W_T) = \sum_{k=0}^{\infty} y_k(T) \mathcal{H}_k^{[T]}(W_T) \quad \text{with} \quad y_k(T) := \frac{1}{\sqrt{T^k k!}} \mathbb{E} \left[ Y(T, W_T) \mathcal{H}_k^{[T]}(W_T) \right]. \quad (3.28)$$

The sequence $(y_k)_{k \in \mathbb{N}}$ belongs to $L^2(\mathbb{N})$ and the convergence of the series is un-
derstood in $L^2_*(\mathcal{F}_T)$ sense. With the same logic as above, we decompose $Y(t, W_t)$ as

$$Y(t, W_t) = \sum_{k=0}^{\infty} y_k(t) \mathcal{H}_k^{[t]}(W_t) \quad \text{with} \quad y_k(t) := \frac{1}{\sqrt{t^k k!}} \mathbb{E} \left[ Y(t, W_t) \mathcal{H}_k^{[t]}(W_t) \right]. \quad (3.29)$$

On the other hand, if we define the process $Y(t, W_t) := \sum_{k=0}^{\infty} y_k(t) \mathcal{H}_k^{[t]}(W_t)$ as the conditional expectation of $Y(T, W_T)$, we obtain

$$Y(t, W_t) = \mathbb{E} \left[ \sum_{k=0}^{\infty} y_k(T) \mathcal{H}_k^{[T]}(W_T) | F_t \right] = \sum_{k=0}^{\infty} y_k(T) \mathcal{H}_k^{[t]}(W_t). \quad (3.30)$$

By identification:

$$y_k(t) = y_k(T), \quad \forall k \in \mathbb{N} \quad \forall t \in [0, T]. \quad (3.31)$$
We can interpret this result as follows: for all \( t \leq T \) the coefficients \( y_k(t) \) of the conditional expectation process \( Y(t, W_t) \) trace a \textit{deterministic} path in our “Hermite space”. Even though the process is stochastic, the coefficients \( y_k(t) \) are deterministic functions of time. For the conditional expectation process, the paths are simply the constant values \( y_k(T) \). In the case of the normalized system of the Hermite polynomials, the result (3.27) leads us to a similar interpretation.

### 3.2 Backward Stochastic Differential Equations in Hermite Spaces

In this section, we consider the solution of BSDE (3.5) in the previous Hermite space. We call Hermite space the set generated by the system of the generalised Hermite Polynomials basis (orthogonal or normalised). Each process of the couple \((Y, Z)\) is Markovian in the state \((t, W_t)\).

In the normalised Hermite basis system \((\tilde{H}_k)^t_{t\in[0,T]} \) (see (3.21)), by assuming that the function \((k, t) \mapsto (k+1)^{1/2} \alpha_{k+1}(t)\) is continuous on \( N \otimes [0, T] \), we represent the solution of the BSDE (3.5) as the following series. For a fixed \( t \) in the interval \([0, T]\),

\[
\begin{align*}
Y_t &= \sum_{k \geq 0} \alpha_k(t) \tilde{H}_k^t(W_t) \quad \text{a.s.}, \\
Z_t &= \sum_{k \geq 0} \beta_k(t) \tilde{H}_k^t(W_t) = \sum_{k \geq 0} \left( \frac{k+1}{t} \right)^{1/2} \alpha_{k+1}(t) \tilde{H}_k^t(W_t) \quad \text{a.s.},
\end{align*}
\]

(3.32)

where \( \alpha, \beta \in L^2(N) \) and \( \beta_k(t) = (k+1)^{1/2} \alpha_{k+1}(t) \). The above series are also assumed to converge in \( L^2_t(F_t) \) sense. In other words for \( Y_t \),

\[
\mathbb{E} \left| Y_t - \sum_{0 \leq k \leq N} \alpha_k(t) \tilde{H}_k^t(W_t) \right|^2 \to 0 \quad \text{as} \quad N \to \infty.
\]

Furthermore, \( g(t, W_t, Y_t, Z_t) \) define a Markovian process that depends on the couple \((t, W_t)\) only and therefore can also be expressed modulo convergence as the series

\[
g(t, W_t, Y_t, Z_t) = \sum_{k=0}^{\infty} \gamma_k(t) \tilde{H}_k^t(W_t) \quad \text{with} \quad \gamma_k(t) := \mathbb{E} \left[ g(t, W_t, Y_t, Z_t) \tilde{H}_k^t(W_t) \right].
\]

(3.33)

For each \( t \in [0, T] \), the coefficients \( \gamma_k(t) \) are deterministic functions (via \( Y_t \) and \( Z_t \)) of the \( \alpha(t) \)'s.
By integrating the equation (3.5) and taking the conditional expectation, we have

$$\mathbb{E}(Y_t|\mathcal{F}_t) - Y_t + \mathbb{E}\left(\int_t^T g(s, W_s, Y_s, Z_s) \, ds \mid \mathcal{F}_t\right) = 0. \quad (3.34)$$

By the previous decomposition of $Y, Z, g$ in the hermite space, we have

$$\sum_{k=0}^{\infty} \left( \frac{\alpha_k(T)}{\sqrt{T^k k!}} - \frac{\alpha_k(t)}{\sqrt{t^k k!}} + \int_t^T \frac{\gamma_k(s)}{\sqrt{s^k k!}} \, ds \right) H_k^{[t]}(W_t) = 0. \quad (3.35)$$

This equation can only be equal to zero for all $H_k^{[t]}$, if each of the coefficients in front of the Hermite basis-functions is equal to zero. Therefore, we obtain the result that the $\alpha_k(t)$’s must be the solution to the following deterministic integral equation

$$\left( \frac{t}{T} \right)^{k/2} \alpha_k(T) - \alpha_k(t) + \int_t^T \left( \frac{t}{s} \right)^{k/2} \gamma_k(s, \alpha_k(s)) \, ds = 0, \quad k = 0, 1, 2, \ldots \quad (3.36)$$

This system can also be expressed as a (countably infinite) system of ordinary differential equations by the partial differentiation of the previous equality. We have

$$t \dot{\alpha}_k(t) - \frac{k}{2} \alpha_k(t) + t \gamma_k(t, \alpha(t)) = 0, \quad \text{with} \quad k = 0, 1, 2, \ldots \quad (3.37)$$

This system (3.37) is defined for $t \in [0, T]$ and the boundary condition $(\alpha_k(T))_{k \in \mathbb{N}}$ at the horizon time $T$. The function $\dot{\alpha}_k$ denotes the time-derivative of the function $\alpha_k$. Furthermore, we use the notation $\gamma_k(t, \alpha(t))$ to emphasise that the $\gamma_k$’s are functions of the $\alpha_k$’s. The couple $(Y, g)$ can be represented by the couple of family $(\alpha_k(t), \gamma_k(t))_{k \in \mathbb{N}}$ at the time instance $t \in [0, T]$. At $t$, the solution $(Y_t, Z_t)$ of the BSDE (3.5) exists if and only if the countable system of ordinary differential equations (3.37) has a solution.

In the system of the generalized and orthogonal Hermite polynomials (see (3.12)), by the same argument of the previous page, the series of coefficients $(y_k(t), g_k(t))_{k \in \mathbb{N}}$ representing $(Y, g)$ at $t \in [0, T]$ solve the following countable system of ordinary differential equations

$$y_k(t) + g_k(t, y(t)) = 0 \quad k = 0, 1, 2, \ldots \quad (3.38)$$

with

$$g(t, X_t, Y_t, Z_t) = \sum_{k=0}^{\infty} g_k(t) H_k^{[t]}(W_t), \quad g_k(t) := \frac{1}{\sqrt{t^k k!}} \mathbb{E}\left[ g(t, X_t, Y_t, Z_t) H_k^{[t]}(W_t) \right] \quad (3.39)$$
and
\[
Y_t = \sum_{k=0}^{\infty} y_k(t) H^{[t]}_k(W_t) \quad \text{with} \quad y_k(t) := \frac{1}{\sqrt{t^k k!}} \mathbb{E} \left[ Y(t, W_t) H^{[t]}_k(W_t) \right]. \tag{3.40}
\]

In the system of the generalized and normalized Hermite polynomials \((\tilde{H}^t)_{t \in [0, T]}\), it turns out that the above system (3.37) introduces a "singular" point at the time instance \(0 \in [0, T]\). We will see later in the convergence study that this singular time instance is in fact a "regular singular" point of the system. Countable systems of ordinary differential equations (CODEs) have been studied extensively. For early references, see [14, 18, 26].\(^2\) For more recent texts, we refer to the books [10] or [25].

3.3 Preliminaries Examples: Linear Case

3.3.1 Orthogonal Decomposition

In the system \((H^t)_{t \in [0, T]}\) of the generalized and orthogonal Hermite polynomials, let us consider the example of a BSDE where the driver is given by the linear function \(g(t, x, y, z) = \mu z\) with \(\mu \in \mathbb{R}\). We then obtain immediately the representation \(\sum_{k=0}^{\infty} \mu(k+1) y_{k+1}(t) H^{[t]}_k(W_t)\) for \(g\) at the time instance \(t\), which leads to the explicit identification \(g_k(t) = \mu(k+1) y_{k+1}(t)\). Hence, for this example we have the system of ordinary differential equations
\[
y_k(t) + \mu(k+1) y_{k+1}(t) = 0 \quad k = 0, 1, 2, \ldots \tag{3.41}
\]

This is a linear CODE, which is row-infinite. Let us consider the truncated system of ordinary differential equations with only \(N\) equations. The \((N+1)\)-dimensional vector \(y^{(N)}(t)\) solves the finite system of ordinary differential equations
\[
y_k^{(N)}(t) + \mu(k+1) y_{k+1}^{(N)}(t) = 0 \quad k = 0, 1, 2, \ldots, N. \tag{3.42}
\]

By setting \(y_{k+1}^{(N)} \equiv 0\), we can solve this system backwardly from \(k = N\) to \(k = 0\). We obtain an explicit solution for the system (3.42),
\[
y_k^{(N)}(t) = \sum_{j=k}^{N} y_j(T) \binom{j}{k} (\mu(T-t))^{j-k} \quad k = 0, 1, 2, \ldots, N. \tag{3.43}
\]

\(^2\)It is interesting to note that [18, Part III] studies the existence and uniqueness of solutions of semi-linear partial differential equations \(u_t = u_{xx} + g(t, x, u, u_x)\) which we would now identify with BSDEs. Lewis considers the Fourier basis to express the solution in terms of a CODE.
Based on this truncated system, we obtain the following representation for an approximate solution \( Y_{t}^{(N)} \) of the corresponding linear backward stochastic differential equations

\[
Y_{t}^{(N)} = \sum_{k=0}^{N} \left( \sum_{j=k}^{N} y_j(T) \binom{j}{k} (\mu(T-t))^{j-k} \right) H_k^{[t]}(W_t)
\]

\[
= \sum_{j=0}^{N} y_j(T) \left( \sum_{k=0}^{j} \binom{j}{k} (\mu(T-t))^{j-k} H_k^{[t]}(W_t) \right)
\]

\[
Y_{t}^{(N)} = \sum_{j=0}^{N} y_j(T) H_j^{[t]}(W_t + \mu(T-t))
\]

where the second line is obtained by interchanging the order of summation, and the third line follows from the addition formula (3.16). The last line shows that we obtain indeed the solution that adds a drift term \( \mu(T-t) \) to the Brownian Motion \( W_t \). However, the solution only contains the first \( (N+1) \) terms.

### 3.3.2 Orthonormal Decomposition

In this part, we assumed that the solution \((Y_t, Z_t)_{t \in [0,T]}\) of the BSDE (3.5) can be represented in the orthonormal system \( (\mathcal{H}^t)_{t \in [0,T]} \). Let us consider the following example of a BSDE where the driver function is given by \( g(t, x, y, z) = \mu y \) with \( \mu \in \mathbb{R} \). For this example, the system of the CODE (3.37), is linear and

\[
\dot{\alpha}_k(t) = (\mu - \frac{k}{2t}) \alpha_k(t) \quad k = 0, 1, 2, \ldots
\]

(3.45)

For each integer \( k \in \mathbb{N} \), the solution of the linear ODE, is given by the following equality

\[
\alpha_k(t) = (\frac{t}{T})^{k/2} \exp(\mu(T-t)) \alpha_k(T).
\]

(3.46)

In this example, the truncated solution \( \tilde{Y}_{t}^{(N)} \) of the BSDE is given by

\[
\tilde{Y}_{t}^{(N)} = \exp(\mu(T-t)) \sum_{j=0}^{N} (\frac{t}{T})^{j/2} \alpha_j(T) \tilde{H}_j^{[t]}(W_t).
\]

(3.47)

For a fixed \( t \), when we consider the limit for \( N \to \infty \), then we see that the truncated solution \( Y_{t}^{(N)} \) converges in the Hilbert space generated by the system
\((\tilde{H}^t)_{t \in [0,T]}\). This convergence is due to the following uniform comparison inequality

\[
\sup_{t \in [0,T]} (\frac{t}{T})^j \alpha_j(T)^2 \leq \alpha_j(T)^2 \quad \text{with} \quad \sum_{j=0}^{\infty} \alpha_j(T)^2 < \infty.
\]

This solution is equivalent to the solution of a BSDE where its driver function is identically null via a change of a probability measure.
4 Convergence and Uniqueness of Solutions

The existence and the uniqueness of the solutions of the BSDE (3.5) is satisfied by the assumptions (H). The books [10] and [25] give sufficient conditions for the existence of solutions of countable systems of ordinary differential equations (I) below or equivalently to the system (3.36). In this part, we study the uniqueness of the solution of the countable systems of ordinary differential equations (I) below and its truncated solution in a finite dimensional subspace which defined the Galerkin approximation of the solution of (I). In the system of the generalized orthonormal Hermite polynomial (\(H^t\))_{t \in [0, T]}), we formulate the following countable backward problem

\[
(I) \begin{cases} 
2t\dot{\alpha}_k(t) = k\alpha_k(t) - 2t\gamma_k(t, \alpha_k(t)) = 0, & k = 0, 1, 2, \ldots, 0 \leq t < T \\
\alpha_k(T) \text{ is the terminal condition}
\end{cases}
\]

The study of regular ordinary differential equations is well understood and developed in the case of a finite dimension. The classical Lipschitz condition and the Nagumo condition (see [23], [20]) are the most known condition to their studies. In an infinite dimensional space, the problem of existence and uniqueness of ordinary differential equations is much technical to achieve.

4.1 Uniqueness of Solutions

The study of the uniqueness or the existence property of the solution of the problem (I) is equivalent to the study of the uniqueness or the existence property of the solution of the system (3.36) introduced at the beginning of the section.

**Proposition 4.2.** If the solution of the problem (I) exists, then the solution is unique on the time interval \([0, T]\).

**Proof.** We remind that the uniqueness of the solution of the system (I) is equivalent to the uniqueness of the solution of the system

\[
\left(\frac{t}{T}\right)^{k/2} \alpha_k(T) - \alpha_k(t) + \int_t^T \left(\frac{t}{s}\right)^{k/2} \gamma_k(s) ds = 0, \quad k = 0, 1, 2, \ldots \quad (4.48)
\]

Let us consider two solutions \((\alpha_k^i(t))_{i=1,2}\) of the previous system. We associate to each deterministic function \(\alpha_k^i(t)\), \(Y_i^t\) solution of the BSDE (3.5) at the time instance \(t\). Let us define the quantities,

\[
\Delta \alpha_k^{1,2}(t) = \alpha_k^1(t) - \alpha_k^2(t), \quad \Delta \beta_k^{1,2}(t) = \left(\frac{k+1}{t}\right)^{1/2} \Delta \alpha_k^{1,2}(t). \quad (4.49)
\]
From the equation of the system (4.48), we have
\[
|\Delta \alpha_{k}^{1,2}(t)| = \left| \int_{t}^{T} \left( \frac{t}{s} \right)^{k/2} (\gamma_{k}^{1}(s) - \gamma_{k}^{2}(s)) ds \right|
= \left| \int_{t}^{T} \left( \frac{t}{s} \right)^{k/2} \mathbb{E} \left[ (g(s, W_{s}, Y_{s}^{1}, Z_{s}^{1}) - g(s, W_{s}, Y_{s}^{2}, Z_{s}^{2})) \tilde{F}_{t}^{[1]}(W_{s}) \right] ds \right|
\]
(4.50)

By the Lipschitz property of the function \( g \),
\[
|\Delta \alpha_{k}^{1,2}(t)| \leq K \int_{t}^{T} \left( \frac{t}{s} \right)^{k/2} |\Delta \alpha_{k}^{1,2}(s)| ds + K \int_{t}^{T} |\Delta \beta_{k}^{1,2}(s)| ds.
\]
(4.51)

By the Gronwall inequality (7.4) and the Cauchy-Schwartz inequality,
\[
|\Delta \alpha_{k}^{1,2}(t)|^{2} \leq K^{2} \exp(2(T - t)) \times \int_{t}^{T} |\Delta \beta_{k}^{1,2}(s)|^{2} ds.
\]
(4.52)

By the Itô’s Lemma applied to \( \Delta Y_{t}^{1,2} = |Y_{t}^{1} - Y_{t}^{2}|^{2} \) and the Lipschitz property of the function \( g \) we obtain,
\[
\mathbb{E} \left( |\Delta Y_{t}^{1,2}|^{2} \right) + \int_{t}^{T} |\Delta Z_{s}^{1,2}|^{2} ds \leq 2K \int_{t}^{T} \mathbb{E} |\Delta Y_{s}^{1,2}|^{2} ds
+ 2K \int_{t}^{T} \mathbb{E} |\Delta Y_{s}^{1,2}| |\Delta Z_{s}^{1,2}| ds.
\]
(4.53)

Form the previous inequality and the Young Inequality, \( (\forall \epsilon > 0, \quad 2ab \leq \frac{1}{\epsilon} a^{2} + \epsilon b^{2} \) with \( a, b \in \mathbb{R} \)), there exists a constant \( C > 0 \) such that
\[
(1 - \epsilon K) \int_{t}^{T} \mathbb{E} |\Delta Z_{s}^{1,2}|^{2} ds \leq K(2 + 1/\epsilon) \int_{t}^{T} \mathbb{E} |\Delta Y_{s}^{1,2}|^{2} ds, \quad (4.54)
\]

By choosing \( \epsilon = 2/K \) in the previous inequality, there exists a positive constant \( C > 0 \) such that
\[
\sum_{k \geq 0} \int_{t}^{T} |\Delta \beta_{k}^{1,2}(s)|^{2} ds \leq C \int_{t}^{T} \sum_{k \geq 0} |\Delta \alpha_{k}^{1,2}(s)|^{2} ds.
\]
(4.55)

By inserting the previous inequality in the relation (4.52) with summation over the variable \( k \), we obtain,
\[
\sum_{k \geq 0} |\Delta \alpha_{k}^{1,2}(t)|^{2} \leq CK^{2} \exp(2(T - t)) \times \int_{t}^{T} \sum_{k \geq 0} |\Delta \alpha_{k}^{1,2}(s)|^{2} ds.
\]
(4.56)
By the Gronwall inequality (7.4) applied to the previous inequality
\[
\sum_{k \geq 0} |\Delta \alpha_k^{1,2}(t)|^2 = 0. \tag{4.57}
\]
The previous equality implies that
\[
\alpha_k^1(t) = \alpha_k^2(t), \quad k = 0, 1, 2, \ldots
\]
\[\Box\]

4.2 Existence and Convergence of the Truncated Solutions

As highlight above, the study of the ordinary differential equations is well understood and developed in the case of a finite dimension. In the case of an infinite dimensional, the problem of existence and uniqueness of ordinary differential equations are much technical to achieve. In the case of a Banach space, the study of CODE is discussed in the following referenced books [14, 18, 26].

With the system of the generalised Hermite polynomial \((\hat{H}^t)_{t \in [0,T]}\), let us consider the family of the orthogonal projection operator \((\mathcal{P}_n)_{n \geq 1}\) in the span of the first \(n\) first normalised basis functions of \((\hat{H}^t)_{t \in [0,T]}\). We formulate the following \(n\)–dimensional countable backward problem of an ordinary differential equation
\[
[I_n] \left\{ \begin{array}{ll}
\dot{\alpha}^n(t) = \mathcal{P}_n f(t, \alpha^n(t)), & 0 \leq t < T \\
\alpha^n(T) = \mathcal{P}_n \alpha(T), & t = T
\end{array} \right. \tag{4.58}
\]
where \(\alpha(T) = (\alpha_k(T))_{k \geq 1}\) and \(f(t, \alpha(t))\) denotes an infinite dimensional vector where each coordinate is defined by \(f_k(t, \alpha(t)) = -\frac{k}{2t} \alpha_k(t) + \gamma_k(t, \alpha(t))\) for each \(k \in \mathbb{N}\). For every \(t \in [0,T]\), \(\alpha^n(t) \in \mathbb{R}^n\). One of the advantage of the problem \((I)\) or \((I_n)\) is that we can inverse the time on the interval \([0,T]\). The result of the following lemma is the cornerstone of the existence result of the truncation solution of countable system of ordinary differential equation \((I)\). The convergence result is based on the result of the theorem 7.1 in Klaus Deimling [10].

4.2.1 Existence of the truncated solutions

The following lemma provides a quadratic inequality of the functional \(f\). This quadratic inequality is a key ingredient to study the existence of the solution of the countable differential equation problem \((I_n)\) or \((I)\).
Lemma 4.1. We assume that on the set \([0, T] \times l^2(\mathbb{N})\), the function \(f\) is continuous. For \(\alpha, \beta \in l^2(\mathbb{N})\), the functional vector \(f(t, \alpha(t))\) satisfies the following quadratic inequality

\[
(f(t, \alpha) - f(t, \beta), \alpha - \beta) \leq K(1 + \frac{K}{2})||\alpha - \beta||^2, \quad \forall t \in [0, T].
\]

(4.59)

where \((.,.)\) denotes the usual inner product associated to the Hilbert space \(l^2(\mathbb{N})\).

Proof. For \(\alpha, \beta \in l^2(\mathbb{N})\), we define the system

\[
\begin{aligned}
    \Delta Y^\alpha_\beta &= Y^\alpha_t - Y^\beta_t \\
    \Delta Z^\alpha_\beta &= Z^\alpha_t - Z^\beta_t \\
    \Delta g^\alpha_\beta &= g(t, Y^\alpha_t, Z^\alpha_t) - g(t, Y^\beta_t, Z^\beta_t)
\end{aligned}
\]

where \(Y^\alpha_t = \sum_{k \geq 0} \alpha_k(t) \tilde{H}^{[k]}_k(W_t)\) and \(Z^\alpha_t = \sum_{k \geq 0} (\frac{k+1}{t})^{1/2} \alpha_{k+1}(t) \tilde{H}^{[k]}_k(W_t)\). The same definition is applied to the couple \((Y^\beta_t, Z^\beta_t)\). From the expression of \(f\) and the theorem of Pythagoras, we have

\[
(f(t, \alpha) - f(t, \beta), \alpha - \beta) = \frac{1}{2} \mathbb{E}[(\Delta Y^\alpha_\beta)^2 + \mathbb{E}((\Delta g^\alpha_\beta) \Delta Y^\alpha_\beta^2)].
\]

(4.60)

Using the Lipschitz property of the function \(g\), we obtain

\[
\mathbb{E}\Delta g^\alpha_\beta \Delta Y^\alpha_\beta \leq K \mathbb{E}|\Delta Y^\alpha_\beta|^2 + K \mathbb{E}|\Delta Z^\alpha_\beta||\Delta Y^\alpha_\beta|.
\]

By the Young’s Inequality

\[
\mathbb{E}(\Delta g^\alpha_\beta \Delta Y^\alpha_\beta) - \frac{1}{2} \mathbb{E}|\Delta Z^\alpha_\beta|^2 \leq K(1 + \frac{K}{2}) \mathbb{E}|\Delta Y^\alpha_\beta|^2.
\]

From the previous inequality and the expression (4.60), we obtain

\[
(f(t, \alpha) - f(t, \beta), \alpha - \beta) \leq K(1 + \frac{K}{2})||\alpha - \beta||^2.
\]

(4.61)

\[\square\]

Proposition 4.3. The solution of the problem \((1)_n\) exists on the time interval \([0, T]\).

Proof. The result is a direct consequence of the previous lemma and the Theorem (7.3). \[\square\]
4.2.2 Convergence of the Truncated Solution

In order to analyse the convergence of the truncated solution of the system (1), we first provide below two key definitions (from the book of Klaus Deimling [10]) used in the analysis of the convergence result of the solution of the countable system of ordinary differential equation (I_n).

**Definition 4.1** (Class $\mathcal{U}$). A function $\omega : [0, a] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is said to be of "class $\mathcal{U}$" if to each $\epsilon$ there exists $\delta > 0$, a sequence $t_i \rightarrow 0^+$ and a sequence of continuous functions $\rho_i : [t_i, a] \rightarrow \mathbb{R}^+$ with

$$\rho_i(t_i) \geq \delta, \quad D^- \rho_i(t) > \omega(t, \rho_i(t)), \quad 0 < \rho_i(t) < \epsilon \quad \in [t_i, a]$$

where $D^- \rho_i(t) = \lim \sup_{t \rightarrow 0} \frac{1}{h} (\rho_i(t) - \rho_i(t - h))$. We will write $\omega \in \mathcal{U}$.

**Definition 4.2** (Class $\mathcal{U}_1$). A function $\omega : [0, a] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is said to be of "class $\mathcal{U}_1$" if $\omega \in \mathcal{U}$ and the function $\rho_i$ from the previous definition satisfy in addition the condition

$$D^- \rho_i(t) \geq \omega(t, \rho_i(t)) + \delta_i \quad \text{for some} \quad \delta_i > 0.$$

We will write $\omega \in \mathcal{U}_1$.

As in the book of Klaus Deimling [10], we remark that the Lipschitz case $\omega(t, x) = Lx$, $L > 0$ and the Nagumo condition $\omega(t, x) = \frac{x}{t}$ are in the class of $\mathcal{U}$ and of $\mathcal{U}_1$.

**Theorem 4.1.** Let us consider the previous family of the orthogonal projection operator $(P_n)_{n \geq 1}$ in the span of the first $n$ first basis functions of the Section 3.1. The truncated solution $\alpha^n$ of the system of the countable ordinary differential equation (I_n) converges uniformly to the true solution on the time interval $[0, T]$, when $n \rightarrow \infty$.

**Proof.** The proof of the result is based on the Theorem 7.1. in Klaus Deimling [10]. To be consistent with their results, one can make the change of variable $t' = T - t$, $t \in [0, T]$. We know from the previous lemma that $f$ satisfies the inequality

$$(f(t', \alpha) - f(t', \beta), \alpha - \beta) \leq \omega(t', |\alpha - \beta|)|\alpha - \beta|, \quad t' \in (0, T) \text{ and } \alpha, \beta \in l^2(\mathbb{N}).$$
The linear function \( (t', \rho) \mapsto \omega(t', \rho) = K(1 + \frac{K}{2})\rho \) is of the class of \( \mathcal{U}_1 \). Let us define the function \( \alpha_n^p(t') = \mathcal{P}_n \alpha(t') \), the error term \( e_n^p(t') = \alpha^p(t') - \alpha_n^p(t') \) and the absolute error \( \phi(t') = |e_n^p(t')| \).

As the projection operator \( \mathcal{P}_n \) is Lipschitz with the Lipschitz constant equal to 1, the previous quadratic inequality is also satisfies for \( \mathcal{P}_n f \). We then have,

\[
(\mathcal{P}_n f(t', \alpha) - \mathcal{P}_n f(t', \beta), \alpha - \beta) \leq \omega(t', |\alpha - \beta|)|\alpha - \beta|.
\]

Furthermore we have

\[
\phi(t') D^- \phi(t') \leq (\overline{(e_n^p(t'))}, e_n^p(t'))
\]

(4.62)

where \( (e_n^p(t')) \) is the derivative with respect to \( t' \) of the error term \( e_n^p(t') \). From the previous inequality and for \( t' \in (0, T] \) we obtain,

\[
\phi(t') D^- \phi(t') \leq (\mathcal{P}_n f(t', \alpha^p(t') - \mathcal{P}_n f(t', \alpha_n^p(t'))), e_n^p(t')) + (|f(t', \alpha^p(t')) - f(t', \alpha(t'))|)\phi(t')
\]

\[
\leq (f(t', \alpha^p(t') - f(t', \alpha_n^p(t'))), e_n^p(t')) + (|f(t', \alpha^p(t')) - f(t', \alpha(t'))|)\phi(t')
\]

\[
\leq \omega(t', \phi(t')) \phi(t') + (|f(t', \alpha_n^p(t')) - f(t', \alpha(t'))|)\phi(t').
\]

(4.63)

We remind that \( f_k(t, \alpha(t)) = -\frac{k}{2} \alpha_k(t) + \gamma_k(t, \alpha(t)) \) and important to point out that, the solution of the BSDE (3.5) is deterministic for \( t = 0 \). This means that the coefficients \( \alpha_k(0) = 0 \) for all \( k \geq 1 \) except \( \alpha_0(0) \). We know from the Section 3 that we can represent the solution of the BSDE (3.5) as \( Y_t = u(t, \mathcal{W}_t) \) for \( t \in [0, T) \) and where \( u \) solve the PDE (\#) in Section 3. In order to solve the singularity problem at \( t = 0 \) of the function \( f_k \), we introduce the following Lemma.

**Lemma 4.2.** Under the assumption (H), the function \( \alpha_k(\cdot) \) solves the following equivalent ODE. For all \( (k, t) \in \mathbb{N} \times [0, T) \),

\[
\dot{\alpha}_k(t) = f_k(t, \alpha(t)) = -E\left( \partial_t F_k(t, \mathcal{W}_t) + \frac{1}{2} \partial_{xx}^2 F_k(t, \mathcal{W}_t) \right)
\]

where \( F_k(t, x) = u(t, x)H_k^{[t]}(x) \) and \( u \) the unique solution of the PDE (\#).

**Proof the Lemma.** By remarking that \( \dot{\alpha}_k(t) = \lim_{h \to 0} \frac{1}{h}(\alpha_k(t - h) - \alpha_k(t)) \), we have easily from of Itô Lemma that

\[
\dot{\alpha}_k(t) = f_k(t, \alpha(t)) = -E\left( \partial_t F_k(t, \mathcal{W}_t) + \frac{1}{2} \partial_{xx}^2 F_k(t, \mathcal{W}_t) \right).
\]

\[\square\]
We know that $u$ solve the PDE $(\ast)$. From the Lemma 4.2, in the neighborhood of $0^+$ and when $t$ goes to $0^+$, from the dominated convergence theorem we have

$$\lim_{t \to 0} f_k(t, \alpha(t)) = E[g(0, 0, Y_0, Z_0)] = \gamma_0(0, \alpha(0)).$$

The function $f_k$ can be extended to $\hat{f}_k$ by continuity from the previous result. We have

$$\hat{f}_k(t) = f_k(t)1_{\{t \neq 0\}} + \gamma_0(0, \alpha(0))1_{\{t = 0\}}, \quad t \in [0, T].$$

By using the function $\hat{f}_k$ and noticing that the function $\alpha_n^P(t') \to \alpha(t')$ as $n \to \infty$, the quantity $|f(t', \alpha_n^P(t')) - f(t', \alpha(t'))| \to 0$ as $n \to \infty$ punctually on $[0, T]$. The following convergence hold

$$\frac{\hat{f}(t')}{t'} \to |P_n f(T, P_n \alpha(T)) - P_n f(T, \alpha(T))| \quad \text{as} \quad t' \to 0^+. \quad (4.64)$$

Furthermore $|P_n f(T, P_n \alpha(T)) - P_n f(T, \alpha(T))| \to 0$ as $n \to \infty$. From the evaluation of the previous result and the result (4.64), there exists $n_\mu$ and $t_\mu > 0$ such that

$$\forall n, n > n_\mu, \quad \phi(t') \leq (1/n + \mu)t' \quad \text{for} \quad t' \in [0, t_\mu].$$

For a given $\epsilon > 0$, we choose the constant $\delta > 0$, $t_1 < t_\mu$, the function $\rho_1$ from the definition 4.2 such that $1/n + \mu \leq \frac{\delta}{2}$ for some $n_0$ such that $n > n_0$. This result implies that

$$\phi(t') \leq \frac{\delta}{2}t' \quad \text{for} \quad t' \leq t_\mu.$$

For $t_1 < t_\mu$ and consider the function $\rho_1$ form the definition 4.2,

$$\phi(t_1) < \rho_1(t_1) \quad \text{for} \quad t' \leq t_\mu.$$

Let us consider the pseudo stopping time $t^*$ defines as

$$t^* = \inf\{t > t_1, \frac{\phi(t)}{t} = \rho_1(t)\}.$$

By the continuity of $\phi$ and the fact that $\phi(t_1) < \rho_1(t_1)$ implies that $\phi(t^*) > 0$. We then have the inequality

$$D^-\phi(t^*) \leq \omega(t^*, \phi(t^*)) + \delta. \quad (4.65)$$

As the function $\omega$ is of the class of $\mathcal{U}_1$ and given the parameter $\delta_1 > 0$ of the definition 4.2, we therefore have by choosing $\delta \leq \delta_1$

$$D^-\phi(t^*) \leq \omega(t^*, \phi(t^*)) + \delta_1 < D^-\phi(t^*). \quad (4.66)$$

The previous double inequality is impossible. Therefore, $\phi(t') \leq \phi(t^*) \leq \epsilon$ for every $\epsilon > 0$. Hence the sequence $\epsilon_n^P(t') \to 0$ as $n \to \infty$.

Since $\alpha_n^P(t') \to \alpha(t')$ as $n \to \infty$, we conclude that the function $\alpha_n^P(t') \to \alpha(t')$, uniformly on $[0, T]$. By the previous change of variable $t' = T - t$, it hold the fact the function $\alpha_n(t) \to \alpha(t)$, uniformly on the interval $[0, T]$. □

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5 Euler Scheme and Convergence Results

Let us consider the unidimensional discrete-time approximation of the equation (3.5), we build an uniform partition \( \pi \) of the interval \([0, T]\) defined as follows: \( 0 = t_0 < t_1 < \ldots < t_N = T, \ \Delta_i := t_{i+1} - t_i \) and \( |\pi| = \max(\Delta_i; 0 \leq i \leq N - 1) \). We will use the decomposition (3.32) in order to solve the backward stochastic differential equations (3.5).

5.1 Euler Scheme

Let us denote \((\tilde{Y}, \tilde{Z})\) the numerical approximation of the exact solution \((Y, Z)\) deduced from the Euler approximation of the countable system of ODE (3.37).

In the hermite basis the couple \((Y, Z)\) is represented by the couple \((\alpha, \beta)\) and its numerical approximation \((\tilde{Y}, \tilde{Z})\) by \((\tilde{\alpha}, \tilde{\beta})\). Following the work of the section 3.2, we have the following decomposition of the couple \((\tilde{Y}_{t_i}, \tilde{Z}_{t_i})\) in our basis for \( t_i \in \pi \).

\[
\begin{align*}
\tilde{Y}_{t_i} &= \sum_{k \geq 0} \tilde{\alpha}_k(t_i) \tilde{H}^{[t_i]}_k(W_{t_i}), \\
\tilde{Z}_{t_i} &= \sum_{k \geq 0} \tilde{\beta}_k(t_i) \tilde{H}^{[t_i]}_k(W_{t_i}) \quad \text{where} \quad \tilde{\beta}_k(t_i) = \left(\frac{k+1}{t_i}\right)^{1/2} \tilde{\alpha}_{k+1}(t_i), \\
\tilde{\gamma}_{t_i} &= \sum_{k \geq 0} \tilde{\gamma}_k(t_i) \tilde{H}^{[t_i]}_k(W_{t_i})
\end{align*}
\]

where \( \tilde{\gamma}_k(t_i) = \mathbb{E}\left[g(t, W_{t_i}, \tilde{Y}_{t_i}, \tilde{Z}_{t_i}) \tilde{H}^{[t_i]}_k(W_{t_i})\right] \). The decomposition of the couple \((Y_{t_i}, Z_{t_i})\) follows the same structure as in the previous Section 3.2. Let us remark that by the decomposition (3.32), the computation of the coefficients \( \tilde{\beta} \) is completely determined by the coefficients \( \tilde{\alpha} \). On the time interval \([t_i, t_{i+1}]\), the Euler scheme is built with the approximation of the integral term in the following system of equations

\[
\alpha_k(t_i) = \left(\frac{t_i}{t_{i+1}}\right)^{k/2} \alpha_k(t_{i+1}) + \int_{t_i}^{t_{i+1}} \left(\frac{t_i}{s}\right)^{k/2} \gamma_k(s, \alpha_k(s)) ds = 0, \quad k = 0, 1, 2, \ldots
\]

We will suppose that we have at our disposal the trajectories of the Brownian motion \( W \) on the discretization grids \( \pi \). It will be enough to compute the couple \((\tilde{\alpha}, \tilde{\gamma})\) on the discretization grid to have the Euler approximation of the couple \((Y, Z)\). We remark that the computation of \( Z \) is completely determined by the
computation of $Y$ and its terminal condition.

---

**Algorithm Description**

- **Initialisation**: Approximate the terminal condition $\tilde{Y}_T = \phi(W_T)$ and we compute the coefficients $\tilde{\alpha}_k(T) = \alpha_k(T)$ and $\tilde{\beta}_k(T) = \beta_k(T) = \alpha_{k+1}(T)(\frac{k+1}{t_i})^{1/2}$ for $k = 0, 1, 2, \ldots$

- For $i = (N - 1)$ to 0, on each sub-interval $[t_i, t_{i+1}] \subset [0, T]$ with $t_i, t_{i+1} \in \pi$,
  - compute the vector $\tilde{Y}^*_i$ by the following projection
    $$\begin{align*}
    \text{Find } \tilde{Y}^*_i &= (\tilde{Y}_1(t_{i+1}), \tilde{Y}_2(t_{i+1}), \tilde{Y}_2(t_{i+1}), \tilde{Y}_3(t_{i+1}), \ldots) \\
    J(\tilde{Y}^*_i) &= \inf_{\xi} \mathbb{E} |\xi \tilde{H}_i(W_{t_{i+1}}) - g(t_{i+1}, W_{t_{i+1}}, \tilde{Y}_{i+1}, \tilde{Z}_{i+1})|^2,
    \end{align*}$$
    where the family of function $\tilde{H}_i := (\tilde{H}_0^{[t_{i+1}]}, \tilde{H}_1^{[t_{i+1}]}, \tilde{H}_2^{[t_{i+1}]}, \ldots)$.

  - compute $\tilde{\alpha}_i$ and $\tilde{\beta}_i(t_i)$
    $$\begin{align*}
    \tilde{\alpha}_k(t_i) &= \left(\frac{t_{i+1}}{t_i}\right)^{k/2} \tilde{\alpha}_k(t_{i+1}) + \Delta_i \left(\frac{t_{i+1}}{t_i}\right)^{k/2} \tilde{\gamma}_k(t_{i+1}) = 0, \quad k = 0, 1, 2, \ldots \\
    \tilde{\beta}_k(t_i) &= \tilde{\alpha}_{k+1}(t_i) \left(\frac{k+1}{t_i}\right)^{1/2} 
    \end{align*}$$

  - compute $\tilde{Y}_i = \sum_{k \geq 0} \tilde{\alpha}_k(t_i) \tilde{H}_k^{[t_i]}(W_{t_i})$ and $\tilde{Z}_i = \sum_{k \geq 0} \tilde{\beta}_k(t_i) \tilde{H}_k^{[t_i]}(W_{t_i})$.

- **End of the algorithm**

The couple of coefficients $(\tilde{\alpha}, \tilde{\gamma})$ are the approximation of the couple $(\alpha, \gamma)$ obtained by the forward Euler approximation of the integral. Note that the computation is done through Monte Carlo simulation.

### 5.2 Convergence Result

The convergence result can be obtained directly from the standard result of the ODE literature if our functional $f$ is Lipschitz in the second variable. In our work, we remind that the functional $f$ is not Lipschitz and satisfies the quadratic inequality (4.61).

**Theorem 5.2.** Under the assumptions (H) and considering the previous uniform subdivision $\pi$ of the interval $[0, T]$, there exists a positive constant $C$ independent of the partition $\pi$ such that

$$\max_{0 \leq i \leq N} \mathbb{E}|Y_{t_i} - \tilde{Y}_{t_i}|^2 + \mathbb{E} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |Z_t - \tilde{Z}_{t_i}|^2 ds \leq C|\pi|. \quad (5.67)$$

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Proof. The proof of the theorem will be deduced from the couple \((\alpha, \beta)\), respectively \((\tilde{\alpha}, \tilde{\beta})\) which represents respectively the couples \((Y, Z)\) and \((\tilde{Y}, \tilde{Z})\).

During the proof the strictly positive constant \(C\) may take different values from line to line, but independent of the partition \(\pi\). With \(0 \leq i \leq N - 1\), we define for every \(t_i \in \pi\) and \(s \in [t_i, t_{i+1}]\), the quantities

\[
\begin{align*}
\Delta \alpha_k(t_i) &= \alpha_k(t_i) - \tilde{\alpha}_k(t_i) \\
\Delta \beta_k(s, t_i) &= \beta_k(s) - \tilde{\beta}_k(t_i) \\
\Delta \gamma(s, t_i) &= \gamma_k(s) - \tilde{\gamma}_k(t_i)
\end{align*}
\]

where \(\tilde{\beta}_k(t_i) = \left(\frac{k+1}{t_i}\right)^{1/2} \tilde{\alpha}_{k+1}(t_i)\) and \(\beta_k(t) = \left(\frac{k+1}{t}\right)^{1/2} \alpha_{k+1}(t)\). We also define the positive quantity

\[U_i = \sum_{k \geq 0} |\Delta \alpha_k(t_i)|^2 + \int_{t_i}^{t_{i+1}} |\Delta \beta_k(s, t_i)|^2 ds.\]

The couple \((\tilde{Y}, \tilde{Z})\) is obtained by the previous approximation of the Euler scheme. From the following system below

\[
\begin{align*}
\left\{ \begin{array}{l}
\left(\frac{t_i}{t_{i+1}}\right)^{k/2} \alpha_k(t_{i+1}) - \int_{t_i}^{t_{i+1}} \left(\frac{t_i}{s}\right)^{k/2} \gamma_k(s) ds = \alpha_k(t_i), \\
\left(\frac{t_i}{t_{i+1}}\right)^{k/2} \tilde{\alpha}_k(t_{i+1}) - \Delta \left(\frac{t_i}{t_{i+1}}\right)^{k/2} \tilde{\gamma}_k(t_{i+1}) = \tilde{\alpha}_k(t_i),
\end{array} \right.
\end{align*}
\]  

we deduced that the error term below

\[
\alpha_k(t_i) - \tilde{\alpha}_k(t_i) = (\alpha_k(t_{i+1}) - \tilde{\alpha}_k(t_{i+1}))\left(\frac{t_i}{t_{i+1}}\right)^{k/2} - \int_{t_i}^{t_{i+1}} \left(\frac{t_i}{s}\right)^{k/2} \gamma_k(s) - \left(\frac{t_i}{t_{i+1}}\right)^{k/2} \tilde{\gamma}_k(t_{i+1}) ds. 
\]  

By adding and subtracting in the integral term of the above equality the term \(\left(\frac{t_i}{s}\right)^{k/2} \tilde{\gamma}_k(t_{i+1})\) and using Jensen inequality,

\[
\begin{align*}
|\alpha_k(t_i) - \tilde{\alpha}_k(t_i)| &\leq |\alpha_k(t_{i+1}) - \tilde{\alpha}_k(t_{i+1})| + \int_{t_i}^{t_{i+1}} \left|\frac{t_i}{s}\right|^{k/2} |\gamma_k(s) - \tilde{\gamma}_k(t_{i+1})| ds \\
&\quad + \int_{t_i}^{t_{i+1}} \left|\frac{t_i}{t_{i+1}}\right|^{k/2} - \left|\frac{t_i}{t_{i+1}}\right|^{k/2} |\tilde{\gamma}_k(t_{i+1})| ds. 
\end{align*}
\]

By the classical Young inequality applied to the right member of the inequality (5.70), for every \(c > 0\), we have
\[
\left\{ \begin{array}{l}
|\Delta \alpha_k(t_i)|^2 + \int_{t_i}^{t_{i+1}} |\Delta \beta_k(s, t_i)|^2 ds \leq (1 + \Lambda_l/e)|\Delta \alpha_k(t_{i+1})|^2 + 2\Delta|\Delta \beta_k(t_i, t_i)|^2 \\
+ (1 + e/\Lambda_l) \left( \int_{t_i}^{t_{i+1}} \left( \frac{t_i}{s} \right)^{k/2} (\gamma_k(s) - \bar{\gamma}_k(t_{i+1})) |ds \right) \\
+ \int_{t_i}^{t_{i+1}} \left( \frac{t_i}{s} \right)^{k/2} - \left( \frac{t_i}{t_{i+1}} \right)^{k/2} ||\bar{\gamma}_k(t_{i+1})|ds \right) + 2 \int_{t_i}^{t_{i+1}} |\beta_k(s) - \bar{\beta}_k(t_i)|^2 ds.
\end{array} \right. \tag{5.71}
\]

By noticing that \( \gamma_k(s) - \bar{\gamma}_k(t_{i+1}) = \gamma_k(s) - \gamma_k(s + \Delta) + \Delta \gamma_k(s + \Delta, t_{i+1}) \) and using the H"older inequality, there exists a positive generic constant \( C \) such that for every \( \epsilon > 0 \), we have

\[
\left\{ \begin{array}{l}
|\Delta \alpha_k(t_i)|^2 + \int_{t_i}^{t_{i+1}} |\Delta \beta_k(s, t_i)|^2 ds \leq (1 + \Lambda_l/e)|\Delta \alpha_k(t_{i+1})|^2 + 2\Delta|\Delta \beta_k(t_i, t_i)|^2 \\
+2 \int_{t_i}^{t_{i+1}} |\beta_k(s) - \bar{\beta}_k(t_i)|^2 ds + (C\Lambda_l + C\epsilon) \left( \int_{t_i}^{t_{i+1}} |\gamma_k(s) - \gamma_k(s + \Delta)|^2 ds \right) \\
+ \int_{t_i}^{t_{i+1}} |\Delta \gamma_k(s + \Delta, t_{i+1})|^2 ds + \Delta_t^2 ||\bar{\gamma}_k(t_{i+1})|^2 .
\end{array} \right. \tag{5.72}
\]

Remark 5.2.

i) \( \int_{t_i}^{t_{i+1}} |\Delta \gamma_k(s + \Delta, t_{i+1})|^2 ds = \int_{t_i}^{t_{i+1}} |\Delta \gamma_k(s, t_{i+1})|^2 ds. \)

ii) \( |\gamma_k(s) - \bar{\gamma}_k(t_{i+1})| \leq K \left( |\alpha_k(s) - \alpha_k(t_{i+1})| + |\Delta \alpha_k(t_{i+1})| + |\Delta \beta_k(s, t_{i+1})| \right). \)

The point ii) is obtained from the Lipschitz property of the driver function \( g \).

Plugging the relations i) and ii) in the inequality (5.72), there exists a positive constant \( C \) such that, for every \( \epsilon > 0 \) and \( \Delta \) small enough,

\[
\left\{ \begin{array}{l}
|\Delta \alpha_k(t_i)|^2 + \int_{t_i}^{t_{i+1}} |\Delta \beta_k(s, t_i)|^2 ds \leq (1 + \Lambda_l/e + C\Lambda_l(\Lambda_l + \epsilon))|\Delta \alpha_k(t_{i+1})|^2 + 2\Delta|\Delta \beta_k(t_i, t_i)|^2 \\
+ 2 \int_{t_i}^{t_{i+1}} |\beta_k(s) - \bar{\beta}_k(t_i)|^2 ds + (C\Lambda_l + C\epsilon) \left( \int_{t_i}^{t_{i+1}} |\gamma_k(s) - \gamma_k(s + \Delta)|^2 ds \right) \\
+ \int_{t_i}^{t_{i+2}} |\alpha_k(s) - \alpha_k(t_{i+1})|^2 ds + \int_{t_i}^{t_{i+1}} |\Delta \beta_k(s, t_{i+1})|^2 ds + \Delta_t^2 ||\bar{\gamma}_k(t_{i+1})|^2 .
\end{array} \right. \tag{5.73}
\]

From the inequality (5.73), taking \( \epsilon = \frac{1}{\pi} \) and \( \pi \) small enough, there exists a positive constant \( C \) independent of \( \pi \) such that
\begin{equation}
\begin{aligned}
U_i \leq (1 + \Delta_i C) U_{i+1} + (C|\pi| + 1) \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\gamma_k(s) - \gamma_k(s + \Delta)|^2 ds \\
+ 2 \sum_{k \geq 0} \Delta_i |\Delta \beta_k(t_i, t_i)|^2 + 2 \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\beta_k(s) - \beta_k(t_i)|^2 ds \\
+ (C|\pi| + 1) \sum_{k \geq 0} \left( \int_{t_{i+1}}^{t_{i+2}} |\alpha_k(s) - \alpha_k(t_{i+1})|^2 ds + \Delta_i^2 |\tilde{\gamma}_k(t_{i+1})|^2 \right).
\end{aligned}
\tag{5.74}
\end{equation}

By the Lipschitz property of \( g \) we deduced the following decomposition,

\[ |\gamma_k(s) - \gamma_k(s + \Delta)| \leq K \left( |\alpha_k(s) - \alpha_k(s + \Delta)| + |\beta_k(s) - \beta_k(s + \Delta)| \right). \]

We then have:

\begin{equation}
\begin{aligned}
U_i \leq (1 + \Delta_i C) U_{i+1} + C(|\pi| + 1) \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\beta_k(s) - \beta_k(s + \Delta)|^2 ds + |\alpha_k(s) - \alpha_k(s + \Delta)|^2 ds \\
+ 2 \sum_{k \geq 0} \Delta_i |\Delta \beta_k(t_i, t_i)|^2 + 2 \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\beta_k(s) - \beta_k(t_i)|^2 ds \\
+ (C|\pi| + 1) \sum_{k \geq 0} \left( \int_{t_{i+1}}^{t_{i+2}} |\alpha_k(s) - \alpha_k(t_{i+1})|^2 ds + \Delta_i^2 |\tilde{\gamma}_k(t_{i+1})|^2 \right)
\end{aligned}
\tag{5.75}
\end{equation}

and by applying the Gronwall Lemma 7.3 to the previous inequality (5.75), there exists the constant \( C > 0 \) such that for \( |\pi| \) small enough,

\[
\max_{0 \leq i \leq N} U_i \leq \sum_{k \geq 0} |\alpha_k(T) - \tilde{\alpha}_k(T)|^2 + \sum_{i=0}^{N-1} A_{t_i}(\alpha) + B_{t_i}(\beta) + 2|\pi| \sum_{i=0}^{N-1} \sum_{k \geq 0} |\Delta \beta_k(t_i, t_i)|^2
\tag{5.76}
\]

with

\[
B_{t_i}(\beta) = C(|\pi| + 1) \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\beta_k(s) - \beta_k(s + \Delta)|^2 ds + 2 \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\beta_k(s) - \beta_k(t_i)|^2 ds
\tag{5.77}
\]

\[
A_{t_i}(\alpha) = C(|\pi| + 1) \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\alpha_k(s) - \alpha_k(s + \Delta)|^2 ds
\]

\[
+ (C|\pi| + 1) \sum_{k \geq 0} \left( \int_{t_i}^{t_{i+1}} |\alpha_k(s) - \alpha_k(t_{i+1})|^2 ds + \Delta_i^2 |\tilde{\gamma}_k(t_{i+1})|^2 \right).
\]

Remark 5.3.
• By the Bessel’s inequality and the Lemma 3.2 in [27], there exists a constant $C > 0$ such that for $|\pi|$ small enough, $\sum_{i=0}^{N-1} A_{i}(\alpha) \leq C|\pi|$.

• Using the $L^2$-regularity result (Lemma 3.1 in [27]), there exists a constant $C > 0$ such that for $|\pi|$ small enough, $\sum_{i=0}^{N-1} B_{i}(\beta) \leq C|\pi|$.

By combining the previous remark, the previous inequality (5.75) and using Gronwall Lemma 7.3, we derive the following inequality:

$$
\max_{0 \leq i \leq N} \sum_{k \geq 0} |\alpha_k(t_i) - \tilde{\alpha}_k(t_i)|^2 \leq C \sum_{k \geq 0} |\alpha_k(T) - \tilde{\alpha}_k(T)|^2 + C|\pi|
$$

$$
+ 2|\pi| \sum_{i=0}^{N-1} \sum_{k \geq 0} |\Delta \beta_k(t_i, t_i)|^2. \quad (5.78)
$$

From the inequality (5.72) and choosing $\varepsilon = \frac{1}{2C}$, there exists a positive constant $C$ independent of $\pi$ such that

$$
\begin{cases}
U_{i-1} + \frac{1}{2} \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\beta_k(s) - \tilde{\beta}_k(t_i)|^2 ds \leq (1 + C\Delta_i)U_i + 2 \sum_{k \geq 0} \Delta_i |\Delta \beta_k(t_{i-1}, t_{i-1})|^2 \\
+ 2 \sum_{k \geq 0} \int_{t_i}^{t_{i-1}} |\beta_k(s) - \beta_k(t_{i-1})|^2 ds + (C\Delta_i + 1) \left( \int_{t_{i-1}}^{t_i} |\gamma_k(s) - \gamma_k(s + \Delta)|^2 ds \right) \\
+ \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\alpha_k(s) - \alpha_k(t_i)|^2 ds + \Delta_i^2 |\tilde{\gamma}_k(t_i)|^2.
\end{cases}
$$

Summing both side of the previous inequality over the variable $i$ for 1 to $N - 1$, and use the remark (5.3) there exists a positive constant $C > 0$ independent of $\pi$ such that

$$
\sum_{i=1}^{N-1} U_{i-1} + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\beta_k(s) - \tilde{\beta}_k(t_i)|^2 ds \leq \sum_{i=1}^{N-1} (1 + C\Delta_i)U_i
$$

$$
+ 2|\pi| \sum_{i=1}^{N-1} \sum_{k \geq 0} |\Delta \beta_k(t_{i-1}, t_{i-1})|^2 + C|\pi|. \quad (5.80)
$$

Applying the inequality (5.76) obtained before to the previous inequality, there exists a constant $C > 0$ independent of $|\pi|$ such that,

$$
\sum_{t=1}^{N-1} \sum_{k \geq 0} \int_{t_i}^{t_{i+1}} |\beta_k(s) - \tilde{\beta}_k(t_i)|^2 ds \leq C(1 + |x|^2)|\pi| + 2|\pi| \sum_{t=1}^{N-1} \sum_{k \geq 0} |\Delta \beta_k(t_{i-1}, t_{i-1})|^2
$$

$$
+ C \sum_{k \geq 0} |\alpha_k(T) - \tilde{\alpha}_k(T)|^2. \quad (5.81)
$$
For the Euler scheme, the terminal condition \( Y_T = \tilde{Y}_T \), the last relation (5.81) and the above inequality (5.76) conclude the proof of the theorem 5.2. \( \square \)
5.3 Numerical Illustrations

In this section we will illustrate the backward Fourier-hermite expansion algorithm with two examples to solve the corresponding Backward SDE’s. For realistic application of BSDE, we refer to the paper of Karaoui et al. [12] and the references therein. The solution of the problem (1) in the previous chapter which solved the countable infinite system of ordinary differential equation dwells in the infinite dimensional space. For numerical purposes, it is desirable to project the solution in a finite dimensional space. In our numerical implementation, we consider the orthogonal projection operator \( (P_k)_{k \geq 1} \) in the span of the \( k \) first orthonormal basis functions as introduced in the section 3.2. Let us consider the unidimensional discrete-time approximation of the equation (3.5). We build an uniform partition \( \pi \) of the interval \([0, T]\) define as follow: \( 0 = t_0 < t_1 < \ldots < t_N = T, \Delta_i := t_{i+1} - t_i \) and \( |\pi| = \max \{\Delta_i ; 0 \leq i \leq N - 1\} \). For our numerical simulation, we define the following parameters:

- \( k \) the number of chosen orthonormal basis functions,
- \( M \) the number of simulated paths \( s \) of the Brownian motion,
- \( N \) the number of the discretization instance on the grid \( \pi \).

Due the propagation of the error during the backward approximation of the solution the BSDE, we will be interested in the initial value of the solution. We will suppose that we have at our disposal the simulated Brownian at the times discretization of the partition \( \pi \).

5.3.1 Application 1

The first example is defined by the following BSDE, inspired from the paper of Ruijter and Oosterlee [24]. The underlying process is the Brownian motion \( (W_t)_{0 \leq t \leq T} \). We consider the system

\[
\begin{cases}
-dY_t = g(s, W_s, Y_s, Z_s)dt - Z_t dW_t, & 0 \leq t < T \\
Y_1 = \phi(W_1), & T = 1,
\end{cases}
\]

where the functions \( f \) and \( \phi \) are defined as follows;

\[
\phi(x) = \cos(x + 1), \quad x \in \mathbb{R},
\]

\[
f(t, X_t, Y_t, Z_t) = Z_t(Y_t + 1) - \frac{1}{2}(Y_t - \sin(2(t + W_t)) + \cos(t + W_t)).
\]

The exact unique solution of the above BSDE is defined by the couple

\[
(Y_t, Z_t) = (\cos(W_t + t), -\sin(W_t + t)), \quad a.s. .
\]
The exact value of the couple \((Y, Z)\) at the time instance \(t = 0\) is \((Y_0, Z_0) = (1, 0)\). By the comparison theorem of BSDE, the couple of process \((Y_t, Z_t)\) is included in the bounded domain \([-1, 1] \times [-1, 1]\). The following figure shows the log-representation of the relative error curve induced by the numerical estimation of the couple \((Y_0, Z_0)\).

![Graph showing error curve](image)

Figure 1: Error curve on the estimation of \((Y_0, Z_0)\) with \(k=8, M=10000\)

### 5.3.2 Application 2

The second example is defined by the following backward stochastic differential equations. As in the first example, the underlying process is the Weiner process \(W\) on the time interval \([0, T]\). We consider the system

\[
\begin{align*}
-dY_t &= g(s, W_s, Y_s, Z_s)dt - Z_t dW_t, \\
Y_T &= \phi(W_T),
\end{align*}
\]

where the terminal condition and the driver functions are defined as follows:

\[
\begin{align*}
\phi(x) &= x \arctan(x) - \ln(\sqrt{1 + x^2}), \\
f(t, X_t, Y_t, Z_t) &= -\frac{1}{2(1 + \tan^2(Z_t))}.
\end{align*}
\]

It is easy to check by the Itô’s Lemma, that the solution of the above system is

\[
(Y_t, Z_t) = (-\frac{1}{2} \ln(1 + W_t^2) + W_t \arctan(W_t), \arctan(W_t)), \quad \text{a.s.}
\]

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By noticing that the function \( x \mapsto \ln(x) \) satisfies the linear growth condition and the function \( x \mapsto \arctan(x) \) is bounded, one has

\[
(Y_t, Z_t)_{0 \leq t \leq T} \in S^2(\mathbb{R}) \times \mathbb{R}^2.
\]

The exact value of the couple \((Y_t, Z_t)\) at the time instance \(t_0\) is \((Y_0, Z_0) = (0, 0)\). The following figure shows the log-representation of the relative error curve induced by the numerical estimation of the couple \((Y_0, Z_0)\). Modulo the choice of \(|\pi|\) and the number of the basis function \(k\), the following numerical illustration shows a stable convergence order regarding the estimation of the couple \((Y_0, Z_0)\).

![Error curve on the estimation of \((Y_0, Z_0)\) with \(k=6, M=10000\)](image)

Figure 2: Error curve on the estimation of \((Y_0, Z_0)\) with \(k=6, M=10000\)

Nonetheless the convergence regarding the estimation of the initial value \(Y_0\) is more stable and quicker than the estimation of the initial value \(Z_0\) in the first example. The convergence of BSDE could be accelerated by some two-step schemes or Runge-Kutta method (see [6], [1], [17], [15]).
6 Conclusion

We have discussed in this paper the numerical approximation of a markovian backward stochastic differential equation on a certain time interval \([0,T]\) with a Gaussian prescribed terminal value. By developing the solution of the BSDE as a Fourier-hermite expansion, we have shown that the problem of solving the backward stochastic differential equation is equivalent to solve a system of countable system of ordinary differential equation. We have derived from the previous connection a numerical algorithm to solve the BSDE via the standard Euler scheme. An interesting remark is the simplicity of the algorithm and its application in high dimensional problem. The two examples in the last section show a stable convergence regarding the computation of the solution of the backward stochastic differential equation. One possible development of this work would be to investigate the case where the forward component of the backward stochastic differential equations is no longer a Brownian motion but a semimartingale.
7 Annexes

Young Inequality. For any $\alpha > 0$ and for any $a, b \in \mathbb{R}$

$$(a + b)^2 \leq (1 + \alpha)a^2 + (1 + \frac{1}{\alpha})b^2$$

We will alter the language slightly and call the previous inequality the Young Inequality. Let us now recall the classical discrete Gronwall Lemma 5.4 in [28].

Lemma 7.3 (Gronwall Inequality (1)). Consider the partition $\pi : 0 = t_0 < \ldots < t_N = T$ of the interval $[0, T]$ and $\Delta_t$ its mesh. Consider the two families $(a_k)_{0 \leq k \leq N}, (b_k)_{0 \leq k \leq N}$ assumed to be non-negative such that for some positive constant $\gamma > 0$ we have:

$$a_{k+1} \leq (1 + \gamma \Delta_t)a_k + b_k, \quad k = 1, \ldots, N.$$  

Then,

$$\max_{0 \leq t \leq N} a_t \leq e^{\gamma t}(a_0 + \sum_{i=1}^{N} b_i).$$

Lemma 7.4 (Gronwall Inequality (2)). Let $y, b, a : \mathbb{R}^+ \mapsto \mathbb{R}$ three continuous functions such that the function $b$ is non-negative,

$$y(t) \leq a(t) + \int_0^t b(s)y(s)ds.$$  

Then,

$$y(t) \leq a(t) + \int_0^t a(s)b(s) \exp \left( \int_s^t b(u)du \right)ds.$$  

More if the function $a$ is non-decreasing or monotone, we have

$$y(t) \leq a(t) \exp \left( \int_0^t b(s)ds \right)$$

Theorem 7.3 (Crouzeix & Michel [9]). Let $I = [t_0, T + t_0]$ be an interval of $\mathbb{R}$ and $f$ a continuous function

$$f : I \times \mathbb{R}^m \mapsto \mathbb{R}^m$$

$$(t, x) \mapsto f(t, x).$$

We assume also that there exists an integrable function $\zeta$ on $I$ such that the function $f$ satisfies

$$\forall (t, x) \in T \times \mathbb{R}^m, \quad (f(t, x), x) \leq \zeta(t)(1 + \|x\|^2).$$

Then there exists a global solution of the following Cauchy problem

$$\begin{cases} 
\dot{y}(t) = f(t, y(t)) & \text{where} \quad (t, y(t)) \in I \times \mathbb{R}^m \\
y(t_0) = y_0, & y_0 \in \mathbb{R}^m.
\end{cases}$$  

(7.85)
References


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