Dynamic Stochastic Programming Tools for Individual Asset Liability Management

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Personal Finance

Household Dilemmas

• What are optimal consumption and saving decisions?
• What are the risks in matching cash inflows from assets with the cash outflows of liabilities?

“Is Personal Finance an exact science? An immediate flat no. ... It is a domain full of ordinary common sense. Common sense is not the same thing as good sense. Good sense in these esoteric puzzles is hard to come by”

Paul Samuelson
Financial Advice

- Regulatory calls (across developed economies) for *transparency* and for a *responsible investment approach* suitable for funds’ specific member profile, liquidity needs and liabilities

- In the UK Financial Advice has changed (Retail Distribution Review) from 1.1.2013
  - Only two types of financial adviser - Independent or Restricted
  - All financial advisory companies are required to clearly set out the charges the client will pay prior to providing any advice - to ensure clients are fully aware of the costs involved: fees only (not commissions) for independent advisers
  - Adviser ‘s duty to clarify on what they are able to, and are not able to, advise

*Unintended consequences*

- Many big names on the high street dramatically cut back on face-to-face financial advisory services, or withdrew them altogether. This is particularly the case for clients with less than £100,000 invested
- Rapid growth of Fund Platforms: *fund wraps and fund supermarkets*
Fund Platforms
Technology risk? Model risk?

- Fund Supermarkets – DIY route to investment in ISA and SIPP
  - Hargreaves Lansdown, Fidelity FundsNetwork, Alliance Trust Savings iNvest (Own), Barclays Stockbrokers MarketMaster (Own), J.P. Morgan Asset Management WealthManager+ (Own), Bestinvest Select (SEI), ...

- Promises of ‘goal-based’ investment; ‘strategic allocations matched to client’s needs’; ...

- These new promises are implemented how?

- It seems that all portfolio asset allocation models are based on some sort of mean-variance optimization (MVO) model

- Is the static view of risks and their dependencies applicable for long-term investment decisions?
‘The myth of risk attitudes’
Daniel Kahneman JPM Fall 2009

“To understand an individual’s complex attitudes towards risk we must know both the size of the loss that may destabilize them, as well as the amount they are willing to put in play for a chance to achieve large gains. Temporary perspectives may be too narrow for the purpose of wealth management. The theories - utility theory and its behavioural alternatives - assume that individuals correctly anticipate their reaction to possible outcomes and incorporate valid emotional prediction into their investment decisions. In fact, people are poor forecasters of their future emotions and future tastes – they need help in this task – and I believe that one of responsibilities of financial advisors should be to provide that help.”

- iALM and Dynamic Stochastic Programming solution
Dynamic Stochastic Optimization Approaches

- There are four theoretical/computational approaches to the optimization of Markov stochastic dynamical systems
  - Discrete state and time dynamic programming using Bellman’s principle of optimality and forward or backward recursion or policy iteration
  - Discrete state and time Markov chains using linear programming techniques pioneered by Howard
  - Continuous state and time dynamic programming solving the Bellman PDE numerically
  - Dynamic stochastic programming in discrete time using mathematical programming algorithms

- Of these only dynamic stochastic programming can handle an arbitrary number of risk factors – the others are restricted to 3 or 4 – and DSP can relax the Markov assumption practically
Dynamic Stochastic Programming

- General idea of dynamic stochastic programming
  - Incorporate many alternative futures in the form of a tree of simulated scenarios for the discrete-time stochastic data process

\[ \omega := \{ \omega_t : t = t_{1,0}, \ldots, t_{T+1,0} \} \]

\[ = \{ \omega_{t_1,0}, \ldots, \omega_{t_1,u} , \omega_{t_2,0}, \ldots, \omega_{t_2,u} , \ldots, \omega_{t_T,0}, \ldots, \omega_{t_T,u} \} \]

- Model future decisions and implement current ones to obtain a forward plan to the problem’s horizon based on the scenario tree sample
  - When the current recommended solution is implemented it takes into account future solutions across all future scenarios generated with equal probability
Scenario Tree Schema

1st stage 2d stage 3d stage

root node leaf node

Scenario 8

A multi-period 3-3-2 scenario tree
Multi-stage Dynamic Stochastic Programme

\[
\min_{x_{t_0}, \ldots, x_{t_1, \omega}} f_1(x_{t_1, \omega}) + \mathbb{E}_{\omega \in \Omega_2} \left\{ \min_{x_{t_2, \omega}} f_2(\omega_{t_2, \omega}, x_{t_2, \omega}) + \ldots + \mathbb{E}_{\omega \in \Omega_{T-1, \omega}} \left[ \min_{x_{t_{T-1}, \omega}} f_{T}(\omega_{t_{T-1}, \omega}, x_{t_{T-1}, \omega}) \right] \right\}
\]

s.t.
\[
A_{11} x_{t_1, \omega} = b_1
\]
\[
A_{21}(\omega_{t_1, \omega}) x_{t_1, \omega} + A_{22}(\omega_{t_1, \omega}) x_{t_1, \omega} = b_2(\omega_{t_1, \omega}) \quad \text{a.s.}
\]
\[
\vdots
\]
\[
A_{T_{u+1}, \omega} x_{t_1, \omega} + \ldots + A_{T_{u+1}, T_{u} + 1}(\omega_{t_{T-1}, \omega}) x_{t_{T-1}, \omega} = b_{T_{u+1}}(\omega_{t_{T-1}, \omega}) \quad \text{a.s.}
\]

Deterministic Equivalent

\[
\min \left\{ f_1(x_{t_1}) + \sum_{\omega_{t_2, \omega} \in \Omega_{t_2, \omega}} p_{t_2, \omega}(\omega_{t_2, \omega}) f_{t_2}(\omega_{t_2, \omega}, x_{t_2, \omega}) + \ldots + \sum_{\omega_{t_{T-1}, \omega} \in \Omega_{t_{T-1}, \omega}} p_{t_{T-1}, \omega}(\omega_{t_{T-1}, \omega}) f_{t_{T-1}}(\omega_{t_{T-1}, \omega}, x_{t_{T-1}, \omega}) \right\}
\]

s.t.
\[
A_{11} x_{t_1, \omega} = b_1
\]
\[
A_{21}(\omega_{t_1, \omega}) x_{t_1, \omega} + A_{22}(\omega_{t_1, \omega}) x_{t_1, \omega} = b_2(\omega_{t_1, \omega}) \quad \omega_{t_1, \omega} \in \Omega_{t_1, \omega}
\]
\[
\vdots
\]
\[
A_{T_{u+1}, \omega} x_{t_1, \omega} + \ldots + A_{T_{u+1}, T_{u} + 1}(\omega_{t_{T-1}, \omega}) x_{t_{T-1}, \omega} = b_{T_{u+1}}(\omega_{t_{T-1}, \omega}) \quad \omega_{t_{T-1}, \omega} \in \Omega_{t_{T-1}, \omega}
\]
Solution Methods

- Scenario history is a possible realization of the random vector and corresponds to a node of the scenario tree.
- Deterministic equivalent of the dynamic stochastic program (DSP) is a very large sparse linear programming (LP) problem.
  - coefficients and right hand sides in the constraints are realizations of the stochastic data process.
- Solution method for linear objectives.
  - nested Benders decomposition and/or interior point.
- The vector process for the optimal stochastic decision process is given by

\[
x := \{x_t : t = t_{1,0}, \ldots, t_{T,u}\} = \{x_{t_{1,0}}, \ldots, x_{t_{1,u}}, x_{t_{2,0}}, \ldots, x_{t_{2,u}}, \ldots, x_{t_{T,0}}, \ldots, x_{t_{T,u}}\}
\]

- Implementable decisions correspond to the root node of the scenario tree.
Stochastic Programming Techniques for iALM

Simulation
- Generation of **stochastic data** with a discrete number of annual observations of a continuous time vector data process branching at specified times (decision times) in the future
- **Scenario tree** is a schema for forward simulation – along each branch a multiple number of stochastic processes are simulated. Some are independent, other may be correlated.
- Simulation **discrete time steps** correspond to the data sampling frequency of the process of interest
- **iALM involves simulation of asset returns and liabilities punctuated by life events**

Optimization
- Discrete time and state optimization giving a **different optimization problem** (given by its objective and constraints) at each node of the scenario tree dependent on both its predecessors and successors
- Major decision time points are stages of the tree
- **Implementable decisions** are at the root node which are the most constrained decisions robust against all alternative scenarios generated while the remainder allow what-if prospective analysis
- **iALM solves a large scale linear optimization problem**
  - Consumption (goal) maximization at each decision time subject to constraints such as risk, budget, cash flow balance and so on annually
  - **Sustainable wealth maximization** across all years and all generated scenarios simultaneously
Modelling Changing Financial Markets

**Investment securities**
- Domestic and International Equities
- Government Bonds
- Corporate Bonds
- Alternatives
- T-bills and all bond coupons
- Treasury Inflation Protected Securities (TIPS)
- Cash
- CPI
- Other fixed assets

**Fundamental financial models**

**Multi-dimensional GBM process**

\[ d \ln X_{i,t} = \mu_i \, dt + \sigma_i \, dW_{i,t} \]

**Geometric Ornstein Uhlenbeck (OU) process**

\[ d \ln r_t = (\alpha - \beta \ln r_t) \, dt + \sigma \, dW_t \]

**OU process**

\[ dr_t = (\alpha - \beta r_t) \, dt + \sigma \, dW_t \]
Annual Returns of the S&P 500 Index
Overview of individual ALM

Gather Individual and Market Data

Econometric and Actuarial Modelling

Scenario Tree Simulation

Optimization Model:
Tailored Portfolios, Goal Spending, Cashflow Balances, etc

Personal data → Market data

Events model

Liabilities model

Model returns on investment classes

Events → Cash out-flow forecasts → Cash in-flow forecasts

Dynamic optimization model for assets-liabilities

Objective: maximize spending on risk managed goal

Visualization of decisions

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Meta-Model Generation with \textit{STOCHASTICS}™
Individual Financial Planning Tool

*iALM*

- The 
  *iALM* system is a decision support tool based on the theory of dynamic stochastic optimization

- Principal ideas are brought together from behavioural and classical finance and from decision theory

- Due to significant uncertainties involved in identification of future feasible consumption the user may interactively re-solve the problem with inputs varied in individual preferences and goal’s priorities. This allows the user to compare the consequences of these changes on the long-term financial plan – the ‘gaming’ aspect of solution process

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**Broad Framing:** overall objective is to provide ‘sustainable spending’ over a household’s lifetime in terms of desired consumption on multiple life goals specified by preferences and their priorities

**Narrow Framing:** maximization of goal consumption at given times (annually)
- Each single goal utility function is defined with respect to reference points chosen by the household in terms of spending on the goal

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Value Function of Prospect Theory

- Recall the Value Function of Prospect Theory

\[ v(x) = x^\alpha \text{ if } x > 0 \]
\[ v(x) = -\lambda(\frac{x^\alpha}{x}) \text{ if } x > 0 \]

with a typical \( \alpha = 0.88 \) and \( \lambda = 2.25 \)

reference point
Individual Goal Utility – Narrow Framing

- **Utility function** for an individual goal is given by **three reference points**
- For each single goal the **level of spending** \( y \) is in the **range between acceptable (s) and desirable (g) and minimum (h)** spending subject to existing and foreseen liabilities. Together with **goal priorities** these values specify the piecewise linear shape of the utility function for each goal
- The **objective** is to maximize **goal spending** with a piecewise linear utility function for the year

\[ u(\text{utility}) = \begin{cases} \alpha_h y & \text{if } y < h \\ \alpha_s (y - s) + s & \text{if } s \leq y \leq g \\ \alpha_g (y - g) + g & \text{if } g < y \end{cases} \]
Utility of Initial Wealth as Lifetime Consumption

Optimal expected utility as function of starting wealth

Utility value

(in 1000£)
**Overall Objective – Broad Framing**

- To provide ‘sustainable spending’
- Optimization problem objective is to **maximize the expected present value** (over all scenarios) of **lifetime consumption**, i.e. spending on goals

\[
E \left[ \sum_{t=1}^{T} 1_{\{\text{any alive},t\}} u_t(C) \right]
\]

where

\[
u_t(C) = \sum_{g \in G} u_{g,t} - \frac{1}{\varphi_t} \left( \pi^{xs} z^{xs}_t + \pi^{rt} i_t^r \right)
\]

Here \(z^{xs}_t\) is excess borrowing, \(i_t^r\) is total tax payment and \(\varphi_t\) is the inflation index at \(t\)

- **Consumption** refers to all “elective” spending on chosen goals

\[
C_t = \sum_{g \in G_m} \alpha^{\text{alive}}_{g,t} \varphi_{g,t} \left( F^d_{g,t} + F^m_{g,t} \right) + \sum_{g \in G \setminus G_m} \alpha^{\text{alive}}_{g,t} \varphi_{g,t} y_{g,t}
\]
Dynamic Stochastic iALM Formulation

- Decision theory type overall objective - optimum resource allocation over the network of income and outcome cash flows
- Optimal management of various portfolios subject to varieties of constraints
  - Management and transaction fee constraints
  - Initial wealth constraints
  - Cashflow balance constraints
- Random time horizon ALM problem
- Stochastic dynamic analysis takes account of many complex dependencies
  - Number of goals
  - Timing of liabilities and goals with significant values
  - Dependency of asset allocation on life expectancy: longevity hedging
  - Health statistics
  - ...
One Goal – Savings for Retirement

Net wealth

Returns

Portfolio

Cash holding

Qualified portfolio

Transaction costs

Retirement Goal

Liabilities

Taxation

Transaction costs (qualified portfolio)

Coupons and dividends

Interest on bank deposits

Regular income

Employer pension contributions

Qualified coupons and dividends

Qualified returns

Contributions qualified

Regular income

Employer pension contributions

Qualified coupons and dividends

Qualified returns

Returns

Coupons and dividends

Interest on bank deposits

Regular income

Employer pension contributions

Qualified coupons and dividends

Qualified returns

Cash holding

Net wealth

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Cash Flow Network

Net wealth

- Returns
- Coupons and dividends
- Interest on bank deposits
- Regular income
- Employer pension contributions
- Qualified coupons and dividends
- Qualified returns
- Portfolio
- Cash holding
- Margin borrowing
- Goal Equity
- Loans secured on assets
- Excess borrowing
- Income borrowing
- Qualified portfolio

Transaction costs
- Interest charges on margin loans
- Interest on goal loans
- Goal consumption (non capital)
- Liabilities
- Taxation
- Unauthorized qualified withdrawal penalty
- Interest charges on secured borrowing
- Interest charges on income loans
- Interest charges on excess borrowing
- Transaction costs (qualified portfolio)
Summary of iALM Features

- Goal-oriented objective – expected life time spend
- More uncertainty accounted for – not just uncertainties of market returns but also uncertainties of personal life events
- DSP methodologies using Stochastics™
- Optimal portfolio decisions correspond to the best feasible desirable consumption subject to existing and future liabilities
- Portfolio risks are managed by
  - constraining portfolio drawdown in each scenario
  - imposing limits on portfolio asset holdings in each scenario
- Simultaneous multiple accounts management with optimal treatment of taxes (rule-based assumptions applicable to various jurisdictions)
iALM Overview

Visual Summary of Profile

Goals

Cash Flows

Portfolio

Wealth
Example I

- A 40 year old couple with 2 dependents

Starting assets:
- Non-Qualified Asset Account: £71,000
- SIPP Account: £15,000
- ISA Account: £35,000
- Tangible Assets (Family home): £400,000

Income: Pre-Retirement Earning: £133,000

- Objective to maintain a desirable level of consumption pre- and after retirement

<table>
<thead>
<tr>
<th>Household Consumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>Priority</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td></td>
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<tr>
<td></td>
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<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Example II

Basic household data as in Example I
Additional Objective: to provide for private education of children

**Liability:** Mortgage 2000-2020 £150,000

### Household Consumption

<table>
<thead>
<tr>
<th>Priority</th>
<th>Name</th>
<th>Minimum</th>
<th>Acceptable</th>
<th>Desirable</th>
<th>Growth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Pre-Retirement</td>
<td>9,200</td>
<td>75,400</td>
<td>86,100</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Post-Retirement</td>
<td>9,200</td>
<td>62,400</td>
<td>68,100</td>
<td></td>
</tr>
</tbody>
</table>

### Education

<table>
<thead>
<tr>
<th>Priority</th>
<th>Beneficiary</th>
<th>StartDate</th>
<th>Years</th>
<th>Minimum</th>
<th>Acceptable</th>
<th>Desirable</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>John</td>
<td>2013-01-01</td>
<td>7</td>
<td>8,600</td>
<td>10,400</td>
<td>12,600</td>
<td>School</td>
</tr>
<tr>
<td>5</td>
<td>Jess</td>
<td>2013-01-01</td>
<td>6</td>
<td>8,600</td>
<td>10,400</td>
<td>12,600</td>
<td>School</td>
</tr>
<tr>
<td>5</td>
<td>John</td>
<td>2020-01-01</td>
<td>4</td>
<td>6,000</td>
<td>7,200</td>
<td>8,800</td>
<td>Uni</td>
</tr>
<tr>
<td>5</td>
<td>Jess</td>
<td>2019-01-01</td>
<td>4</td>
<td>6,000</td>
<td>7,200</td>
<td>8,800</td>
<td>Uni</td>
</tr>
</tbody>
</table>

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Example I - Retirement Planning Solution

Living Expenses
(Fallen short of desirable level on average)

Retirement Spending
(Met or exceeded desirable level on average)

Wealth Evolution: Expected

Dynamic Asset Allocation: Expected

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Example II – Multiple Goals Solution

Living Expenses
(Fallen short of desirable level on average)

Retirement Spending
(Met or exceeded desirable level on average)

Wealth Evolution: Expected

Dynamic Asset Allocation: Expected

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Sensitivity of Investment Decisions to Goals

Number of Life Goals

<table>
<thead>
<tr>
<th>Year</th>
<th>Initial Assets</th>
<th>Spending</th>
<th>Size</th>
<th>Return</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Now</td>
<td>£521,000</td>
<td>£86,100</td>
<td>129,376</td>
<td>5.7%</td>
<td>9.6%</td>
</tr>
<tr>
<td>2013</td>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2019</td>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2020</td>
<td>65</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>65</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Sensitivities for Expected Retirement

- Sensitivities for Education-Uncertainty
  - Spending: £86,100
  - Size: 96,692
  - Return: 6.0%
  - Risk: 10.6%

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Sensitivity of Investment Decisions to Goals

Timing of Goal

<table>
<thead>
<tr>
<th>Goal</th>
<th>Size</th>
<th>Return</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Now</td>
<td>106,433</td>
<td>4.1%</td>
<td>5.9%</td>
</tr>
<tr>
<td>Retirement</td>
<td>106,246</td>
<td>6.1%</td>
<td>10.9%</td>
</tr>
<tr>
<td>Now</td>
<td>97,647</td>
<td>6.0%</td>
<td>10.5%</td>
</tr>
</tbody>
</table>

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Sensitivity of Investment Decisions to Uncertainties of Life Events

First Allocation

SA actuarial life table
(Return: 5.45%, Vol: 8.58%)

UK actuarial life table
(Return: 5.91%, Vol: 10.24%)

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Sensitivity of Investment Decisions to Uncertainties of Life Events

Stochasticity of Life Expectancy

Dynamic Asset Allocation Expected

SA actuarial life table

UK actuarial life table
Dynamic iALM

Static (MVO type)
Dynamic vs Static Consumption Benefit

Increase in expected spending

<table>
<thead>
<tr>
<th></th>
<th>Dynamic</th>
<th>Static</th>
</tr>
</thead>
<tbody>
<tr>
<td>School fees John</td>
<td>10,400</td>
<td>10,400</td>
</tr>
<tr>
<td>School fees Jess</td>
<td>10,400</td>
<td>10,400</td>
</tr>
<tr>
<td>University Fees John</td>
<td>7,306</td>
<td>7,200</td>
</tr>
<tr>
<td>University Fees Jess</td>
<td>7,306</td>
<td>7,200</td>
</tr>
<tr>
<td>Living Expenses</td>
<td>82,617</td>
<td>77,972</td>
</tr>
<tr>
<td>Retirement Spending</td>
<td>72,154</td>
<td>63,670</td>
</tr>
<tr>
<td>Terminal Wealth</td>
<td>167,647</td>
<td>99,293</td>
</tr>
</tbody>
</table>
Technology & HPC Technique Impact

![Bar chart showing CPU time (sec.) from 2005 to 2013]
Technology & HPC Technique Impact

- The improvement in run time from 2005 to 2012 is solely due to hardware improvement.
- The run time in 2013 (approx. half of 2012) is the result of the use of optimization algorithms that benefit from a parallel computing environment.
- Benders decomposition type algorithms such as Stochastics’ GNBS typically benefit greatly from HPC techniques.
- Asset return models’ Markov property allows efficient implementation and numerical solution.
**iALM – Summary**

- **iALM** provides **optimum values** for many decision variables – spending, borrowing, saving, etc -- **across time simultaneously** for multiple scenarios of random processes representing uncertain markets and life circumstances.

- Current **iALM model** includes **20 random processes** that vary over the client’s lifetime and around 200 **mathematically formulated conditions (constraints)** per node.

- Average **solution time** is currently **around 1 minute** (for problem size 3mln to 5mln non-zero entries).

- Further significant improvements are being tested which combine **algorithmic techniques** and new **parallel HPC techniques**.
References

- Dempster et al. (2009). Risk profiling defined benefit pension schemes. *Journal of Portfolio Management* **35.4** 76-93