A Unified Risk Minimization Framework

Hassan Jawaid

Aarhus University

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Research Project:

Project title: A unified risk minimization framework (ESR 1).
Description: Companies offering variable annuities are exposed to financial risks, i.e., equity performance, volatility risk, interest rate risk, and credit risks, in addition to insurance risk (mortality, longevity,and surrender). The aim of our research project is:

- To integrate insurance models with financial models for overall risk management with variable annuities.
- To apply risk minimization strategies for variable annuities.

Working Papers

1. Effective Methods for Pricing Variable Annuities with Guaranteed Minimum Withdrawal Benefits.
2. Pricing and Hedging of the European Option Linked to Target Volatility Portfolio.
3. An Analysis of Guaranteed Lifetime Withdrawal Benefits Linked to Target Volatility Portfolio.

Agenda

- Managing equity exposure by the help of risk-based re-balancing strategy.
- The target volatility portfolios are gaining popularity as an important asset class.
- Currently, insurance companies are trending to provide pension products with the target volatility portfolios.
- Pricing of the financial derivatives linked to target volatility portfolio is not straight forward.
- Price of the financial derivatives linked to target volatility portfolio depends critically on modeling assumption, rebalancing frequency and historical volatility estimation.
- Market indexes: FTSE 100 Risk target Excess Return, EuroStoxx 50 Risk Control Index, Dow Jones Volatility Control Index, S&p Risk Control Indexes.

Basic Idea

- The main purpose of using the target volatility approach is to keep the realized volatility of the portfolio under a certain level. We can achieve constant realized volatility by a dynamic allocation between the equity asset and fixed income asset. In addition, dynamic allocation depends on the realized historical volatility of risky asset.
- The target volatility approach reduces the realized volatility of the underlying portfolio.
- Financial derivatives linked to target volatility portfolios are less expensive compared to financial derivatives linked to standard equity or equities index.

The Financial Model

- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Also $\mathcal{G}_t = \sigma\left(S(s) : 0 \leq s \leq t\right)$ is a subfiltration of $\mathcal{F}_t$ containing all information about financial market.

$$\frac{dS_j(t)}{S_j(t)} = \zeta_j \cdot dt + \sum_{k=1}^{d} \sigma_{j,k}(t) \cdot dW_k(t),\ S_j(t) \geq 0,\ j = 1, 2, \cdots, N, \tag{1}$$

and the dynamics of self-financing portfolio Z without consumption is expressed as:

$$\frac{dZ(t)}{Z(t)} = \alpha_0(t) \cdot \frac{dS_0(t)}{S_0(t)} + \sum_{j=1}^{N} \alpha_j(t) \cdot \frac{dS_j(t)}{S_j(t)}, \tag{2}$$

where $S_j(t)$ is the price of equity asset and $\alpha_j(t)$ is a relative weight of asset j and $\alpha_0(t) = 1 - \sum_{j=1}^{N} \alpha_j(t)$. Moreover, we will not borrowed the amount $\alpha_0(t_j)$ in the risk-free asset $(\alpha_0(t_j) \geq 0)$.

The Financial Model

- Consider a financial market consists of two assets, an index with price process $S(t)$ and a risk free bond $S_0(t)$.The dynamics of the risk-free asset $S_0(t)$ is defined as:

$$\frac{dS_0(t)}{S_0(t)} = r(t) \cdot dt, \quad S_0(0) = 1, \tag{3}$$

In our setup, we assume that the market consists of one equity asset $S(t)$ and one risk-free asset $S_0(t)$ and we name the relative portfolio Z as the target volatility portfolio (TVP):

$$\frac{dZ(t)}{Z(t)} = (\alpha_1(t) \cdot (\zeta - r(t)) + r(t)) \cdot dt + \alpha_1(t) \cdot (\sigma(t) \cdot dW(t)) . \tag{4}$$

where $W(t)$ is a $\mathbb{P}-$Wiener process.

- let define a random variable U which will depend upon a past price trajectory of $\{S(s); t \geq s\}$ and consider $\alpha_1(t)$ and $\alpha_2(t)$ are stochastic process and their value depends upon the realized historical volatility of a risky asset S,

$$\alpha_1(t) = \begin{cases} \min\left(\frac{VT}{U(u_k)}, 1\right) & \text{if } u_k \leq t < u_{k+1}; \\ \min\left(\frac{VT}{U(u_M)}, 1\right) & \text{otherwise.} \end{cases} \tag{5}$$

$$= \alpha_1(0)\mathbf{1}_{\{0 \leq t < u_1\}} + \sum_{k=1}^{M-1} \min\left(\frac{VT}{U\left(u_k\right)}, 1\right)\mathbf{1}_{\{u_k \leq t < u_{k+1}\}}$$

$$+ \min\left(\frac{VT}{U\left(u_M\right)}, 1\right)\mathbf{1}_{\{t \leq u_M < T\}},$$

$$\alpha_2(t) = 1 - \alpha_1(t),$$

where VT is a volatility target and $U\left(u_k\right)$ is estimated historical volatility of a risky asset S at the re-balancing time u_k.

Discretization of a Target Volatility Portfolio

- The allocation of weights depends on the value of an annualized historical volatility of risky asset under the assumption of 252 business days and we take a time step of one day. Moreover, we estimate a annualized daily historical volatility of risk asset by EWMA method:

$$R(t_j) = \ln\left(\frac{S(t_{j+1})}{S(t_j)}\right),$$

where $\Delta t_j = t_{j+1} - t_j$, $t_0 = 0 < t_1 < t_2 \cdots < t_{N-1} = T$,

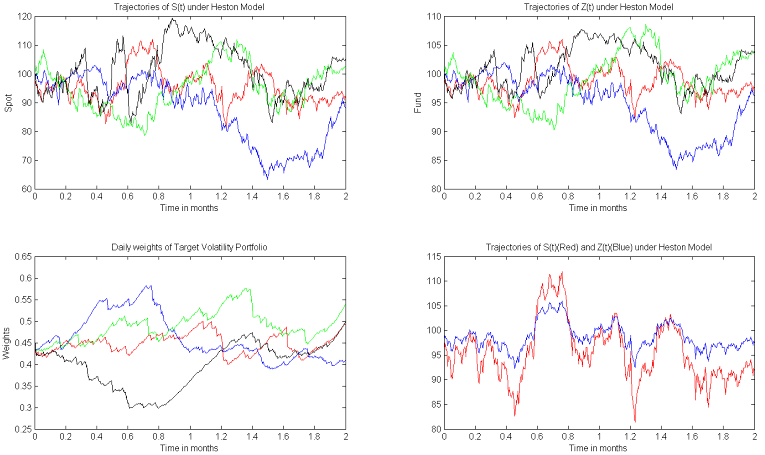
$$\widetilde{\sigma}^2_{t_{j+1}} = \lambda \cdot \widetilde{\sigma}^2_{t_j} + (1-\lambda) \cdot \frac{1}{\Delta t_j} R^2(t_j).$$

- The weights are re-balanced on u_k date and u_k can be chosen daily, weekly or monthly.

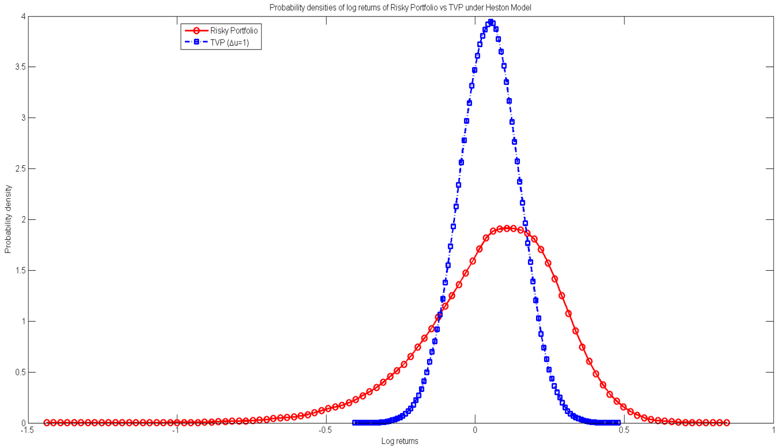
$$U(u_k) = \widetilde{\sigma}_{t_k}, \tag{6}$$

$$\alpha_1(t_j) = \min\left(\frac{VT}{U(u_k)}, 1\right), \ \textit{if} \ \ u_k \leq t_j < u_{k+1} \ \& \ \alpha_2(t_j) = 1 - \alpha_1(t_j).$$

Paths of Target volatility portfolio under Heston Model



Distributional Properties of Target Volatility Portfolio under Heston Model (Daily re-balancing)



Unified Model for Unit-linked insurance products (Variable annuities)

- We extended the existing literature on GLWB pricing and hedging by incorporating simultaneously:

1. Equity risk
2. Mortality risk
3. Volatility risk

- A unified model that manages the equity risk (target volatility strategy) also incorporates mortality risk.

Stochastic Model for Mortality

- In our setup, $\mathcal{I}_t = \mathcal{G}_t \vee \mathcal{M}_t$ where $\mathcal{G}_t = \sigma\left(S(s) : 0 \leq s \leq t\right)$ and $\mathcal{M}_t = \sigma\left(\mu_{x+s}(s) : 0 \leq s \leq t\right)$ contain all information about the financial market and mortality rates. **The Non mean reverting CIR process** (Luciano (2005)):

$$d\mu_{x+t}(t) = a \cdot \mu_{x+t}(t) + \sigma_\mu \sqrt{\mu_{x+t}(t)} \cdot dW_x(t), \;\; \mu_x(0) > 0, \qquad (7)$$

$$\mu_x(0) = \alpha_\mu + \beta_\mu \cdot \gamma_\mu^x, \qquad \alpha_\mu, \beta_\mu, \gamma_\mu > 0.$$

We set the market price of the mortality risk equals to zero.
Benchmark parameters for the mortality model:[1]

a	σ_μ	α_μ	β_μ	γ_μ	*error*
0.0963	0.0100	0.0030	1.32×10^{-5}	1.1094	0.0075

[1] We calibrate the model with the survival probabilities ${}_tp_{65}(0)$ of American male aged $x = 65$.

Stochastic Model for Mortality

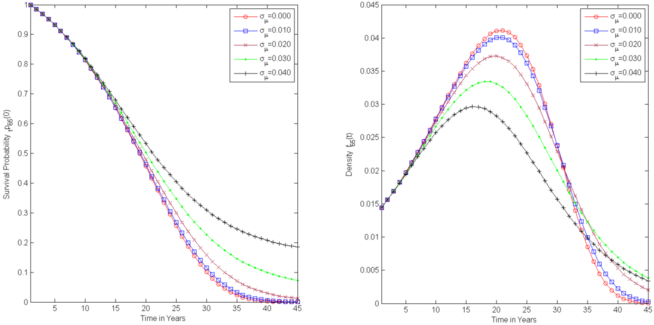


Figure: Survival probabilities and densities of remaining lifetime of the policyholder aged 65.

Policyholder Wealth Process

- Let $I(t)$ denotes the policyholder's investment account. The investment account is depleted by continuous consumption at the rate $g(t)$ and the insurance fee α_f. The dynamics of the investment account $I(t)$ is described as:

$$dI(t) = -\alpha_f \cdot I(t) \cdot dt + I(t) \cdot \frac{dZ(t)}{Z(t)} - \gamma(t) \cdot dt, \ \ I(t) \geq 0, \qquad (8)$$

Lifetime ruin probability (LPoR) is defined as the probability that the policyholder's investment account, defined in equation (8), will hit zero while the policyholder is still alive:

$$LPoR = \mathbb{P}\left[\inf_{0 \leq t \leq \tau_x} I(t) = 0 \mid I(0) = P\right] \qquad (9)$$

Results (Ruin probabilities)

Retirement age	Expected age at death	Annual Withdrawal Rate per 100$			
		2%	4%	6%	8%
65	**83.48**	**0.01%**	**3.13%**	**21.24%**	**46.33%**
70	84.21	0.00%	1.48%	13.29%	34.39%
75	86.07	0.00%	0.48%	6.82%	21.82%
80	88.25	0.00%	0.14%	2.52%	10.74%

Table: Ruin probability approximation for the target volatility portfolio at different retirement ages.

Sensitivity Analysis with Respect to Financial Risk

Long-term Variance (Volatility Risk)	Withdrawal Rate 5%	5.5%	6%
$\theta^* = 0.198^2$, $r = 4\%$	0.14%	0.31%	0.68%
$\theta^* = 0.220^2$, $r = 4\%$	**0.15%**	**0.32%**	**0.69%**
$\theta^* = 0.251^2$, $r = 4\%$	0.15%	0.33%	0.69%
$\theta^* = 0.220^2$, $r = 3\%$	0.49%	0.92%	1.88%
$\theta^* = 0.220^2$, $r = 5\%$	0.07%	0.15%	0.31%

Table 8: Fair fee rate $\alpha_f^*(\%)$ per annum for GLWB linked to target volatility portfolio under different market prices of volatility risk and interest rates.

Sensitivity Analysis with Respect to Mortality Risk

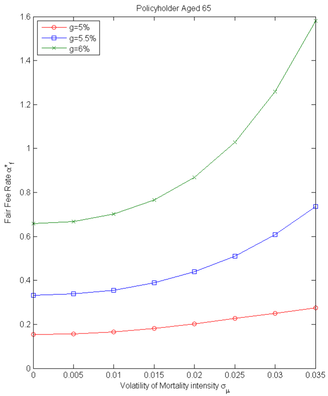


Figure: Sensitivity of the fair fee rate α_f^* with respect to the volatility of mortality intensity σ_μ.

Conclusion

- We find that the distributions of the daily, weekly and monthly re-balanced target volatility portfolios become more normalized under the Heston model.
- The benefits of the target volatility portfolio can be achieved with weekly re-balancing frequency and it can save transaction costs and computation time for the product designer.
- The target volatility strategy significantly reduces the ruin probabilities for withdrawal rate ranges from $4 - 6\%$ per annum.
- The fair fee rates for GLWB linked to TVP is around $0.14 - 0.68\%$ per annum in contrast to GLWB linked to equity portfolio $(0.68 - 1.84\%)$.
- The P&L distributions show that Value-at-Risk for GLWB linked to TVP exponentially increases with low interest rate levels.

Challenges

- The target volatility strategy does not entirely remove the impact of stochastic volatility.
- More research is needed on the impact of different volatility measures e.g.
  Econometric models (GARCH, NGARCH etc.)
  Implied Volatility measures (VIX, VIXC etc.)
- Finally yet importantly, a hybrid model is needed that also incorporates stochastic interest rates for valuing long-term insurance products linked to TVP.

Any Questions

Literature

- Guido. G (2012), Optimal Design of Risk-Controlled Investment Strategies .
- S. Albeverio , V. Steblovskaya , K. Wallbaum (2013), Investment instruments with volatility target mechanism.
- Ewald, C.-O., Poulsen, R., Schenk-Hoppe, K.R. (2007), Risk Minimization in Stochastic Volatility Models: Model Risk and Empirical Performance.
- Jawaid, Hassan, Pricing and Hedging of the European Option Linked to Target Volatility Portfolio (August 1, 2015). Available at SSRN: http://ssrn.com/abstract=2743699.
- Jawaid, Hassan, An Analysis of Guaranteed Lifetime Withdrawal Benefits Linked to Target Volatility Portfolio (February 1, 2016). Available at SSRN: http://ssrn.com/abstract=2743780.
- Luciano, E. and Vigna, E. (2005). Non mean reverting affine processes for stochastic mortality.