(RP13) Efficient numerical methods on high-performance computing platforms for the underlying financial models: Series Solution and Option Pricing

Jun Hu

Tampere University of Technology

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Formula

Please check out solutions for

- European option for stochastic volatility models (Heston, GARCH and 3/2).
- American option for the Black-Scholes model.

at

`https://github.com/optionseries`

Results

Publication:

- Asymptotic expansion of European options with mean-reverting stochastic volatility dynamics, *Finance Research Letters*, **14**, 1–10
- Limit order book models and market phenomenology, Project report
- Doctoral dissertation

Code:

- European option for stochastic volatility models
- American option for the Black-Scholes model

Series solution to ODE

Consider the ordinary differential equation

$$\frac{d}{dx}V(x) = V(x), \tag{1}$$

$$V(0) = 1. \tag{2}$$

The solution is $V(x) = e^x$. Suppose we do not know the answer and try to obtain the series solution

$$V(x) = \sum_{i=0}^{\infty} a_i x^i, \tag{3}$$

$$\frac{d}{dx}V(x) = \sum_{i=0}^{\infty} i a_i x^{i-1} = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i. \tag{4}$$

Then the ODE becomes

$$\sum_{i=0}^{\infty}(i+1)a_{i+1}x^i = \sum_{i=0}^{\infty} a_i x^i, \tag{5}$$

$$a_0 = 1. \tag{6}$$

Equivalently

$$a_{i+1} = \frac{1}{i+1}a_i, \tag{7}$$

$$a_0 = 1. \tag{8}$$

The result is

$$a_i = \frac{1}{i!} \Rightarrow V(x) = \sum_{i=0}^{\infty} \frac{1}{i!}x^i = e^x. \tag{9}$$

Exactly the same result as we expect. Series solutions has the *potential* to be exact.

Series solution and pricing PDE (simplified)

The pricing PDE usually looks like

$$\mathcal{B}V = p\mathcal{L}V, \tag{10}$$

$$V(p=0) = S\,\mathrm{N}(\cdot) - Ke^{-rt}\,\mathrm{N}(\cdot). \tag{11}$$

In comparison

$$V(x) = \frac{d}{dx}V(x), \tag{12}$$

$$V(0) = 1. \tag{13}$$

Similarly, we propose

$$V(p) = \sum_{i=0}^{\infty} V_i(x, t, \ldots)p^i. \tag{14}$$

The original PDE

$$\mathcal{B}V = p\mathcal{L}V, \tag{15}$$
$$V(p=0) = S\,\mathrm{N}(\cdot) - Ke^{-rt}\,\mathrm{N}(\cdot), \tag{16}$$

becomes

$$\sum_{i=0}^{\infty} \mathcal{B}V_i p^i = p\mathcal{L}\sum_{i=0}^{\infty} V_i p^i = \sum_{i=1}^{\infty} \mathcal{L}V_{i-1} p^i, \tag{17}$$
$$V_0 = S\,\mathrm{N}(\cdot) - Ke^{-rt}\,\mathrm{N}(\cdot), \tag{18}$$

equivalently

$$\mathcal{B}V_0 = 0, \tag{19}$$
$$\mathcal{B}V_i = \mathcal{L}V_{i-1} \Rightarrow V_i = \mathcal{B}^{-1}\mathcal{L}V_{i-1}, \tag{20}$$
$$V_0 = S\,\mathrm{N}(\cdot) - Ke^{-rt}\,\mathrm{N}(\cdot). \tag{21}$$

Difference

Iteration for ODE

$$a_{i+1} = \frac{1}{i+1} a_i, \tag{22}$$

$$a_0 = 1, \tag{23}$$

leads to

$$a_i = \frac{1}{i!}. \tag{24}$$

Iteration for PDE

$$V_i = \mathcal{B}^{-1} \mathcal{L} V_{i-1}, \tag{25}$$

$$V_0 = S\,\mathrm{N}(\cdot) - K e^{-rt}\,\mathrm{N}(\cdot), \tag{26}$$

is hard to generalize, because the iteration usually involves "complicated" integration, e.g.

$$V_1 = e^{ay+bs} \int_0^s dt \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi v t}} \exp\left(-\frac{(x-y)^2}{2v(s-t)} - ax - bt \right) \times \partial_v \left[S\,\mathrm{N}(\cdot) - K e^{-rt}\,\mathrm{N}(\cdot) \right]. \tag{27}$$

Better choice of basis

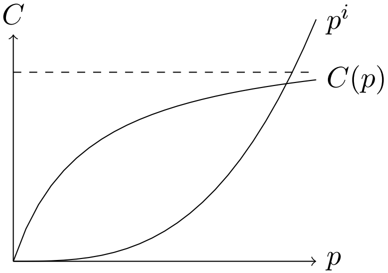
The option price is bounded

$$0 < C(p) < S, \quad \text{for} \quad q \in (0, \infty). \tag{28}$$

However p^i is not, therefore

$$\sum_{i=0}^{N} V_i p^i \tag{29}$$

always blows up at $p \to \infty$.

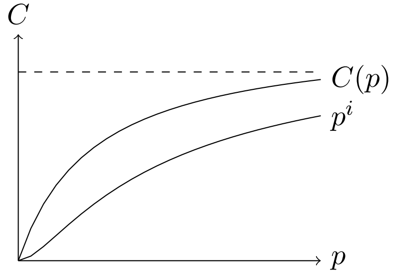


The solution: basis function with better asymptotic behavior

$$\left(\frac{p}{1+p}\right)^i, \tag{30}$$

because

$$\lim_{p\to 0}\left(\frac{p}{1+p}\right)^i = p^i, \qquad \lim_{p\to\infty}\left(\frac{p}{1+p}\right)^i = 1. \tag{31}$$



$$\boxed{C(p) = \sum_{i=0}^{\infty} V_i p^i = \sum_{i=0}^{\infty} \bar{V}_i \left(\frac{p}{1+p}\right)^i} \tag{32}$$

Best expansion (so far)

European option under stochastic volatility models looks like

$$C = \sum_{i,j=0}^{\infty} u_{ij} \left(\frac{\eta}{1+\eta}\right)^i \left(\frac{v-\theta}{1+v-\theta}\right)^j . \tag{33}$$

Error is bounded

$$\lim_{\eta,v\to\infty} \sum_{i,j=0}^{N} u_{ij} \left(\frac{\eta}{1+\eta}\right)^i \left(\frac{v-\theta}{1+v-\theta}\right)^j = \sum_{i,j=0}^{N} u_{ij} < \infty, \tag{34}$$

$$\lim_{\eta,v\to\infty} \sum_{i,j=0}^{N} u_{ij}\eta^i (v-\theta)^j \to \infty. \tag{35}$$

Scale invariance

Scale invariance for the Heston model

$$V(t, x, r, v, \theta, \rho, \eta, \kappa) = V\left(\frac{t}{\lambda}, x, \lambda r, \lambda v, \lambda\theta, \rho, \lambda\eta, \lambda\kappa\right), \tag{36}$$

because

$$\begin{aligned}\lambda\partial_t V - \frac{\lambda v}{2}\left(\partial_x^2 - \partial_x\right)V - \lambda r\left(\partial_x - 1\right)V \\ - \rho\lambda\eta\lambda v\partial_x\frac{1}{\lambda}\partial_v V - \frac{1}{2}\lambda^2\eta^2\lambda v\frac{1}{\lambda^2}\partial_v^2 V - \lambda\kappa\lambda(v-\theta)\frac{1}{\lambda}\partial_v V \end{aligned} \tag{37}$$

$$\begin{aligned}= \lambda\Big(\partial_t V - \frac{v}{2}\left(\partial_x^2 - \partial_x\right)V - r\left(\partial_x - 1\right)V \\ -\rho\eta v\partial_x\partial_v V - \frac{1}{2}\eta^2 v\partial_v^2 V - \kappa(v-\theta)\partial_v V\Big) = 0.\end{aligned} \tag{38}$$

Scale invariance for the Heston model

$$V(t, x, r, v, \theta, \rho, \eta, \kappa) = V\left(\frac{t}{\lambda}, x, \lambda r, \lambda v, \lambda\theta, \rho, \lambda\eta, \lambda\kappa\right) \tag{39}$$

Scale invariance for the GARCH model

$$V(t, x, r, v, \theta, \rho, \eta, \kappa) = V\left(\frac{t}{\lambda}, x, \lambda r, \lambda v, \lambda\theta, \rho, \sqrt{\lambda}\eta, \lambda\kappa\right) \tag{40}$$

Scale invariance for the 3/2 model

$$V(t, x, r, v, \theta, \rho, \eta, \kappa) = V\left(\frac{t}{\lambda}, x, \lambda r, \lambda v, \lambda\theta, \rho, \eta, \kappa\right) \tag{41}$$

Breaking of symmetry (Scale invariance)

Scale invariance for the Heston model is

$$V(t,x,r,v,\theta,\rho,\eta,\kappa) = V\left(\frac{t}{\lambda}, x, \lambda r, \lambda v, \lambda\theta, \rho, \lambda\eta, \lambda\kappa\right). \tag{42}$$

Symmetry is broken possibly (in fact it is not!) for

$$\sum_{i,j=0}^{N} u_{ij}\eta^i (v-\theta)^j. \tag{43}$$

Because $\frac{\lambda\eta}{1+\lambda\eta} \neq \lambda\frac{\eta}{1+\eta}$, symmetry is broken definitely for

$$\sum_{i,j=0}^{N} u_{ij}\left(\frac{\eta}{1+\eta}\right)^i \left(\frac{v-\theta}{1+v-\theta}\right)^j. \tag{44}$$

We have a new degree of freedom to manipulate convergence.

Comparison

	$v = 0.04$	0.24	0.44	0.64	0.84
$\eta = 0 \quad (\eta, v)$	0.0993	0.1265	0.1597	0.2738	0.6822
$(\eta + 1, v + 1)$	0.0993	0.1263	0.1479	0.1671	0.1850
FFT	0.0993	0.1263	0.1476	0.1658	0.1819
0.5	0.0985	0.1248	0.1531	0.2517	0.6241
	0.0986	0.1249	0.1459	0.1641	0.1808
	0.0985	0.1249	0.1458	0.1637	0.1794
1	0.0956	0.1197	0.1367	0.2009	0.4998
	0.0955	0.1216	0.1422	0.1599	0.1759
	0.0954	0.1212	0.1419	0.1596	0.1752
1.5	0.0936	0.1117	0.1078	0.1157	0.3001
	0.0918	0.1177	0.1382	0.1556	0.1712
	0.0915	0.1166	0.1370	0.1545	0.1700
2	0.0967	0.1003	0.0607	-0.0150	0.0084
	0.0881	0.1139	0.1344	0.1517	0.1670
	0.0876	0.1118	0.1318	0.1490	0.1643

American option

PDE (after front-fixing)

$$\partial_t V - \frac{v}{2}\left(\partial_x^2 - \partial_x\right)V - r(\partial_x - 1)V = d'(t)\partial_x V, \tag{45}$$

defined on $(t,x) \in (0,\infty) \times (0,\infty)$, with

$$V(0,x) = 0, \tag{46}$$

$$V(t,0) = 1 - e^{d(t)}, \tag{47}$$

$$\partial_x V(t,0) = -e^{d(t)}. \tag{48}$$

Difficulties

- $d(t)$ and $V(t,x)$ are coupled and should be solved simultaneously
- $d'\partial_x V$ is non-linear
- No obvious starting point $p = 0$
- Integration $\mathcal{B}^{-1}V$

Special functions

We denote the set of following functions Σ_1

$$e^{ax+bt}\operatorname{erfc}\left(\frac{x}{\sqrt{2vt}}\right)\sum_{i,j}A_{ij}x^i\sqrt{t}^j+\exp\left(ax+bt-\frac{x^2}{2vt}\right)\sum_{i,j}B_{ij}x^i\sqrt{t}^j, \tag{49}$$

the set of following functions Σ_2

$$\sum_i C_i\sqrt{t}^i. \tag{50}$$

If we propose the expansion

$$V=\sum_{i=0}^{\infty}V_i(t,x)p^i, \quad d=\sum_{i=0}^{\infty}d_i(t)p^i. \tag{51}$$

Then iteration can go on

$$V_i(t,x)\in\Sigma_1, \quad d_i(t)\in\Sigma_2. \tag{52}$$

Advanced models

The PDE looks like

$$\partial_t V - \frac{v}{2}\left(\partial_x^2 - \partial_x\right) V - r(\partial_x - 1)V = d'(t)\partial_x V + \mathcal{O}V, \tag{53}$$

defined on $(t, x) \in (0, \infty) \times (0, \infty)$, with

$$V(0, x) = 0, \tag{54}$$

$$V(t, 0) = 1 - e^{d(t)}, \tag{55}$$

$$\partial_x V(t, 0) = -e^{d(t)}. \tag{56}$$

Actually, we are solving

$$\partial_t V - \frac{\theta}{2}\left(\partial_x^2 - \partial_x\right) V - r(\partial_x - 1)V = d'(t)\partial_x V + \frac{v - \theta}{2}\left(\partial_x^2 - \partial_x\right) V. \tag{57}$$

It has better convergence when $r \gg v$, when we set $\theta = r$.

Numerical results

Time	Boundary	-0.2	-0.1	0	0.1	0.2
0.03	Price	76.801	83.773	92.584	102.32	113.08
	Tree	24.199	16.227	7.4473	1.0484	0.0140
	Series	24.224	16.252	7.4410	0.7727	0.0036
0.1	Price	71.879	79.439	87.793	97.027	107.23
	Tree	28.121	20.561	12.223	4.9207	1.1108
	Series	28.170	20.610	12.256	4.6009	0.8297
0.3	Price	67.146	74.208	82.013	90.638	100.17
	Tree	32.854	25.792	17.987	10.644	5.2976
	Series	32.904	25.842	18.037	10.512	4.9747
1	Price	61.874	68.382	75.574	83.522	92.306
	Tree	38.126	31.618	24.426	17.427	11.838
	Series	38.109	31.602	24.410	17.331	11.439
3	Price	59.037	65.246	72.108	79.692	88.073
	Tree	40.963	34.754	27.892	21.459	16.281
	Series	41.072	34.863	28.001	21.142	15.097

HPC

There are two types of calculations involved in expansion methods.

- Calculation of formula. It is very computationally intensive, therefore HPC is needed. The computation is parallelizable, e.g. $\mathcal{B}^{-1}(V_1 + V_2) = \mathcal{B}^{-1}V_1 + \mathcal{B}^{-1}V_2$. The formula can be calculated and published. In research, Mathematica is used, however more efficient languages can be used.
- Calculation of price. It is within the power of a PC, and HPC is not needed.

Summary

- Expansion methods have the "potential" to solve complicated option under complicated models, which is dimensionally more difficult for finite-difference methods.

	European	American
Black-Scholes	trivial	✓
Beyond Black-Scholes	✓	✓

- Expansion methods are easy to implement. HPC is not needed for end-users.
- Greeks are straight-forward to calculate.
- Pricing option families is efficient.

Please try the code.
Thank you!
Questions?