Option Pricing with Expansion Methods

TAMPEREEN TEKNILLINEN YLIOPISTO
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New Approaches to Advanced Stochastic Volatility Models and American Options

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Abstract

In the thesis, we aim to develop a new framework for pricing advanced options quickly and accurately. Specifically, we price European options under stochastic volatility models and American options under the Black−Scholes model using expansion methods, which are widely used in physical sciences.

Efficient and accurate pricing of option contracts has long been the central problem of mathematical finance. Apart from the classic Black−Scholes model, which has a closed-form solution, there is no universally accepted method for pricing of options. The Monte Carlo simulation is general but slow, while finite-difference and numerical integration (Fourier transform) methods are comparably accurate but are not always applicable to exotic options and non-affine models. Furthermore, although those popular numerical methods can provide decent estimates of option prices, their discrete nature makes it difficult for them to achieve efficiency and accuracy simultaneously. While considerable amount of research has been devoted to these methods, we believe that the potential of expansion methods are underestimated.

The expansion methodology, which is widely used in mathematics and physics, divides an unknown quantity into an infinite and converging series whose neighbouring terms are related by algebraic or differential equations. Therefore, starting from the known leading terms, we can work out the iteration equations and obtain any number of terms in the series, as long as the iteration equations are exact and explicitly solvable. In the context of finance, the option price can be expanded as an infinite series of analytical functions, which are related by the pricing partial differential equation (PDE). Besides the flexibility to deal with different models and options, the biggest advantage of expansion methods is that, users can evaluate the formulae derived by the author by plugging in parameter values, which greatly reduces computational intensity.

First, we show that European options under stochastic volatility models can be expanded with various pairs of parameters in the volatility process, such as initial volatility, speed of mean-reversion, volatility of volatility and long-term volatility. The methods use powers of parameters as basis functions, and work with small parameter values. To achieve better performance, a modified version of expansion with initial volatility and volatility of volatility is proposed to reduce the pricing error when the parameters are large. The new method uses bounded basis functions, rather than the unbounded power series, and the numerical results confirm that the promotion from unbounded to bounded greatly improves the ability of expansion methods to approximate option prices. Moreover, symmetry considerations are also helpful for expansion methods. When the scale invariance is broken, we are equipped with one more degree of freedom to fine-tune the convergence, which is not proven or guaranteed.

Then, we show that the non-linear problem of American options under the Black−Scholes model can be solved as a series of special functions that we defined earlier. Such special functions remain in the same family under many operations, making explicit expression of the option prices possible. We formally demonstrate two of the many ways of expansion, which work numerically except in the case of low volatility and high interest rate. Thus, an improved version, is proposed. It treats the Black−Scholes model as an advanced model with an additional operator. The improved method is able to deal with reasonable values of volatility, interest rate, moneyness and maturity. Finally, we outline the possibility of combining advanced models with advanced option types. American options can be treated similarly under many popular models, as long as the extra operators preserve the closedness of the special functions.

The main contribution of the thesis is the demonstration that expansion methods can be used efficiently with non-affine stochastic volatility models and American options, which have no closed-form solutions. Additionally, explicit formulae, instead of formal relations in terms of integrals, are derived and available for reproduction.

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Contents

1 Introduction

1.1 Background and motivation

An option is a financial instrument, that gives its holder the opportunity to trade something in the future. With a European option, the holder can buy or sell an asset in the future at a predefined price. It is like an insurance contract against market risk, as it protects the holders (i.e. financial institutions or retail investors) from big losses when the market moves against their prediction.

Like any insurance, protection comes with a price. Although millions of option contracts are traded around the world everyday, the way in which they are priced is far from unanimous or easy.

Even though the price of an option is agreed upon by the buyers and sellers, either in the exchange or over the counter, the payoff of the option solely depends on the future price movement of the underlying asset. Therefore, we can obtain a rough estimate of the option price by assuming the future evolution of the underlying asset. Such an assumption is called an option pricing model, the most famous of which is the Black−Scholes model [1]. This Nobel prize-winning model is surprisingly simple, yet it explains most market dynamics. It serves as a role model and benchmark for the more complicated models that follow. However, there are some market characteristics that the Black−Scholes model is unable to account for, such as the smile and time inhomogeneity of volatility. Advanced models are thus needed to bridge the gap between theory and reality. Unfortunately, closed-form solutions, such as in the European Black−Scholes case, are not common among advanced models. Rather, numerical methods are usually required for option pricing in most cases.

From both industrial and academic perspectives, there is great interest in a unified framework for pricing different types of options under different types of models. This thesis attempts to provide a new perspective, in which different option types and different models are treated equally with expansion methods.

1.1.1 Stochastic volatility models

In the classical Black−Scholes model [1], the underlying asset is assumed to have a constant volatility, which controls the amplitude of the geometric Brownian motion

$$\mathrm{d}S_t = S_t(\mu\,\mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t), \tag{1.1}$$

where μ is a constant drift, $\sqrt{v_t}$ is usually denoted by a constant σ and W_t is a Wiener process. However, what we observed in the real market is that volatility is far from constant. Sometimes the market moves much more dramatically than normal, such as during the financial crisis in 2008 [2]. One way of dealing with this problem is through the introduction of stochastic volatility.

In a continuous-time setting, the variance (square of volatility) process of stochastic volatility models is governed by an independent SDE, usually written in the following form

$$\mathrm{d}v_t = f(v_t)\,\mathrm{d}t + g(v_t)\,\mathrm{d}W_t, \tag{1.2}$$

where W_t is a Wiener process and $g(v_t)$ is a non-negative function. With extra terms, this representation can be extended to include jumps [3, 4] and several coupled Wiener processes [3, 5].

Another phenomenon observed in the market is that, although volatility is not constant microscopically (in days), it does move around a certain level within a macroscopic time frame (in years). Therefore, $f(v_t)$ in (1.2) is often proposed to be of the form

$$f(v_t) = \kappa(\theta - v_t)v_t^{\alpha}, \tag{1.3}$$

where $\kappa > 0$. In such a setting, $f(v_t)$ is positive when $v_t < \theta$ and negative when $v_t > \theta$. In other words, there is a stochastic 'force' pulling the volatility back to its long-term mean θ when v_t moves away from θ due to the contribution $g(v_t)\,\mathrm{d}W_t$, which usually takes the form

$$g(v_t) = \eta v_t^{\beta}, \tag{1.4}$$

where $\eta > 0$. With the introduction of this process, the mean-reverting stochastic volatility models (1.1) and (1.2) greatly improve option pricing performance, with respect to the real market [6−10].

While (1.1) and (1.2) cover a family of models, the Heston model [11] ($\alpha = 0$ and $\beta = \frac{1}{2}$) is particularly interesting due to the positivity of v_t (under Feller's condition) and the tractability of the model. The option price can be written as an inverse Fourier transform, and the fast Fourier transform (FFT) technique can be applied [12−14]. Unfortunately, the FFT only applies to the Heston and 3/2 models, while other cases still rely on Monte Carlo simulation.

At the same time, considerable research, with strong empirical evidence, suggests that non-affine volatility models, for example, the continuous-time GARCH model ($\alpha = 0$ and $\beta = 1$) and the 3/2 model ($\alpha = 1$ and $\beta = 3/2$), outperform the affine Heston volatility specification with time-series and option price data in continuous time [15−18] and in discrete time [19, 20]. In fact, according to [17], the simple continuous-time GARCH diffusion model performs even better than the affine jump-diffusion model. Unfortunately, due to the lack of known characteristic function of the GARCH model, the efficient FFT methods cannot be applied. Therefore, in Chapter 3, prices produced using Monte Carlo methods for all models and FFT for the Heston and 3/2 models are provided as a reference for the methods developed in this thesis.

1.1.2 American options

The American option differs from the European option in one additional feature, an American option can be exercised at any time before maturity. It is this simple addition that makes the American option significantly more difficult to price than the European version.

The valuation of American options has long been a problem in financial research. The first and most intuitive attempt to price American option was (binomial) tree methods [21]. By going backwards in time, at every node of the tree, the expected value of holding and the payoff of immediate exercise can be easily compared. If volatility is constant, the number of nodes at each point in time grows as $n+1$, instead of 2^n otherwise. When the time step goes to zero, the discrete-time approximation converges to the continuous-time value. The biggest drawback of this type of methods is that, although they produce the finest result for the Black−Scholes model, they cannot be applied to advanced models.

From a PDE point of view, there are two representations of American option pricing. The first corresponds to a linear complementary problem

$$\mathcal{B}V\,[V-g(x)]=0, \quad \text{with} \quad \mathcal{B}V \geq 0, \quad V-g(x)\geq 0, \tag{1.5a}$$

$$V(0,x)=g(x), \quad g(x)=(e^x-K)^+. \tag{1.5b}$$

In this representation, the equation in (1.5a) governs the domain $(t,x)\in[0,\infty)\times(-\infty,\infty)$, and therefore there is no need to explicitly calculate the moving boundary. This representation is widely used in finite-difference methods [22−26].

The second representation bridges the 'hold' and 'exercise' regions with the 'smooth-pasting' condition on the moving boundary.

$$\mathcal{B}V=0, \quad \text{with} \quad V(0,x)=0, \quad B(0)=0, \tag{1.6a}$$

$$V(t,B(t))=1-e^x, \quad \partial_x V(t,B(t))=-e^x. \tag{1.6b}$$

The equations above only govern the 'hold' region $x\geq B(t)$. The coupled option price $V(t,x)$ and the moving boundary $B(t)$ should be solved simultaneously. The expansion methods developed in the thesis are based on this representation.

The Monte Carlo simulation is widely used to price European options, but much research was required to determine how it could be implemented for American options [27]. By regressing the expected payoff of the next time step on the current stock price, we can calculate the average 'expected payoff' of all the paths in the money rather than that of a single path. The method works for most models; however, like most Monte Carlo methods, a large number of paths must be generated to yield acceptable results. In addition, the least-squares regressions at every time step are very computationally intensive.

American option pricing can also be treated as an optimal stopping problem of a stochastic process [28, 29]. For an American put option, when the price is above the moving exercise boundary, it is optimal to hold the option. When the price is below the moving boundary, the put option should be exercised. However, although the stopping time approach is conceptually important, it is not very helpful in terms of numerical computation. Early exercise premium (EEP) [30−32] is another way to present the problem. American option prices can be written as the sum of the corresponding European option and a premium, which accounts for the early exercise feature. From a mathematical point of view, EEP is a financial realisation of the Riesz decomposition of a supermartingale. With the possibility of exercising at any time, the expected value of an American option is no longer a martingale, as in the European case, but rather a supermartingale, which is the Snell envelope of the payoff. This relationship was established [33] and applied [34] to the American option under the Black−Scholes model. In order to calculate option prices, we must solve the recursive integral equation for the unknown moving boundary and then substitute the boundary to the formula for the EEP.

There are many techniques for improving the performance of the basic ideas discussed above. [35−38] provide a comprehensive representation of the methods.

1.1.3 Numerical methods

The most popular and widely used numerical method is the Monte Carlo simulation [39]. The idea is to first generate paths, according to the model's dynamical stochastic differential equations (SDEs), for price and other variables in small time steps, until maturity. Then, the discounted average payoff provides a good estimate of the option price. Due to their intuitiveness, Monte Carlo methods are usually the first tool we use to tackle a new problem when we know little about it. However, these methods are often not optimal when we know enough about the problem, as they are inaccurate and slow due to discretisation errors and the need for a large number of paths.

Finite-difference methods [40−42] solve the option pricing problem from the PDE perspective. Differentiation in the PDE is replaced with the difference of a few adjacent points on a grid in variable space:

$$\partial_x V(x) = \lim_{\delta x \to 0} \frac{V(x+\delta x) - V(x)}{\delta x} \approx \frac{V(x+\Delta x) - V(x)}{\Delta x}, \tag{1.7}$$

where Δx takes a small value, and higher order differentials can be constructed similarly. The pricing PDE is then transformed to an algebraic equation relating to values in a small neighbourhood. The price at any time point can be obtained by solving the grid backwards in time. The methods work well for low dimensional problems, because the number of nodes in the mesh grows exponentially with the dimensionality.

The price of a European options can be written as an inverse Fourier transform:

$$V(k) \propto \int_{-\infty}^{\infty} e^{\mathrm{i}ku} g(u)\, \mathrm{d}u, \tag{1.8}$$

where $g(u)$ involves the model characteristic function. The integral can be evaluated numerically by carefully choosing the step of u-discretization, so that $e^{\mathrm{i}ku}$ is periodic therefore computation is reduced [11, 12]. Such (semi)-analytical solutions are often regarded as closed-form in finance community. In fact, they are not so different than other numerical methods, in terms of discretization error. Furthermore, processes that have an explicit characteristic function only cover a small part of models that we are interested in.

1.2 Research questions and methodology

The idea of parameter expansion is heavily used in the physical sciences [43, 44], where the solution to a problem can be written as an infinite power series of some model parameters. Although, in many cases, the solution is wrapped up as a simple special function, credit for solving the problem should be given to expansion methods, as special functions often have an infinite series representation.

Option pricing problems share one important feature of a physical problem. The value of many options is determined by a PDE with boundary conditions at maturity and elsewhere. Therefore, it is natural to assume that the expansion methods, which have been proven to be highly successful in physical sciences, can be used similarly in a finance context.

In this thesis, we attempt to answer the following questions:

- How well can expansion methods be applied to option pricing problems beyond the European option under the Black−Scholes model?
- Expansion methods usually work well as expansion parameters go to zero. However, do the methods work for reasonably large parameters that are realistic in the actual market?
- Are the series solutions convergent? If not, how efficient and accurate are the finite-term approximations?

The general methodology used in the research can be described as follows. For the sake of simplicity, we take a model with one more parameter than the Black−Scholes model as an example.

- Propose an expansion form. Assume we can choose a physical or dummy variable p, such that the pricing PDE can be written as

  $$\mathcal{B}V = p\mathcal{L}V, \tag{1.9}$$

  where $\mathcal{B}$ is the Black−Scholes operator and $\mathcal{L}$ is the operator associated with p. The solution can therefore be written as

  $$V = \sum_{i=0}^{\infty} V_i p^i. \tag{1.10}$$

- Derive the iteration formula for V_i. After bringing the ansatz (1.10) back to (1.9) and equating the coefficients of p^i, we arrive at the relation of the form

  $$V_i = \mathcal{B}^{-1}\mathcal{L}V_{i-1}. \tag{1.11}$$

- Calculate the *explicit* form of V_i-terms in the above equation, up to a predefined order N

  $$V \approx \sum_{i=0}^{N} V_i p^i. \tag{1.12}$$

- Calculate the option prices using (1.12) with specific parameter values and compare the results with those produced by reference methods (Monte Carlo methods and/or FFT). Determine whether the formula works in all reasonable scenarios.
- If the results are not satisfying, we repeat the above steps attempting to find a new expansion form with better numerical properties.

The real problems we solve in later chapters follow the same methodology, though the technicalities involved are much more complicated.

1.3 Related literature and contribution of the thesis

Previous papers have explored the possibility of applying the expansion idea to particular models with some success. Hagan et al. [45] obtained option price asymptotics for the SABR model. Lewis [46] derived the two series expansion for the Heston and 3/2 models. Pagliarani and Pascucci [47] and Lorig et al. [48] introduced heat kernel expansion to local volatility models. Papanicolaou et al. [49] and Chiu et al. [50] applied singular perturbation methods to the family of Ornstein−Uhlenbeck driven stochastic volatility models. Park and Kim [51−53] and Leung [54] solved the constant elasticity of variance model (CEV) and Heston models using the homotopy analysis method (HAM), which was originally developed by Liao [55, 56]. Zhu [57] solved American options under the Black−Scholes model with the HAM. The HAM is an elegant and powerful framework for solving complicated PDE problems. However, we find the homotopy concept is unnecessary in some cases. The homotopy parameter p serves as an indicator for differentiating orders of expansion, and such a role can also be played by model parameters.

The expansion methods described in Chapters 3 and 4 are the main contribution of the thesis.

1.3.1 European options under stochastic volatility models

First, we consider mean-reverting stochastic volatility models, which are governed by SDEs:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}W_t, \tag{1.13a}$$

$$\mathrm{d}v_t = \kappa(\theta - v_t)v_t^{\alpha}\,\mathrm{d}t + \eta v_t^{\beta}\,\mathrm{d}Z_t, \tag{1.13b}$$

$$\mathrm{d}[W,Z]_t = \rho\,\mathrm{d}t, \tag{1.13c}$$

where r is the risk-free interest rate, κ is the mean-reverting rate and η is the volatility of volatility. The option prices under such models are solutions to the PDE:

$$\partial_t u - \frac{v}{2}\partial_x^2 u + \left(\frac{v}{2} - r\right)\partial_x u + ru - \rho\eta v^{\beta+\frac{1}{2}}\partial_x\partial_v u - \frac{1}{2}\eta^2 v^{2\beta}\partial_v^2 u - \kappa(\theta - v)v^{\alpha}\partial_v u = 0, \tag{1.14}$$

where $x = \ln S$ is the log-price and t denotes the time to maturity. In the current literature, several papers have explored the possibility of writing the solution as an infinite series, from the PDE point of view. Park and Kim [51] derived expansion series for European, barrier and lookback options under the CEV model. In [52], Park and Kim derived European and barrier options under stochastic volatility models. Leung [54] solved lookback options under the Heston model.

Their methods can be summarised as follows. With the pricing PDE generally written as

$$\mathcal{B}V + p\mathcal{O}V = 0, \tag{1.15}$$

where $\mathcal{B}$ is the Black−Scholes operator and $\mathcal{O}$ denotes the remaining operators

$$\mathcal{B} = \partial_t - \frac{v}{2}\partial_x^2 + \left(\frac{v}{2} - r\right)\partial_x + r, \tag{1.16}$$

they propose that the option price can be written as

$$V = \sum_{i=0}^{\infty} V_i p^i, \tag{1.17}$$

where p is a homotopy parameter. When p varies continuously from 0 to 1, the solution V varies accordingly from the Black−Scholes solution to the advanced model solution, which in this case is the Heston model. Substituting the ansatz (1.17) into the PDE (1.15), they manage to derive iterative relations between V_i and V_{i-1} in integral form

$$\begin{aligned} V_i(s,y) &= \mathcal{B}^{-1}\mathcal{O}V_{i-1}(t,x) \\ &= e^{ay+bs}\int_0^t \frac{\mathrm{d}t}{\sqrt{2\pi v(s-t)}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{2v(s-t)} - ax - bt\right)\mathcal{O}V_{i-1}(t,x)\,\mathrm{d}x, \end{aligned} \tag{1.18}$$

where $\mathcal{B}^{-1}$ involves the double integral of x and t. However, the expansion terms V_i are not given explicitly, nor is it proven that the integral (1.18) can always be analytically evaluated. Since numerical analysis is given very briefly in those papers, we assume that the integrals are evaluated numerically. However, numerical integration, which introduces an additional discretisation error, can greatly undermine the accuracy and efficiency of the methods.

In Chapter 3, we show that the option price of a stochastic volatility model can be expanded in four different ways. More importantly, we contribute to the current literature by proving that the iteration relations of the form (1.18) can be analytically evaluated (i.e. an explicit formula, rather then a number, can be obtained as a result of the double integral).

Expansion	Form
(κ,η)	$V = \sum_{i,j=0}^{\infty} u_{ij}\kappa^i\eta^j$
(η,v)	$V = \sum_{i,j=0}^{\infty} u_{ij}\eta^i(v-\theta)^j$
(η,θ)	$V = \sum_{i,j=0}^{\infty} u_{ij}\eta^i(\theta-v)^j$
(κ,v)	$V = \sum_{i,j=0}^{\infty} u_{ij}\left(\frac{1}{\kappa}\right)^i(v-\theta)^j$

Table 1.1: Four ways of decomposing option price of stochastic volatility models.

It can be shown that the (κ,η)-expansion is equivalent to the results obtained by the HAM [52] because the terms derived from both methods match when the sum is taken to infinity and p is set to 1. By abandoning the HAM framework, which is widely used in the literature, we are able to find three new methods ((η,v), (η,θ) and (κ,v)), which are not equivalent to the HAM and, according to numerical analysis, outperform it for long-term options. The results up to now have been published in our paper [58], for which I took primary responsibility for the concept, computation and presentation.

Apart from the unbounded power series (e.g. η^i) used in previous methods, we find that the bounded power series (e.g. $\eta^i/(1+\eta)^i$) can greatly reduce pricing errors when the expansion parameters go to infinity. This is due to the fact that, with respect to model parameters, option prices are also bounded. Therefore, they can be approximated better with bounded power series, which have similar asymptotic properties. All the power series in the four methods above can be promoted to the corresponding bounded version

$$\sum_{i,j=0}^{\infty} u_{ij} P^i Q^j \to \sum_{i,j=0}^{\infty} \bar{u}_{ij} \left(\frac{P}{1+P}\right)^i \left(\frac{Q}{1+Q}\right)^j, \tag{1.19}$$

where P and Q can be any expansion parameter κ, η or $v-\theta$. Without loss of generality, we only show the results for $(\eta+1, v+1)$-expansion

$$V = \sum_{i,j=0}^{\infty} u_{ij} \left(\frac{\eta}{1+\eta}\right)^i \left(\frac{v-\theta}{1+v-\theta}\right)^j, \tag{1.20}$$

which is modified from (η, v)-expansion. Although convergence is not guaranteed, the error is bounded when $\eta \to \infty$ and $v \to \infty$. These are the best results we have obtained so far, and the formula is available at [59].

1.3.2 American options under the Black–Scholes model

In Chapter 4, we consider American put options under the Black−Scholes model. The renowned result for American option expansion was proposed by Zhu [57], who used the HAM idea to construct the American option PDE:

$$(1-p)\mathcal{B}\left[V(t,x,p) - V_0(t,x)\right] = -p\mathcal{A}[V(t,x,p), B(t,p)], \tag{1.21a}$$

$$V(0,x,p) = (1-p)V_0(0,x), \tag{1.21b}$$

$$V(t,0,p) + B(t,p) = 1, \tag{1.21c}$$

$$\partial_x V(t,0,p) + B(t,p) = (1-p)\left[1 + \partial_x V_0(t,0) - V_0(t,0)\right], \tag{1.21d}$$

where $\mathcal{B}$ is the Black−Scholes operator,

$$\mathcal{A}[V(t,x,p), B(t,p)] = \mathcal{B}V(t,x,p) - \frac{B'(t,p)}{B(t,p)}\partial_x V(t,x,p) \tag{1.22}$$

and $B(t)$ denotes the moving boundary. The solution can also be expanded as

$$V(t,x,p) = \sum_{i=0}^{\infty} V_i(t,x)p^i, \quad B(t,p) = \sum_{i=0}^{\infty} B_i(t)p^i, \tag{1.23}$$

with the European Black-Scholes formula being the leading term V_0 and the American option price being the sum of the infinite series $\sum_{i=0}^{\infty} V_i$. Similarly to the case in the previous subsection, the expansion terms V_i are also expressed in integral form (Equation (23) in [57]). The option prices were obtained by numerical integration. Without the explicit formula for V_i, the method is hard to implement in practice.

In Chapter 4, we will show that, with additional structures, the iteration relation (similar to Equation (23) in [57]) can be evaluated analytically, and therefore explicit formulae can be obtained.

In order to prove the computability of integrals involved in later derivations, we introduce new types of special functions that are closed under many operations including the inverse Black−Scholes double integral. We also show that by carefully dealing with the boundary conditions, the expansion terms can be written as special functions that we defined. Therefore, when evaluating option prices, we only need to plug in values for the parameters, which greatly improves computational efficiency.

For American options, because the option price is expanded as a power series of ‘dummy’ parameters, which serves as a formal indication of expansion order, there are many ways to expand an option price. In Chapter 4, two expansion methods (ABS-I and II) are proposed to show that boundary conditions can be dealt with in various ways. However, numerical analysis shows that neither of them works in cases with low volatility and a high interest rate. Therefore a modified method (ABS-III) is proposed to deal with this particular occasion. This method also shows that American options under advanced models can be expanded in the same framework, with minor modifications. Because this method works in most cases, it is recommended for practical use [59].

In summary, our primary contribution in the thesis is that we showed that expansion methods not only make formal sense, but are also practically usable. The proofs and derivations of the explicit formulae for expansion terms of various options are given, and their numerical validity is demonstrated.

1.4 Outline

The thesis is composed of six chapters. Chapter 1 briefly reviews the background of stochastic volatility models, American options and expansion methods in finance, as well as the motivation and contribution of the thesis. Chapter 2 defines a number of mathematical notions that will be used in later chapters and generally compares the procedure of series expansion to ODE and PDE. Chapter 3 applies the series expansion methods to European options under stochastic volatility models. Chapter 4 applies the series expansion methods to American options under the Black−Scholes model. Chapter 5 discusses a few validity issues and Chapter 6 concludes the thesis.

I hereby declare that I am the sole author of this thesis.

2 Mathematical preliminaries

Several mathematical results defined in this chapter will be used repeatedly in later chapters. While the idea of series expansion has been applied to various option pricing problems in the past [51, 52, 56, 57], the existing literature seems to ignore one crucial aspect: computability. The main focus of this thesis and its main contribution is, the procedure for obtaining expansion terms. The analytical form of expansion terms are rigorously derived and expressed in the special functions defined in this chapter.

2.1 Solutions to the Black–Scholes equation

According to the Black−Scholes model, the price of the underlying asset is assumed to follow the SDE, under the risk-neutral measure,

$$\mathrm{d}S_t = S_t \left(r \,\mathrm{d}t + \sqrt{v}\,\mathrm{d}W_t \right), \tag{2.1}$$

where W_t is a Wiener process, v is the constant variance, and r is the risk-free interest rate. The SDE (2.1) describes the simplest and most widely used diffusion process, and thus it is of fundamental importance in mathematical finance. Many advanced models, such as the stochastic volatility models used in Chapter 3, are extensions of the Black−Scholes model, in the sense that one or more constants in (2.1) are assumed to have their own dynamics.

In order to obtain pricing PDE for the European option price $V(S,t)$, the time evolution (derivative) of the option should be determined. With chain rule and Itô's Lemma,

$$\begin{aligned} \mathrm{d}V(S,t) &= \partial_t V \,\mathrm{d}t + \partial_S V \,\mathrm{d}S_t + \frac{1}{2}\partial_S^2 V \,\mathrm{d}[S,S]_t \\ &= \partial_t V \,\mathrm{d}t + \partial_S V S \left(r\,\mathrm{d}t + \sqrt{v}\,\mathrm{d}W_t \right) + \frac{1}{2} v S^2 \partial_S^2 V \,\mathrm{d}t. \end{aligned} \tag{2.2}$$

Given the fact that W_t is a martingale

$$\mathbb{E}\left[\frac{\mathrm{d}W_t}{\mathrm{d}t} \right] = 0 \tag{2.3}$$

and the assumption of no arbitrage, the expected value of an option grows at the risk-free interest rate r

$$\mathbb{E}\left[\frac{\mathrm{d}V(S,t)}{\mathrm{d}t} \right] = rV(S,t), \tag{2.4}$$

for any realized stock price S at time t. Therefore, the celebrated Black−Scholes equation is obtained:

$$\partial_t V + \frac{v}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = 0. \tag{2.5}$$

When the equation is being solved, the option price $V(t,x)$ is often expressed as a function of time to maturity t and log-price $x = \ln S$. As a result, the Black–Scholes equation becomes

$$\partial_t V - \frac{v}{2}\partial_x^2 V + \left(\frac{v}{2} - r\right)\partial_x V + rV = 0. \tag{2.6}$$

For the sake of argument, we might denote the Black–Scholes operator as follows.

Notation 2.1.1.

$$\mathcal{B}_v = \partial_t - \frac{v}{2}\partial_x^2 + \left(\frac{v}{2} - r\right)\partial_x + r. \tag{2.7}$$

With this notation, (2.6) can be abbreviated as $\mathcal{B}_v V = 0$. To simplify, we make the following ansatz

$$V(t,x) = e^{ax+bt}u(t,x), \quad a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2, \tag{2.8}$$

and the Black–Scholes equation (2.6) becomes the heat equation

$$\partial_t u - \frac{v}{2}\partial_x^2 u = 0. \tag{2.9}$$

Since the heat equation can be solved on various boundary (initial) conditions and is related to the Black–Scholes equation as in (2.8), corresponding solutions can be obtained for the Black–Scholes equation.

For European options, the underlying price moves on $(0,\infty)$. In log-price, the options are defined on the half-plane $(t,x) \in [0,\infty) \times (-\infty,\infty)$.

Lemma 2.1.2. *For the homogeneous Black–Scholes equation*

$$\left[\frac{\partial}{\partial t} - \frac{v}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{v}{2} - r\right)\frac{\partial}{\partial x} + r\right] g(t,x) = 0, \tag{2.10}$$

with initial condition

$$g(0,x) = h(x), \quad -\infty < x < \infty, \tag{2.11}$$

the solution is

$$g(t,x) = e^{ax+bt}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi vt}} e^{-\frac{(x-y)^2}{2vt}} e^{-ay} h(y)\,\mathrm{d}y, \tag{2.12}$$

where

$$a = \frac{1}{2} - \frac{r}{v}, \qquad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2.$$

Proof. Define

$$g(t,x) = e^{ax+bt}u(t,x), \quad a = \frac{1}{2} - \frac{r}{v}, \qquad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2. \tag{2.13}$$

Then (2.10) with (2.11) becomes the homogeneous heat equation

$$\left(\partial_t - \frac{v}{2}\partial_x^2\right) u(t,x) = 0, \tag{2.14}$$

with

$$u(0,x) = e^{-ax}h(x), \quad -\infty < x < \infty. \tag{2.15}$$

The solution for the heat equation above is [60]

$$u(t,x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi vt}} e^{-\frac{(x-y)^2}{2vt}} e^{-ay} h(y)\,\mathrm{d}y, \tag{2.16}$$

and therefore

$$g(t,x) = e^{ax+bt}u(t,x) = e^{ax+bt} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi vt}} e^{-\frac{(x-y)^2}{2vt}} e^{-ay} h(y)\,\mathrm{d}y. \tag{2.17}$$

□

Lemma 2.1.3. *For the inhomogeneous Black–Scholes equation*

$$\left[\frac{\partial}{\partial t} - \frac{v}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{v}{2} - r\right)\frac{\partial}{\partial x} + r + \kappa\right] g(t,x) = f(t,x), \tag{2.18}$$

with initial condition

$$g(0,x) = 0, \quad -\infty < x < \infty, \tag{2.19}$$

the solution is

$$g(t,x) = e^{ax+bt-\kappa t} \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t-s)}} e^{-\frac{(x-y)^2}{2v(t-s)}} e^{-ay-bs+\kappa s} f(s,y)\,\mathrm{d}y\,\mathrm{d}s. \tag{2.20}$$

Proof. Define

$$g(t,x) = e^{ax+bt-\kappa t}u(t,x), \quad a = \frac{1}{2} - \frac{r}{v}, \qquad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2. \tag{2.21}$$

Then, (2.10) with (2.11) becomes the inhomogeneous heat equation

$$\left(\partial_t - \frac{v}{2}\partial_x^2\right) u(t,x) = e^{-ax-bt+\kappa t} f(t,x), \tag{2.22}$$

with

$$u(0,x) = 0, \quad -\infty < x < \infty. \tag{2.23}$$

The solution for the inhomogeneous heat equation above is [60]

$$u(t,x) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t-s)}} e^{-\frac{(x-y)^2}{2v(t-s)}} e^{-ay-bs+\kappa s} f(s,y)\,\mathrm{d}y\,\mathrm{d}s \tag{2.24}$$

and therefore

$$\begin{aligned} g(t,x) &= e^{ax+bt-kt}u(t,x) \\ &= e^{ax+bt-\kappa t} \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t-s)}} e^{-\frac{(x-y)^2}{2v(t-s)}} e^{-ay-bs+\kappa s} f(s,y)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{2.25}$$

□

The above two lemmas will be used extensively in Chapter 3 for European options under stochastic volatility models. In order to compute the first term of the series, the payoff often serves as the initial condition, taking the role of $h(x)$ in Lemma 2.1.2. Higher order terms are then calculated by Lemma 2.1.3 after the iteration relation is obtained by breaking up the pricing PDE.

For American options, the price domain is different than for European options. In the case of an American put option without dividends, the pricing PDE only governs the price region above the undetermined exercise boundary $x \in [B(t), \infty)$. Fortunately, with the front-fixing technique [61], the unknown boundary can be reduced to a straight line $x = 0$. Written in log-price, the option price is defined on the quarter-plane $(t, x) \in [0, \infty) \times [0, \infty)$. The following lemma deals with the additional boundary conditions on $x = 0$.

Lemma 2.1.4. *The solution of*

$$\left[\frac{\partial}{\partial t} - \frac{v}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{v}{2} - r\right)\frac{\partial}{\partial x} + r\right] g(t, x) = 0, \tag{2.26a}$$

$$g(0, x) = 0, \tag{2.26b}$$

$$g(t, 0) = h(t), \tag{2.26c}$$

is

$$g(t, x) = e^{ax+bt} \int_0^t \frac{x}{\sqrt{2\pi v(t-s)^3}} \exp\left[-\frac{x^2}{2v(t-s)} - bs\right] h(s)\,\mathrm{d}s. \tag{2.27}$$

Proof. According to ansatz (2.8), the heat equation for $u(t, x)$ is

$$\left(\partial_t - \frac{v}{2}\partial_x^2\right) u(t, x) = 0, \tag{2.28a}$$

$$u(0, x) = 0, \tag{2.28b}$$

$$u(t, 0) = e^{-bt}h(t). \tag{2.28c}$$

The boundary conditions in (2.28) can be dealt with using the Laplace transform. Suppose the solution is transformed as

$$\bar{u}(p, x) = \mathcal{L}[u(t, x)] = \int_0^\infty e^{-pt}u(t, x)\,\mathrm{d}t. \tag{2.29}$$

Then, heat equation (2.28) is transformed to

$$p\bar{u} - \frac{v}{2}\partial_x^2\bar{u} = 0, \tag{2.30a}$$

$$\bar{u}(p, 0) = \mathcal{L}[e^{-bt}h(t)]. \tag{2.30b}$$

The general solution to the ODE (2.30a) is

$$\bar{u} = Ae^{\sqrt{\frac{2p}{v}}x} + Be^{-\sqrt{\frac{2p}{v}}x}, \tag{2.31}$$

where A and B are constants to be determined. Because $u(t, \infty) = 0$, the first term in the above solution vanishes $A = 0$. By setting $x = 0$, B can be determined

$$\bar{u}(p, x) = \mathcal{L}[e^{-bt}h(t)]e^{-\sqrt{\frac{2p}{v}}x}. \tag{2.32}$$

According to the convolution theorem, the Laplace inverse u can be expressed as

$$\begin{aligned} u = \mathcal{L}^{-1}[\bar{u}] &= \mathcal{L}^{-1}\left[\mathcal{L}[e^{-bt}h(t)]e^{-\sqrt{\frac{2p}{v}}x}\right] = \mathcal{L}^{-1}\left[\mathcal{L}[e^{-bt}h(t)]\mathcal{L}\left[\mathcal{L}^{-1}\left[e^{-\sqrt{\frac{2p}{v}}x}\right]\right]\right] \\ &= e^{-bt}h(t) * \mathcal{L}^{-1}\left[e^{-\sqrt{\frac{2p}{v}}x}\right] = e^{-bt}h(t) * \frac{x}{\sqrt{2\pi v t^3}}e^{-\frac{x^2}{2vt}}, \end{aligned} \tag{2.33}$$

where $*$ indicates convolution

$$[f(t) * g(t)](s) = \int_0^s f(s-t)g(t)\,\mathrm{d}t. \tag{2.34}$$

Therefore the solution to heat equation (2.28) is

$$u(s,y) = \int_0^s \frac{y}{\sqrt{2\pi v(s-t)^3}} \exp\left(-\frac{y^2}{2v(s-t)} - bt\right) h(t)\,\mathrm{d}t. \tag{2.35}$$

Then, the Black–Scholes version is

$$g(s,y) = e^{ay+bs}u(s,y). \tag{2.36}$$

□

The following result is also important for manipulating the boundary conditions at $x = 0$. We state the lemma without proof because it can be checked by direct computation.

Lemma 2.1.5. *The solution of*

$$g(t,x) - \partial_x g(t,x) = h(t,x), \tag{2.37a}$$

$$g(t,\infty) = 0, \tag{2.37b}$$

is

$$g(t,x) = e^x \int_x^\infty e^{-y}h(t,y)\,\mathrm{d}y. \tag{2.38}$$

2.2 Black–Scholes special functions

The key to ensuring the computability of the expansion terms is to find a family of functions that are closed under the operations involved in the computation. In other words, we should not only write down the iteration relations for the expansion terms but also ensure that they are integrable and expressible in terms of analytical functions.

In the series expansion context, the solution of a PDE (e.g. in a one-dimensional case) is written generally as

$$V = \sum_{i=0}^{\infty} p^i V_i, \tag{2.39}$$

with the iteration relation

$$V_i = \mathcal{B}^{-1} f(V_{i-1}, \ldots, V_0). \tag{2.40}$$

The operator $\mathcal{B}^{-1}$ denotes the inverse Black–Scholes operator with various boundary conditions, the expressions of which are given in the previous section. The following special functions are defined with respect to the Black–Scholes equation (2.10).

Definition 2.2.1. *A Black–Scholes exponential (BSE) function is a function of the form*

$$\exp\left(-\frac{x^2}{2vt} + ax + bt\right) \sum_{i,j} A_{ij} x^i t^{j-\frac{1}{2}}, \tag{2.41}$$

where A_{ij} are constants, $i, j \in \mathbb{N}$ and

$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2. \tag{2.42}$$

The set of all such functions is denoted as Σ_1.

Definition 2.2.2. *A Black–Scholes damped exponential (BSDE) function is a function of the form*

$$\exp\left(-\frac{x^2}{2vt} + ax + bt\right) \sum_{i,j,k} A_{ijk} x^i t^{j-\frac{1}{2}} e^{-k\kappa t}, \tag{2.43}$$

where A_{ijk} are constants, $i, j, k \in \mathbb{N}$ and

$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2. \tag{2.44}$$

The set of all such functions is denoted as Σ_1'.

As defined above, a BSE function is also a BSDE function with $\kappa = 0$:

$$\Sigma_1 \subset \Sigma_1'. \tag{2.45}$$

BSDE functions will be used for the pricing of European options under stochastic volatility models.

Definition 2.2.3. *A Black–Scholes complementary error (BSCE) function is a function of the form*

$$\exp\left(-\frac{x^2}{2vt} + ax + bt\right) \sum_{i,j} A_{ij} x^i t^{\frac{j}{2}} + e^{ax+bt} \operatorname{erfc}\left(\frac{x}{\sqrt{2vt}}\right) \sum_{k,l} B_{kl} x^k t^{\frac{l}{2}}, \tag{2.46}$$

where A_{ij} and B_{ij} are constants, $i, j, k, l \in \mathbb{N}$ and

$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2. \tag{2.47}$$

The set of all such functions is denoted as Σ_2.

Obviously, a BSE function is also a BSCE function with $B_{kl} = 0$:

$$\Sigma_1 \subset \Sigma_2. \tag{2.48}$$

BSCE functions will be used to construct solutions to American options.

Definition 2.2.4. *A Black–Scholes time (BST) function is a function of the form*

$$\sum_i C_i t^{\frac{i}{2}}, \tag{2.49}$$

where C_i are constants, $i \in \mathbb{N}$ and the set of all such functions is denoted as Σ_3.

BST functions are useful for describing the moving boundary of American options.

The Black–Scholes special functions have some asymptotic properties that make them suitable for series expansion. For European calls under advanced models (extensions of the Black–Scholes model), the Black–Scholes formula

$$C(t,x) = e^x \, \mathrm{N}(d_+) - Ke^{-rt} \, \mathrm{N}(d_-), \quad d_\pm = \frac{x - \ln K + \left(r \pm \frac{v}{2}\right) t}{\sqrt{vt}} \tag{2.50}$$

serves as the leading term V_0. Asymptotically, European call option price $C(t,x)$ under any model should satisfy the same conditions

$$C(0,x) = (e^x - K)^+, \quad C(t,\infty) = e^x - Ke^{-rt}, \quad C(t,-\infty) = 0. \tag{2.51}$$

Therefore, the higher order terms V_i, $i > 0$ should vanish in the above cases. It is possible to verify that the BSDE functions indeed satisfy those conditions.

Lemma 2.2.5. *If $g(t,x) \in \Sigma_1'$,*

$$g(0,x) = 0, \quad g(t,-\infty) = 0, \quad g(t,\infty) = 0. \tag{2.52}$$

For an American put option under any model, the Black–Scholes formula for put options

$$P(t,x) = Ke^{-rt} \, \mathrm{N}(-d_-) - e^x \, \mathrm{N}(-d_+), \quad d_\pm = \frac{x - \ln K + \left(r \pm \frac{v}{2}\right) t}{\sqrt{vt}} \tag{2.53}$$

may seem like a good candidate for the leading term V_0. However, (2.53) is not closed under the double integral in the inverse Black–Scholes operator. Therefore, a new function form is needed for V_0 due to this difficulty. Details will be covered in Chapter 4. Regardless of V_0, the higher order terms V_i, $i > 0$ should satisfy the asymptotic properties:

$$V_i(0,x) = 0, \quad V_i(t,\infty) = 0. \tag{2.54}$$

Simple calculation confirms that the above properties are satisfied by BSCE function.

Lemma 2.2.6. *If $g(t,x) \in \Sigma_2$ and $x > 0$,*

$$g(0,x) = 0, \quad g(t,\infty) = 0. \tag{2.55}$$

Furthermore, BSCE functions reduce to BST functions at $x = 0$.

Lemma 2.2.7. *If $g(t,x) \in \Sigma_2$, then $e^{-bt} g(t,0) \in \Sigma_3$.*

This lemma relates the bulk and boundary of the 'hold' region of American put options. The most desirable feature of BSDE and BSCE functions is that they are closed under many operations.

Lemma 2.2.8. *If $g(t,x) \in \Sigma_1'$, then $\partial_t g,\ \partial_x g,\ \mathcal{B}g,\ x^n t^{\frac{m}{2}} g \in \Sigma_1'$.*

Lemma 2.2.9. *If $g(t,x) \in \Sigma_2$, then $\partial_t g,\ \partial_x g,\ \mathcal{B}g,\ x^n t^{\frac{m}{2}} g \in \Sigma_2$.*

For differential operations in the above lemmas, the proof of closeness is trivial. However, the closeness under integration due to the inverse Black−Scholes operator is not so obvious.

Lemma 2.2.10. *For inhomogeneous Black−Scholes equation*

$$\left[\frac{\partial}{\partial t} - \frac{v}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{v}{2} - r\right)\frac{\partial}{\partial x} + r + \kappa\right] f(t,x) = g(t,x), \tag{2.56}$$

with initial condition

$$f(0,x) = 0. \tag{2.57}$$

If $g(t,x) \in \Sigma_1'$, then $f(t,x) \in \Sigma_1'$.

Proof. Without loss of generality, we define

$$g(t,x) = x^n t^{m-\frac{1}{2}} e^{-p\kappa t} \exp\left(-\frac{x^2}{2vt} + ax + bt\right) \tag{2.58}$$

and

$$G(n) = \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{2v(s-t)} - \frac{x^2}{2vt}\right] x^n \, \mathrm{d}x. \tag{2.59}$$

Because

$$\partial_y \exp\left[-\frac{(x-y)^2}{2v(s-t)}\right] = \frac{x-y}{v(s-t)} \exp\left[-\frac{(x-y)^2}{2v(s-t)}\right], \tag{2.60}$$

it is easy to show that

$$G(n+1) = v(s-t)\partial_y G(n) + yG(n). \tag{2.61}$$

For $n = 0$, the integral (2.59) can be computed explicitly:

$$G(0) = \sqrt{\frac{2\pi vt(s-t)}{s}} \exp\left(-\frac{y^2}{2vs}\right). \tag{2.62}$$

If we assume

$$G(n) = \sqrt{\frac{2\pi vt(s-t)}{s}} \exp\left(-\frac{y^2}{2vs}\right) \sum_{i,j,k} A_{ijk} y^i t^j s^k, \tag{2.63}$$

then by Lemma 2.2.8, $G(n+1)$ admits the same form as $G(n)$. By induction, $G(n)$ can be written in the general form for arbitrary n:

$$G(n) = \sqrt{\frac{2\pi vt(s-t)}{s}} \exp\left(-\frac{y^2}{2vs}\right) \sum_{i,j,k} A_{ijk} y^i t^j s^k. \tag{2.64}$$

Therefore,

$$\begin{aligned} f(s,y) &= e^{ay+bs-\kappa s}\int_0^s \frac{1}{\sqrt{2\pi v(s-t)}} G(n) t^m e^{-p\kappa t}\,\mathrm{d}t \\ &= \exp\left(-\frac{y^2}{2vs}+ay+bs-\kappa s\right)\sum_{i,j,k} A_{ijk} y^i s^{k-\frac{1}{2}} \int_0^s t^{m+j} e^{(1-p)\kappa t}\,\mathrm{d}t. \end{aligned} \tag{2.65}$$

According to Lemma 2.2.13, the integral in the above equation can be written as

$$\int_0^s t^{m+j} e^{(1-p)\kappa t}\,\mathrm{d}t = \frac{1}{[(p-1)\kappa]^{m+j+1}}\left\{(m+j)!-(m+j)!\,e^{(1-p)\kappa s}\sum_{i=0}^{m+j}\frac{[(p-i)\kappa s]^i}{i!}\right\}. \tag{2.66}$$

Combing (2.65) and (2.66), the solution can be expressed as

$$f(s,y) = \exp\left(-\frac{y^2}{2vs}+ay+bs\right)\left(\sum_{i,j} A_{ij} y^i s^{j-\frac{1}{2}} e^{-p\kappa s} + \sum_{k,l} B_{kl} y^k s^{l-\frac{1}{2}} e^{-\kappa s}\right), \tag{2.67}$$

where $i,j,k,l \in \mathbb{N}$. Therefore,

$$f(t,x) \in \Sigma_1'. \tag{2.68}$$

□

Lemma 2.2.11. *For inhomogeneous Black–Scholes equation*

$$\left[\frac{\partial}{\partial t} - \frac{v}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{v}{2}-r\right)\frac{\partial}{\partial x} + r\right] f(t,x) = g(t,x), \tag{2.69}$$

with initial condition

$$f(0,x) = 0. \tag{2.70}$$

If $g(t,x) \in \Sigma_2$, then $f(t,x) \in \Sigma_2$.

Proof. Without loss of generality, we define

$$g_1(t,x) = x^n t^{\frac{m}{2}} \exp(ax+bt)\,\mathrm{erfc}\left(\frac{x}{\sqrt{2vt}}\right), \tag{2.71a}$$

$$g_2(t,x) = x^p t^{\frac{q}{2}} \exp\left(-\frac{x^2}{2vt}+ax+bt\right), \tag{2.71b}$$

and

$$G_1(n) = \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{2v(s-t)}\right] \mathrm{erfc}\left(\frac{x}{\sqrt{2vt}}\right) x^n\,\mathrm{d}x, \tag{2.72a}$$

$$G_2(n) = \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{2v(s-t)}\right] \exp\left(-\frac{x^2}{2vt}\right) x^n\,\mathrm{d}x. \tag{2.72b}$$

Since

$$\partial_y \exp\left[-\frac{(x-y)^2}{2v(s-t)}\right] = \frac{x-y}{v(s-t)} \exp\left[-\frac{(x-y)^2}{2v(s-t)}\right], \tag{2.73}$$

it is easy to show that

$$G_i(n+1) = v(s-t)\partial_y G_i(n) + yG_i(n), \quad i = 1, 2. \tag{2.74}$$

For $n = 0$, the integral (2.72) can be computed explicitly

$$G_1(0) = \sqrt{2\pi v(s-t)}\,\mathrm{erfc}\left(\frac{y}{\sqrt{2vs}}\right), \quad G_2(0) = \sqrt{\frac{2\pi vt(s-t)}{s}}\exp\left(-\frac{y^2}{2vs}\right). \tag{2.75}$$

If we assume

$$G_1(n) = \sqrt{2\pi v(s-t)}\left[\exp\left(-\frac{y^2}{2vs}\right)\sum_{i,j,k} A_{ijk} y^i t^{\frac{j}{2}} s^{\frac{k}{2}} + \mathrm{erfc}\left(\frac{y}{\sqrt{2vs}}\right)\sum_{i,j,k} B_{ijk} y^i t^{\frac{j}{2}} s^{\frac{k}{2}}\right], \tag{2.76a}$$

$$G_2(n) = \sqrt{\frac{2\pi vt(s-t)}{s}}\exp\left(-\frac{y^2}{2vs}\right)\sum_{i,j,k} C_{ijk} y^i t^{\frac{j}{2}} s^{\frac{k}{2}}, \tag{2.76b}$$

then by Lemma 2.2.9, $G_i(n+1)$ admits the same form as $G_i(n)$, $i = 1, 2$. Using induction, (2.76) is the general form for arbitrary n.

Therefore,

$$\begin{aligned} f(s,y) &= e^{ay+bs}\int_0^s \frac{1}{\sqrt{2\pi v(s-t)}}\left[G_1(n)t^{\frac{m}{2}} + G_2(p)t^{\frac{q}{2}}\right]\mathrm{d}t \\ &= \exp\left(-\frac{y^2}{2vs} + ay + bs\right)\left(\sum_{i,j,k}\frac{2A_{ijk}y^i s^{\frac{j+k+m+2}{2}}}{j+m+2} + \sum_{i,j,k}\frac{2C_{ijk}y^i s^{\frac{j+k+q+2}{2}}}{j+q+3}\right) \\ &\quad + e^{ay+bs}\,\mathrm{erfc}\left(\frac{y}{\sqrt{2vs}}\right)\sum_{i,j,k}\frac{2}{j+m+2}B_{ijk}y^i s^{\frac{j+k+m+2}{2}}, \end{aligned} \tag{2.77}$$

and

$$f(t,x) \in \Sigma_2. \tag{2.78}$$

□

The above two lemmas not only guarantee the closeness of Σ_1' and Σ_2 under operation $\mathcal{B}^{-1}$ but also show how to calculate the integral in $G(n)$ from 0 to n by induction. However, in some cases, the following general form is more convenient than the induction form.

Lemma 2.2.12.

$$\begin{aligned} G(n) &= \int_{-\infty}^{\infty}\exp\left[-\frac{(sx-ty)^2}{2vst(s-t)}\right]x^n\,\mathrm{d}x \\ &= \sqrt{\frac{2\pi vt(s-t)}{s}}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n!}{i!\,(n-2i)!}\left(\frac{vt(s-t)}{2s}\right)^i\left(\frac{ty}{s}\right)^{n-2i}. \end{aligned} \tag{2.79}$$

The following result can be used to calculate BSDE functions. The time integration preserves the form $e^{kt}t^n$ and, consequently, the form of the BSDE functions.

Lemma 2.2.13.

$$\int_0^s \exp(kt)t^n \, \mathrm{d}t = (-k)^{-n-1}\left[n! - n! \exp(ks) \sum_{i=0}^{n} \frac{(-ks)^i}{i!}\right], \qquad k \neq 0. \tag{2.80}$$

The following theorem is the most important tool to obtain the series solution of American put options.

Theorem 2.2.14. *For*

$$\left[\frac{\partial}{\partial t} - \frac{v}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{v}{2} - r\right)\frac{\partial}{\partial x} + r\right] V(t,x) = 0, \tag{2.81a}$$

$$V(0,x) = 0, \tag{2.81b}$$

$$V(t,0) - \partial_x V(t,0) = e^{bt} \sum_i \lambda_i t^{\frac{i}{2}}, \tag{2.81c}$$

where

$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2, \tag{2.82}$$

$V \in \Sigma_2$ *if*

$$\sum_i \left(\frac{2}{v}\right)^{\frac{i}{2}} \frac{\lambda_i \Gamma\left(1 + \frac{i}{2}\right)}{(a-1)^i} = 0. \tag{2.83}$$

Proof. Denote $h = V - \partial_x V$. For

$$\mathcal{B}h(t,x) = 0, \tag{2.84a}$$

$$h(t,0) = e^{bt} \sum_i \lambda_i t^{\frac{i}{2}}, \tag{2.84b}$$

the solution is

$$\begin{aligned}
h(s,x) &= e^{ax+bs} \int_0^s \frac{x}{\sqrt{2\pi v (s-t)^3}} \exp\left(-\frac{x^2}{2v(s-t)} - bt\right) e^{bt} \sum_i \lambda_i t^{\frac{i}{2}} \, \mathrm{d}t \\
&= \sum_i \lambda_i e^{ax+bs} \frac{x}{\sqrt{2\pi v}} \int_0^s \frac{t^{\frac{i}{2}}}{(s-t)^{\frac{3}{2}}} \exp\left(-\frac{x^2}{2v(s-t)}\right) \mathrm{d}t \\
&= \sum_i \lambda_i e^{ax+bs} \frac{x}{\sqrt{2\pi v}} \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\exp\left(-\frac{x^2}{2vt}\right) t^{-\frac{3}{2}}\right\} \mathcal{L}\left\{t^{\frac{i}{2}}\right\}\right\} \\
&= \sum_i \lambda_i \Gamma\left(1 + \frac{i}{2}\right) e^{ax+bs} \mathcal{L}^{-1}\left\{\exp\left(-\sqrt{\frac{2p}{v}} x\right) p^{-\left(1+\frac{i}{2}\right)}\right\},
\end{aligned} \tag{2.85}$$

where $\mathcal{L}$ is short for Laplace transformation operator $\mathcal{L}\{\cdot\}(p)$ and $\mathcal{L}^{-1}$ for $\mathcal{L}^{-1}\{\cdot\}(t)$. Denote

$$H_i = e^{y+bs} \int_y^\infty e^{(a-1)x} \mathcal{L}^{-1}\left\{\exp\left(-\sqrt{\frac{2p}{v}} x\right) p^{-\left(1+\frac{i}{2}\right)}\right\} \mathrm{d}x. \tag{2.86}$$

The above equation can be evaluated using integration by parts,

$$
\begin{aligned}
H_i &= -\sqrt{\frac{v}{2}} e^{y+bs} \int_y^\infty e^{(a-1)x} \mathcal{L}^{-1}\left\{ \exp\left(-\sqrt{\frac{2p}{v}}x\right) p^{-\left(1+\frac{i+1}{2}\right)} \left(-\sqrt{\frac{2p}{v}}\right) \right\} \mathrm{d}x \\
&= -\sqrt{\frac{v}{2}} e^{y+bs} \int_y^\infty e^{(a-1)x} \partial_x \mathcal{L}^{-1}\left\{ \exp\left(-\sqrt{\frac{2p}{v}}x\right) p^{-\left(1+\frac{i+1}{2}\right)} \right\} \mathrm{d}x \\
&= -\sqrt{\frac{v}{2}} e^{y+bs+(a-1)x} \mathcal{L}^{-1}\left\{ \exp\left(-\sqrt{\frac{2p}{v}}x\right) p^{-\left(1+\frac{i+1}{2}\right)} \right\}\Bigg|_y^\infty \\
&\quad + \sqrt{\frac{v}{2}} e^{y+bs} \int_y^\infty \left[\partial_x e^{(a-1)x}\right] \mathcal{L}^{-1}\left\{ \exp\left(-\sqrt{\frac{2p}{v}}x\right) p^{-\left(1+\frac{i+1}{2}\right)} \right\} \mathrm{d}x \\
&= \sqrt{\frac{v}{2}} e^{ay+bs} \mathcal{L}^{-1}\left\{ \exp\left(-\sqrt{\frac{2p}{v}}y\right) p^{-\left(1+\frac{i+1}{2}\right)} \right\} + (a-1)\sqrt{\frac{v}{2}} H_{i+1}.
\end{aligned}
\tag{2.87}
$$

The inverse Laplace transform in the last equality can be calculated for $i \in \mathbb{N}$ as

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{ \exp\left(-\sqrt{\frac{2p}{v}}y\right) p^{-\left(1+\frac{i+1}{2}\right)} \right\} &= \frac{t^{\frac{i+1}{2}}}{\Gamma\left(\frac{i+3}{2}\right)} {}_1\mathrm{F}_1\left[-\frac{i+1}{2}; \frac{1}{2}; -\frac{y^2}{2vs}\right] \\
&\quad - \sqrt{\frac{2}{v}} \frac{t^{\frac{i}{2}} y}{\Gamma\left(\frac{i}{2}+1\right)} {}_1\mathrm{F}_1\left[-\frac{i}{2}; \frac{3}{2}; -\frac{y^2}{2vs}\right] \in \Sigma_2,
\end{aligned}
\tag{2.88}
$$

where ${}_1\mathrm{F}_1[a; b; x]$ denotes confluent hypergeometric function, and therefore

$$
H_{i+1} - \sqrt{\frac{2}{v}} \frac{H_i}{a-1} \in \Sigma_2, \tag{2.89}
$$

By induction,

$$
H_i - \left(\frac{2}{v}\right)^{\frac{i}{2}} \frac{H_0}{(a-1)^i} \in \Sigma_2. \tag{2.90}
$$

Since for $V - \partial_x V = h$ and $V(t, \infty) = 0$, the difference

$$
\begin{aligned}
&V(s, y) - \sum_i \lambda_i \Gamma\left(1 + \frac{i}{2}\right) \left(\frac{2}{v}\right)^{\frac{i}{2}} \frac{H_0}{(a-1)^i} \\
&= e^y \int_y^\infty e^{-x} h(s, x)\, \mathrm{d}x - \sum_i \lambda_i \Gamma\left(1 + \frac{i}{2}\right) \left(\frac{2}{v}\right)^{\frac{i}{2}} \frac{H_0}{(a-1)^i} \\
&= \sum_i \lambda_i \Gamma\left(1 + \frac{i}{2}\right) \left[H_i - \left(\frac{2}{v}\right)^{\frac{i}{2}} \frac{H_0}{(a-1)^i}\right] \in \Sigma_2,
\end{aligned}
\tag{2.91}
$$

and $H_0 \notin \Sigma_2$, therefore, $V \in \Sigma_2$ if

$$
\sum_i \left(\frac{2}{v}\right)^{\frac{i}{2}} \frac{\lambda_i \Gamma\left(1 + \frac{i}{2}\right)}{(a-1)^i} = 0. \tag{2.92}
$$

□

The following notation and lemmas are used for deriving American option expansions. To differentiate from $\mathcal{B}^{-1}g := V$, which is the solution of the inhomogeneous Black–Scholes equation

$$\mathcal{B}V = g, \quad V(0,x) = 0, \tag{2.93}$$

we introduce a different notation.

Notation 2.2.15.

$$\mathcal{I}^{-1}g := V(t,x), \tag{2.94}$$

where

$$\left[\frac{\partial}{\partial t} - \frac{v}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{v}{2} - r\right)\frac{\partial}{\partial x} + r\right] V(t,x) = 0, \tag{2.95a}$$

$$V(0,x) = 0, \tag{2.95b}$$

$$V(t,0) - \partial_x V(t,0) = g, \tag{2.95c}$$

Notation 2.2.16.

$$L_j\left(\sum_{i=0}^{N} \lambda_i t^{\frac{i}{2}}\right) := \lambda t^j, \tag{2.96}$$

such that

$$\sum_{i=0}^{N}\left(\frac{2}{v}\right)^{\frac{i}{2}} \frac{\lambda_i \Gamma\left(1+\frac{i}{2}\right)}{(a-1)^i} + \left(\frac{2}{v}\right)^j \frac{\lambda \Gamma(1+j)}{(a-1)^{2j}} = 0. \tag{2.97}$$

In other words, $L_i(g)$ denotes the t^j-term needed to make

$$\mathcal{I}^{-1}\left[e^{bt}(g + L_i(g))\right] \in \Sigma_2, \quad \text{where } g \in \Sigma_3. \tag{2.98}$$

Notation 2.2.17.

$$\mathcal{D}_p^n f(p) = \left.\frac{1}{n!}\frac{\partial^n}{\partial p^n} f(p)\right|_{p=0} \tag{2.99}$$

$\mathcal{D}_p^n$ picks out the coefficient of n-th power of the p-polynomial:

$$a_n = \mathcal{D}_p^n\left(\sum_{i=0}^{\infty} a_i p^i\right). \tag{2.100}$$

If $f(p)$ is regular at $p = 0$, the sum of operators can be written as:

Lemma 2.2.18.

$$\sum_{i=0}^{\infty} p^i \mathcal{D}_p^i f(p) = f(p). \tag{2.101}$$

Likewise, a similar relation can be proven for exponential functions.

Lemma 2.2.19.

$$d_i + \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right) = \mathcal{D}_p^i \exp\left(\sum_{j=1}^{\infty} p^j d_j\right). \tag{2.102}$$

Proof. Because when $\mathcal{D}_p^i$ is applied to

$$\begin{aligned}\exp\left(\sum_{j=1}^{\infty} p^j d_j\right) &= \exp\left(\sum_{j=1}^{i-1} p^j d_j\right)\exp\left(\sum_{j=i}^{\infty} p^j d_j\right)\\ &= \exp\left(\sum_{j=1}^{i-1} p^j d_j\right)\left[1+\sum_{j=i}^{\infty} p^j d_j + \frac{1}{2}\left(\sum_{j=i}^{\infty} p^j d_j\right)^2 + \cdots\right],\end{aligned} \tag{2.103}$$

it picks out the coefficient of p^i, and those terms that are higher order than i provide no contribution. Therefore,

$$\begin{aligned}\mathcal{D}_p^i \exp\left(\sum_{j=1}^{\infty} p^j d_j\right) &= \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right)\left(1+\sum_{j=i}^{\infty} p^j d_j\right)\\ &= \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right) + \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right)\sum_{j=i}^{\infty} p^j d_j\\ &= \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right) + d_i.\end{aligned} \tag{2.104}$$

The last equality holds because $d_i p^i$ is the lowest order p-power term. □

2.3 Series solutions to ODE and PDE revisited

Series solution is a general method for solving differential equations. Most differential equations do not have a closed-form solution, whereas within a particular parameter set, a series solution yields highly accurate numerical values. In fact, many special functions, which are solutions to a differential equation, are defined as an infinite sum of elementary functions:

$$V(x, y, \ldots) = \sum_{i=0}^{\infty} V_i(x, y, \ldots), \tag{2.105a}$$

$$V_i(x, y, \ldots) = f(i, x, y, \ldots). \tag{2.105b}$$

Solution (2.105) is often considered closed-form, although it is expressed as an infinite sum.

However, in most cases, it is only possible to obtain V_i in terms of an iterative relation regarding lower order terms:

$$V(x, y, \ldots) = \sum_{i=0}^{\infty} V_i(x, y, \ldots), \tag{2.106a}$$

$$V_i(x, y, \ldots) = f(V_{i-1}, V_{i-2}, \ldots, V_0). \tag{2.106b}$$

Solution (2.106) is not as good as (2.105) because:

- the convergence of V_i in (2.105) is much easier to prove;
- the expression of V_i in (2.106) is usually complicated when i gets bigger;
- when the iterative relation in (2.106) is written as an integral, the integrability is not ensured.

However, iterative relation (2.106) is often the first step towards the explicit expression (2.105). In practice, (2.106) is often good enough for calculation, when the expansion form is carefully chosen and the parameter values are modestly large.

The following simple example demonstrates the logic and general steps of series solution methods. Consider the ODE

$$\partial_x V(x) - V(x) = 0, \tag{2.107a}$$
$$V(0) = 1. \tag{2.107b}$$

The solution is $V(x) = e^x$. However, suppose we do not know the answer and try to obtain the series solution. First, a leading term must be chosen because it serves as the starting point V_0 of the expansion. The simplest candidate could be a constant, which satisfies the following ODE:

$$\partial_x V(x) = 0, \tag{2.108a}$$
$$V(0) = 1. \tag{2.108b}$$

The solution is easily obtainable:

$$V(y) = \int_0^y \partial_x V(x)\,\mathrm{d}x + V(0) = \int_0^y 0\,\mathrm{d}x + 1 = 1. \tag{2.109}$$

Then, the solution of (2.107) can be regarded as the solution of

$$\partial_x V(x) - pV(x) = 0, \tag{2.110a}$$
$$V(0) = 1, \tag{2.110b}$$

if $p = 1$. The solution is written as a power series of p:

$$V(x) = \sum_{i=0}^{\infty} V_i(x) p^i. \tag{2.111}$$

Now, substitute (2.111) back into (2.110), and we have

$$\sum_{i=0}^{\infty} \partial_x V_i(x) p^i - p \sum_{i=0}^{\infty} V_i(x) p^i = \sum_{i=0}^{\infty} \partial_x V_i(x) p^i - \sum_{i=1}^{\infty} V_{i-1}(x) p^i = 0, \tag{2.112a}$$
$$\sum_{i=0}^{\infty} V_i(0) p^i = 1. \tag{2.112b}$$

Because p is defined continuously on $[0, 1]$, the coefficients of p^i terms should equal. The ODE (2.112) can be decomposed into two cases:

- When $i = 0$

$$\partial_x V_0(x) = 0, \tag{2.113a}$$
$$V_0(0) = 1. \tag{2.113b}$$

- When $i > 0$

$$\partial_x V_i(x) - V_{i-1}(x) = 0, \tag{2.114a}$$
$$V_i(0) = 0. \tag{2.114b}$$

In (2.109), we have chosen the leading term $V_0(x) = 1$. Higher order terms can be obtained by integrating the previous term

$$V_i(y) = \int_0^y V_{i-1}(x)\,\mathrm{d}x + V_i(0) = \int_0^y V_{i-1}(x)\,\mathrm{d}x. \tag{2.115}$$

The leading higher order terms are

$$V_1(y) = \int_0^y 1\,\mathrm{d}x = y, \qquad V_2(y) = \int_0^y x\,\mathrm{d}x = \frac{y^2}{2}, \tag{2.116a}$$
$$V_3(y) = \int_0^y \frac{x^2}{2}\,\mathrm{d}x = \frac{y^3}{3}, \qquad V_4(y) = \int_0^y \frac{x^3}{3}\,\mathrm{d}x = \frac{y^4}{4}. \tag{2.116b}$$

Because we know that the set of power terms is closed under integration, the result of integration (2.115) is always a power term, if we start the iteration with a power term $V_0(x) = 1$. In this case, it is easy to guess the general form for V_i:

$$V_i(x) = \frac{1}{i!}x^i. \tag{2.117}$$

Therefore, the solution of (2.110) can be written as

$$V(x) = \sum_{i=0}^{\infty} V_i(x)p^i = \sum_{i=0}^{\infty} \frac{1}{i!}(xp)^i = e^{xp}. \tag{2.118}$$

Finally, if we set $p = 1$, the same solution $V(x) = e^x$ we obtained earlier by guessing is recovered. When $p = 0$, $V(x) = 1$, as we proposed in (2.108). Even if we cannot obtain the general expression (2.118), leading terms (2.116) are good enough when we calculate $V(x)$ in the proximity of $x = 0$.

The convergence of (2.118) can be checked,

$$\lim_{i\to\infty} \frac{V_{i+1}}{V_i} = \lim_{i\to\infty} \frac{x}{i+1} = 0, \tag{2.119}$$

with $x < \infty$. Therefore, the series is convergent for any x.

PDEs are expanded in the same way as in the above example, with more complicated inverse operations. Suppose we want to solve the simplest advanced model beyond the Black−Scholes with just one additional parameter p:

$$\mathcal{B}_v V + p\partial_v V = 0, \tag{2.120a}$$
$$V(0, x) = (e^x - 1)^+, \tag{2.120b}$$

where $\mathcal{B}_v$ is the Black−Scholes operator. When $p = 0$, the solution reduces to the Black−Scholes formula. Therefore, the series solution is expanded as

$$V = \sum_{i=0}^{\infty} V_i p^i, \tag{2.121a}$$

$$V_0 = e^x \, \mathrm{N}(d_+) - K e^{-rt} \, \mathrm{N}(d_-), \quad d_\pm = \frac{x - \ln K + \left(r \pm \frac{v}{2}\right) t}{\sqrt{vt}}. \tag{2.121b}$$

After substituting (2.121) into (2.120) and equating the corresponding coefficients of p^i, we have

$$\mathcal{B}_v V_i = -\partial_v V_{i-1}. \tag{2.122}$$

Higher order terms can be calculated iteratively. Direct calculation yields

$$V_1 = \mathcal{B}_v^{-1} \partial_v V_0 = \frac{1}{4} \sqrt{\frac{t^3}{2\pi v}} \exp\left(-\frac{x^2}{2vt} + ax + bt\right) \in \Sigma_1. \tag{2.123}$$

As we have shown in Lemma 2.2.8 and Lemma 2.2.10, BSE functions are closed under ∂_v and $\mathcal{B}_v^{-1}$, and therefore higher order terms are also BSE functions:

$$V_i \in \Sigma_1, \quad i > 1. \tag{2.124}$$

For the simple example, higher order terms are not only calculable; they are proven to be certain types of functions (i.e. BSE functions). However, because the operation $\mathcal{B}_v^{-1}$ is far more complicated than the simple integration in the ODE case, we are not able to derive the general form of the expansion terms. Convergence is hard to prove with iteration relation (2.122).

The example above summarises the methods in the current literature [50−52, 54], which replace the ∂_v operator with additional operators other than $\mathcal{B}_v$. In the case of the Heston model,

$$\partial_v \to -\rho\eta v \partial_x \partial_v - \frac{1}{2}\eta^2 v \partial_v^2 - \kappa(\theta - v)\partial_v. \tag{2.125}$$

The method is valid in some cases, especially for small parameter values, as demonstrated by numerical examples. However, there are more ways to expand the solution, which have better numerical accuracy and we will present them in the next chapter.

3 European options under stochastic volatility models

3.1 Stochastic volatility models

Empirical evidence shows that equity volatility is far from constant, while the classical Black−Scholes model assumes universal volatility. Stochastic volatility is an attempt to account for this phenomenon by allowing the volatility to vary over time. Among these models, mean-reversion is another desirable feature since it keeps the volatility around a certain level, as it is observed in the market. In this chapter, the following setting is employed. The stock price S_t and the variance process v_t satisfy the following set of SDEs, under a risk-neutral measure:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}W_t, \tag{3.1a}$$

$$\mathrm{d}v_t = \kappa(\theta - v_t)v_t^{\alpha}\,\mathrm{d}t + \eta v_t^{\beta}\,\mathrm{d}Z_t, \tag{3.1b}$$

$$\mathrm{d}[W, Z]_t = \rho\,\mathrm{d}t, \tag{3.1c}$$

where r is the risk-free interest rate, κ is the mean-reverting rate and η is the volatility of volatility. The setting (3.1) covers several popular models with different choices of α and β, as demonstrated in Table 3.1. More information on these models can be found in [16, 62, 63].

	α	β
Heston model	0	1/2
GARCH model	0	1
3/2 model	1	3/2

Table 3.1: Stochastic volatility model specifications.

According to standard hedging arguments [63], the pricing PDE for vanilla options under the model (3.1) is

$$\partial_t u - \frac{v}{2}\partial_x^2 u + \left(\frac{v}{2} - r\right)\partial_x u + ru - \rho\eta v^{\beta+\frac{1}{2}}\partial_x\partial_v u - \frac{1}{2}\eta^2 v^{2\beta}\partial_v^2 u - \kappa(\theta - v)v^{\alpha}\partial_v u = 0, \tag{3.2}$$

where $x = \ln S$ is the log-price and t denotes the time to maturity. The PDE (3.2) is tractable in the case of European options under the Heston and 3/2 models, by inverse

Fourier transform [11, 12]. For the GARCH model, because of its lack of characteristic function, Monte Carlo methods are heavily relied on to produce option prices.

In order to solve the PDE (3.2), an initial condition should be imposed. The form of this initial condition corresponds to the type of option. In this chapter, only European call options are considered. Therefore,

$$u(0, x) = (e^x - 1)^+, \quad x \in (-\infty, \infty). \tag{3.3}$$

A put option can be solved using put-call parity. The strike is set to 1 for simplicity. Given the option price u_1 for strike 1, the option price u_K for strike K can be obtained by

$$u_K(t, x) = K u_1(t, x - \ln K). \tag{3.4}$$

3.2 Power series expansions

After deriving the pricing PDE (3.2) from the model SDEs (3.1) and imposing the boundary condition (3.3) for the option, we can start to write down the expansion form, as we did in the previous chapter. First, we need to find a set of parameters (variables) p_i, such that when they are set to specific values $p_i = c_i$, the Black−Scholes model is recovered. In the case of model (3.1), there are several ways to recover the Black−Scholes model. The variance process v_t is constant in the following cases.

1. $\kappa = 0$ and $\eta = 0$. In this case, the stochastic part ηv_t^β and the mean-reverting part $\kappa(\theta - v_t)v_t^\alpha$ of the variance process vanish $\mathrm{d}v_t = 0$. This means that the variance process remains at its initial value $v_t = v$. The Black−Scholes model with volatility $\sqrt{v}$ is recovered. Therefore, model (3.1) admits (κ, η)-expansion

$$u(t, x) = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij} \kappa^i \eta^j. \tag{3.5}$$

2. $\eta = 0$ and $v = \theta$. Since the stochastic part $\eta v_t^\beta = 0$, there is nothing that deviates the variance process from long-term variance θ once the initial variance starts from the long-term mean $v_0 = v = \theta$. From the PDE perspective, initial variance v and long-term variance θ are both variables. Likewise, (η, v)-expansion

$$u(t, x) = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(t, x; \theta) \eta^i (v - \theta)^j \tag{3.6}$$

and (η, θ)-expansion

$$u(t, x) = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(t, x; v) \eta^i (\theta - v)^j \tag{3.7}$$

are equally valid. The two expansion methods seem the same, however, the difference of them will be explained in detail later in the chapter.

3. $\kappa = \infty$. Since the mean-reverting rate is infinitely large, the variance process returns to the long-term volatility θ instantaneously:

$$v_t = \begin{cases} v, & t = 0, \\ \theta, & t > 0. \end{cases} \tag{3.8}$$

This process stays constant, except in the initial instance $t = 0$. It seems that the option price can be expanded as a power series of κ^{-1}:

$$u = \sum_{i=0}^{\infty} \frac{u_i(t,x)}{\kappa^i}. \tag{3.9}$$

However, it turns out that the coefficient u_i of the above expansion can not be uniquely determined. Therefore, the two-parameter (κ, v)-expansion

$$u(t,x) = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(t,x,\theta) \frac{(v-\theta)^j}{\kappa^i} \tag{3.10}$$

is proposed.

Now that the forms of the expansion have been determined for each case, the iteration relations for higher order terms can be derived.

3.2.1 (κ, η)-expansion

The first proposition shows how to derive expansion terms in (3.5).

Proposition 3.2.1. *The solution of* (3.2) *with initial condition* (3.3) *can be written as*

$$u(t,x) = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(t,x)\kappa^i \eta^j, \tag{3.11}$$

with

$$a = \frac{1}{2} - \frac{r}{v}, \qquad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2, \tag{3.12a}$$

$$u_{00}(t,x) = \frac{e^x}{2}\operatorname{erfc}\left[-\frac{x+(1-a)vt}{\sqrt{2vt}}\right] - \frac{e^{-rt}}{2}\operatorname{erfc}\left[-\frac{x-avt}{\sqrt{2vt}}\right], \tag{3.12b}$$

$$\begin{aligned} u_{ij}(s,y) = e^{ay+bs} &\int_0^s \frac{\mathrm{d}t}{\sqrt{2\pi v(s-t)}} \int_{-\infty}^{\infty} \mathrm{d}x \exp\left[-\frac{(x-y)^2}{2v(s-t)} - ax - bt\right] \\ &\times \left[\rho v^{\beta+\frac{1}{2}} \partial_x \partial_v u_{i(j-1)} + \frac{1}{2} v^{2\beta} \partial_v^2 u_{i(j-2)} + (\theta - v) v^{\alpha} \partial_v u_{(i-1)j}\right]. \end{aligned} \tag{3.12c}$$

Proof. Since (3.2) reduces to the Black–Scholes equation with volatility $\sqrt{v}$ when $\eta = 0$ and $\kappa = 0$, the solution of (3.2) with initial condition (3.3) can be written as

$$u(t,x) = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(t,x)\kappa^i \eta^j, \tag{3.13}$$

$$u_{00}(t,x) = \frac{e^x}{2}\operatorname{erfc}\left[-\frac{x+(1-a)vt}{\sqrt{2vt}}\right] - \frac{e^{-rt}}{2}\operatorname{erfc}\left[-\frac{x-avt}{\sqrt{2vt}}\right]. \quad (3.14)$$

With ansatz (3.13), (3.2) can be written as

$$\sum_{\substack{i=0\\j=0}}^{\infty} \mathcal{B}_v u_{ij}\kappa^i\eta^j - \sum_{\substack{i=0\\j=0}}^{\infty} \rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{ij}\kappa^i\eta^{j+1}$$
$$- \sum_{\substack{i=0\\j=0}}^{\infty} \frac{1}{2}v^{2\beta}\partial_v^2 u_{ij}\kappa^i\eta^{j+2} - \sum_{\substack{i=0\\j=0}}^{\infty}(\theta-v)v^{\alpha}\partial_v u_{ij}\kappa^{i+1}\eta^j = 0, \quad (3.15)$$

where $\mathcal{B}_v$ is the Black−Scholes operator defined in Notation 2.1.1. In order to collect coefficients of $\kappa^i\eta^j$, the starting value of dummy indexes i and j must be shifted:

$$\sum_{\substack{i=0\\j=0}}^{\infty} \mathcal{B}_v u_{ij}\kappa^i\eta^j - \sum_{\substack{i=0\\j=1}}^{\infty} \rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{i(j-1)}\kappa^i\eta^j$$
$$- \sum_{\substack{i=0\\j=2}}^{\infty} \frac{1}{2}v^{2\beta}\partial_v^2 u_{i(j-2)}\kappa^i\eta^j - \sum_{\substack{i=1\\j=0}}^{\infty}(\theta-v)v^{\alpha}\partial_v u_{(i-1)j}\kappa^i\eta^j = 0. \quad (3.16)$$

The above equation holds if, for any i and j, the coefficients of $\kappa^i\eta^j$ satisfy the following inhomogeneous Black−Scholes equation

$$\mathcal{B}_v u_{ij} = \rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{i(j-1)} + \frac{1}{2}v^{2\beta}\partial_v^2 u_{i(j-2)} + (\theta-v)v^{\alpha}\partial_v u_{(i-1)j}, \quad (3.17)$$

with the initial condition

$$u_{ij}(0,x) = \begin{cases} (e^x-1)^+, & \text{for } i=0 \text{ and } j=0,\\ 0, & \text{otherwise.}\end{cases} \quad (3.18)$$

It should be noted that since the option price is regular at $\eta = 0$ and $\kappa = 0$, coefficients with at least one negative index vanish

$$u_{ij} = 0, \quad \text{for } i<0 \text{ or } j<0. \quad (3.19)$$

Therefore, in some cases ($i < 1$ or $j < 2$), the right-hand side of (3.17) may involve fewer terms, due to negative indexes ($i-1 < 0$ or $j-2 < 0$).

The $(0,0)$-component of (3.17) can be solved by Lemma 2.1.2

$$u_{00}(s,y) = e^{ay+bs}\int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi vs}} e^{-\frac{(x-y)^2}{2vs}} e^{-ax}(e^x-1)^+$$
$$= \frac{e^x}{2}\operatorname{erfc}\left[-\frac{x+(1-a)vt}{\sqrt{2vt}}\right] - \frac{e^{-rt}}{2}\operatorname{erfc}\left[-\frac{x-avt}{\sqrt{2vt}}\right]. \quad (3.20)$$

The result agrees with the intuitive result (3.14). Higher order components of (3.17) can be solved by Lemma 2.1.3

$$u_{ij}(s,y) = e^{ay+bs}\int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi v(s-t)}} \exp\left[-\frac{(x-y)^2}{2v(s-t)} - ax - bt\right]$$

$$\times \left[\rho v^{\beta+\frac{1}{2}} \partial_x \partial_v u_{i(j-1)} + \frac{1}{2} v^{2\beta} \partial_v^2 u_{i(j-2)} + (\theta - v) v^{\alpha} \partial_v u_{(i-1)j} \right]. \tag{3.21}$$

□

Although the iterative expression for u_{ij} is given, the coefficients should be calculated in a particular order to ensure that when calculating u_{ij}, all the terms involved (Figure 3.1) have been calculated.

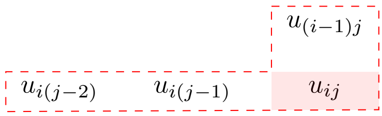

Figure 3.1: Terms involved in the calculation of u_{ij}.

Order 3.2.2. *Let $i, j, n \in \mathbb{N}$ and $O(1) = (0, 0)$. If $O(n) = (i, j)$, then*

$$O(n+1) = \begin{cases} (0, i+1), & j = 0, \\ (i+1, j-1), & j \neq 0. \end{cases} \tag{3.22}$$

For (κ, η)-expansion, Order 3.2.2 illustrated in Figure 3.2 is recommended.

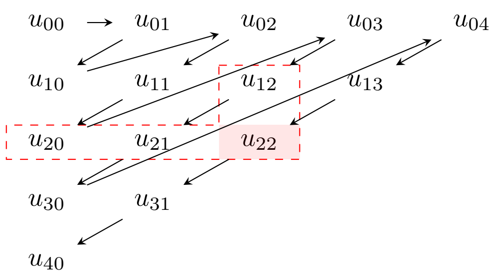

Figure 3.2: Order of calculation for (κ, η), (η, v) and (η, θ)-expansion.

The following corollary shows the general form of the expansion terms u_{ij} for (κ, η)-expansion.

Corollary 3.2.3. *For solution* (3.11)*, the coefficients u_{ij}, except the leading term u_{00}, are BSE functions,*

$$u_{ij} \in \Sigma_1, \quad i + j > 0. \tag{3.23}$$

Proof. Define

$$w = \frac{1}{2\sqrt{2\pi v t}} \exp\left(-\frac{x^2}{2vt} + ax + bt \right), \tag{3.24}$$

$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2} \left(\frac{1}{2} + \frac{r}{v} \right)^2. \tag{3.25}$$

Since $w \in \Sigma_1$, the leading few terms of the expansion, obtained by direct computation, are

$$u_{10} = \mathcal{B}_v^{-1}(\theta - v)v^\alpha \partial_v u_{00} = \frac{1}{2} w v^\alpha t^2(\theta - v) \in \Sigma_1 \tag{3.26a}$$

$$u_{01} = \mathcal{B}_v^{-1}\rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{00} = \frac{1}{2} w\rho v^{\beta-\frac{1}{2}} t(avt - x) \in \Sigma_1 \tag{3.26b}$$

$$\begin{aligned}
u_{02} &= \mathcal{B}_v^{-1}\left(\rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{01} + \frac{1}{2}v^{2\beta}\partial_v^2 u_{00}\right) \\
&= w\Bigg(\frac{1}{12}a^4\rho^2t^4v^{2\beta+1} - \frac{1}{12}a^3\rho^2t^4v^{2\beta+1} - \frac{1}{3}a^3\rho^2t^3xv^{2\beta} - \frac{5}{12}a^2\rho^2t^3v^{2\beta} \\
&\quad + \frac{1}{6}a^2\beta\rho^2t^3v^{2\beta} + \frac{1}{12}a^2t^3v^{2\beta} + \frac{1}{4}a^2\rho^2t^3xv^{2\beta} + \frac{1}{2}a^2\rho^2t^2x^2v^{2\beta-1} + \frac{1}{4}a\rho^2t^3v^{2\beta} \\
&\quad - \frac{1}{12}at^3v^{2\beta} - \frac{1}{4}a\rho^2t^2x^2v^{2\beta-1} + \frac{5}{6}a\rho^2t^2xv^{2\beta-1} - \frac{1}{3}a\beta\rho^2t^2xv^{2\beta-1} - \frac{1}{6}at^2xv^{2\beta-1} \\
&\quad - \frac{1}{3}a\rho^2tx^3v^{2\beta-2} + \frac{1}{6}\rho^2t^2v^{2\beta-1} - \frac{1}{6}\beta\rho^2t^2v^{2\beta-1} - \frac{1}{12}t^2v^{2\beta-1} - \frac{1}{4}\rho^2t^2xv^{2\beta-1} \\
&\quad + \frac{1}{12}t^2xv^{2\beta-1} + \frac{1}{12}\rho^2tx^3v^{2\beta-2} - \frac{5}{12}\rho^2tx^2v^{2\beta-2} + \frac{1}{6}\beta\rho^2tx^2v^{2\beta-2} + \frac{1}{12}tx^2v^{2\beta-2} \\
&\quad + \frac{1}{12}\rho^2x^4v^{2\beta-3}\Bigg) \in \Sigma_1.
\end{aligned} \tag{3.26c}$$

Using Lemma 2.2.8 and 2.2.10, without explicit calculation, we also have

$$u_{20} = \mathcal{B}_v^{-1}(\theta - v)v^\alpha\partial_v u_{10} \in \Sigma_1, \tag{3.27}$$

$$u_{11} = \mathcal{B}_v^{-1}\left[(\theta - v)v^\alpha\partial_v u_{01} + \rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{10}\right] \in \Sigma_1. \tag{3.28}$$

Since if

$$u_{ij} \in \Sigma_1, \quad \text{for } i + j = k - 1 \text{ and } k - 2, \tag{3.29}$$

we have

$$u_{ij} = \mathcal{B}_v^{-1}\left[\rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{i(j-1)} + \frac{1}{2}v^{2\beta}\partial_v^2 u_{i(j-2)} + (\theta - v)v^\alpha\partial_v u_{(i-1)j}\right] \in \Sigma_1, \quad \text{for } i + j = k. \tag{3.30}$$

By induction,

$$u_{ij} \in \Sigma_1, \quad \text{for } i + j > 0. \tag{3.31}$$

□

3.2.2 (η, v)-expansion

The following results deal with the second expansion method, which is defined as in (3.6).

Proposition 3.2.4. *The solution of* (3.2) *with initial condition* (3.3) *can be written as*

$$u(t, x) = \sum_{\substack{i=0\\j=0}}^{\infty} u_{ij}(t, x, \theta)\eta^i(v - \theta)^j, \tag{3.32}$$

with

$$a = \frac{1}{2} - \frac{r}{\theta}, \quad b = -\frac{\theta}{2}\left(\frac{1}{2} + \frac{r}{\theta}\right)^2, \tag{3.33a}$$

$$u_{00}(t,x) = \frac{e^x}{2}\operatorname{erfc}\left[-\frac{x+(1-a)\theta t}{\sqrt{2\theta t}}\right] - \frac{e^{-rt}}{2}\operatorname{erfc}\left[-\frac{x-a\theta t}{\sqrt{2\theta t}}\right]. \tag{3.33b}$$

For the Heston model,

$$\begin{aligned} u_{ij}(s,y) = e^{ay+bs-j\kappa s} &\int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left[-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt + j\kappa t\right] \\ \times &\left[\frac{1}{2}\left(\partial_x^2 - \partial_x\right) u_{i(j-1)} + \rho j \partial_x u_{(i-1)j} + \rho\theta(j+1)\partial_x u_{(i-1)(j+1)} \right. \\ &\left. + \frac{1}{2} j(j+1) u_{(i-2)(j+1)} + \frac{\theta}{2}(j+1)(j+2) u_{(i-2)(j+2)}\right]. \end{aligned} \tag{3.34}$$

For the GARCH model,

$$\begin{aligned} u_{ij}(s,y) = e^{ay+bs-j\kappa s} &\int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left[-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt + j\kappa t\right] \\ \times &\left[\frac{1}{2}\left(\partial_x^2 - \partial_x\right) u_{i(j-1)} + \frac{1}{2} j(j-1) u_{(i-2)(j+2)} + j(j+1)\theta u_{(i-2)(j+1)} \right. \\ &+ \frac{1}{2}(j+1)(j+2)\theta^2 u_{(i-2)(j+2)} \\ &\left. + \sum_{n=0}^{j+1} \frac{3\rho(j+1-n)(2n-5)!!}{(2n)!!\sqrt{\theta}} \left(-\frac{1}{\theta}\right)^{n-2} \partial_x u_{(i-1)(j+1-n)}\right]. \end{aligned} \tag{3.35}$$

For the 3/2 model,

$$\begin{aligned} u_{ij}(s,y) = e^{ay+bs-j\kappa\theta s} &\int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left[-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt + j\kappa\theta t\right] \\ \times &\left[\frac{1}{2}\left(\partial_x^2 - \partial_x\right) u_{i(j-1)} + \rho(j-1)\partial_x u_{(i-1)(j-1)} + 2\rho\theta j \partial_x u_{(i-1)j} \right. \\ &+ \rho\theta^2(j+1)\partial_x u_{(i-1)(j+1)} + \frac{1}{2}(j-1)(j-2) u_{(i-2)(j-1)} \\ &+ \frac{3}{2}\theta j(j-1) u_{(i-2)j} + \frac{3}{2}\theta^2 j(j+1) u_{(i-2)(j+1)} \\ &\left. + \frac{1}{2}\theta^3 (j+1)(j+2) u_{(i-2)(j+2)} - (j-1)\kappa u_{i(j-1)}\right]. \end{aligned} \tag{3.36}$$

Proof. From model definition (3.1), the Black–Scholes model can be recovered by setting $\eta = 0$ and $v = \theta$; therefore, the solution can be written as

$$u = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(\theta)\eta^i(v-\theta)^j, \tag{3.37a}$$

$$u_{00} = \frac{e^x}{2}\operatorname{erfc}\left[-\frac{x+(1-a)\theta t}{\sqrt{2\theta t}}\right] - \frac{e^{-rt}}{2}\operatorname{erfc}\left[-\frac{x-a\theta t}{\sqrt{2\theta t}}\right]. \tag{3.37b}$$

In the above expansion, v-derivatives in PDE (3.2) act only on the power terms $(v-\theta)^j$, while other derivatives act on the coefficients u_{ij}

$$\partial_v u = \sum_{\substack{i=0\\j=0}}^{\infty} j u_{ij}\eta^i (v-\theta)^{j-1} = \sum_{\substack{i=0\\j=0}}^{\infty} (j+1) u_{i(j+1)}\eta^i (v-\theta)^{j}, \tag{3.38a}$$

$$\partial_v^2 u = \sum_{\substack{i=0\\j=0}}^{\infty} j(j-1) u_{ij}\eta^i (v-\theta)^{j-2} = \sum_{\substack{i=0\\j=0}}^{\infty} (j+1)(j+2) u_{i(j+2)}\eta^i (v-\theta)^{j}, \tag{3.38b}$$

$$\partial_z^n u = \sum_{\substack{i=0\\j=0}}^{\infty} \partial_z^n u_{ij}\eta^i (v-\theta)^{j}, \tag{3.38c}$$

where z denotes variables other than v. The v-terms in (3.2) should also be expanded in terms of $(v-\theta)^j$

$$\mathcal{L}_v = \mathcal{L}_\theta - \frac{v-\theta}{2}\left(\partial_x^2 - \partial_x\right), \tag{3.39a}$$

$$v = (v-\theta) + \theta, \tag{3.39b}$$

$$v^2 = (v-\theta)^2 + 2\theta(v-\theta) + \theta^2, \tag{3.39c}$$

$$v^3 = (v-\theta)^3 + 3\theta(v-\theta)^2 + 3\theta^2(v-\theta) + \theta^3, \tag{3.39d}$$

$$v^{\frac{3}{2}} = \sum_{n=0}^{\infty} \frac{3(2n-5)!!}{(2n)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{n-2}(v-\theta)^n. \tag{3.39e}$$

Then, (3.37), (3.38) and (3.39) are substituted back into (3.2), and the starting values of indexes i and j are shifted in order to have terms of the same form $\eta^i(v-\theta)^j$. For the Heston model, (3.2) becomes

$$\begin{aligned}
&\sum_{\substack{i=0\\j=0}}^{\infty} \mathcal{B}_\theta u_{ij}\eta^i(v-\theta)^j - \sum_{\substack{i=0\\j=1}}^{\infty} \frac{1}{2}\left(\partial_x^2 - \partial_x\right) u_{i(j-1)}\eta^i(v-\theta)^j - \sum_{\substack{i=1\\j=1}}^{\infty} \rho j \partial_x u_{(i-1)j}\eta^i(v-\theta)^j \\
&\quad - \sum_{\substack{i=1\\j=0}}^{\infty} \rho\theta(j+1)\partial_x u_{(i-1)(j+1)}\eta^i(v-\theta)^j - \sum_{\substack{i=2\\j=1}}^{\infty} \frac{1}{2} j(j+1) u_{(i-2)(j+1)}\eta^i(v-\theta)^j \\
&\quad\quad - \sum_{\substack{i=2\\j=0}}^{\infty} \frac{\theta}{2}(j+1)(j+2) u_{(i-2)(j+2)}\eta^i(v-\theta)^j + \sum_{\substack{i=0\\j=1}}^{\infty} j\kappa u_{ij}\eta^i(v-\theta)^j = 0.
\end{aligned} \tag{3.40}$$

For the GARCH model, (3.2) becomes

$$\begin{aligned}
&\sum_{\substack{i=0\\j=0}}^{\infty} \mathcal{B}_\theta u_{ij}\eta^i(v-\theta)^j - \sum_{\substack{i=0\\j=1}}^{\infty} \frac{\partial_x^2 - \partial_x}{2} u_{i(j-1)}\eta^i(v-\theta)^j - \sum_{\substack{i=2\\j=2}}^{\infty} \frac{j(j-1)}{2} u_{(i-2)j}\eta^i(v-\theta)^j \\
&\quad - \sum_{\substack{i=2\\j=1}}^{\infty} j(j+1)\theta u_{(i-2)(j+1)}\eta^i(v-\theta)^j - \sum_{\substack{i=2\\j=0}}^{\infty} \frac{\theta^2(j+1)(j+2)}{2} u_{(i-2)(j+2)}\eta^i(v-\theta)^j \\
&\quad\quad - \sum_{\substack{i=1\\j=0}}^{\infty}\sum_{n=0}^{j+1} \frac{3\rho(j+1-n)(2n-5)!!}{(2n)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{n-2} \partial_x u_{(i-1)(j+1-n)}\eta^i(v-\theta)^j
\end{aligned}$$

$$+\sum_{\substack{i=0\\j=1}}^{\infty} j\kappa u_{ij}\eta^i(v-\theta)^j = 0. \quad (3.41)$$

For the 3/2 model, (3.2) becomes

$$\begin{aligned}
&\sum_{\substack{i=0\\j=0}}^{\infty} \mathcal{B}_\theta u_{ij}\eta^i(v-\theta)^j - \sum_{\substack{i=0\\j=1}}^{\infty} \frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{i(j-1)}\eta^i(v-\theta)^j \\
&\quad - \sum_{\substack{i=1\\j=2}}^{\infty}(j-1)\rho\partial_x u_{(i-1)(j-1)}\eta^i(v-\theta)^j - \sum_{\substack{i=1\\j=1}}^{\infty} 2j\rho\theta\partial_x u_{(i-1)j}\eta^i(v-\theta)^j \\
&\quad - \sum_{\substack{i=1\\j=0}}^{\infty}(j+1)\rho\theta^2\partial_x u_{(i-1)(j+1)}\eta^i(v-\theta)^j - \sum_{\substack{i=2\\j=3}}^{\infty}\frac{1}{2}(j-1)(j-2)u_{(i-2)(j-1)}\eta^i(v-\theta)^j \\
&\quad - \sum_{\substack{i=2\\j=2}}^{\infty}\frac{3\theta}{2}j(j-1)u_{(i-2)j}\eta^i(v-\theta)^j - \sum_{\substack{i=2\\j=1}}^{\infty}\frac{3\theta^2}{2}j(j+1)u_{(i-2)(j+1)}\eta^i(v-\theta)^j \\
&\quad - \sum_{\substack{i=2\\j=0}}^{\infty}\frac{\theta^3}{2}(j+1)(j+2)u_{(i-2)(j+2)}\eta^i(v-\theta)^j + \sum_{\substack{i=0\\j=1}}^{\infty} j\kappa\theta u_{ij}\eta^i(v-\theta)^j \\
&\quad + \sum_{\substack{i=0\\j=2}}^{\infty}(j-1)\kappa u_{i(j-1)}\eta^i(v-\theta)^j = 0. \quad (3.42)
\end{aligned}$$

The above equations hold if the coefficients of $\eta^i(v-\theta)^j$ satisfy the following inhomogeneous Black−Scholes equations. For the Heston model,

$$\begin{aligned}
\mathcal{B}_\theta u_{ij} + j\kappa u_{ij} &= \frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{i(j-1)} + \rho j\partial_x u_{(i-1)j} + \rho\theta(j+1)\partial_x u_{(i-1)(j+1)} \\
&\quad + \frac{1}{2}j(j+1)u_{(i-2)(j+1)} + \frac{\theta}{2}(j+1)(j+2)u_{(i-2)(j+2)}. \quad (3.43)
\end{aligned}$$

For the GARCH model,

$$\begin{aligned}
\mathcal{B}_\theta u_{ij} + j\kappa u_{ij} &= \frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{i(j-1)} + \frac{1}{2}j(j-1)u_{(i-2)j} + j(j+1)\theta u_{(i-2)(j+1)} \\
&\quad + \frac{1}{2}(j+1)(j+2)\theta^2 u_{(i-2)(j+2)} \\
&\quad + \sum_{n=0}^{j+1}\frac{3\rho(j+1-n)(2n-5)!!}{(2n)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{n-2}\partial_x u_{(i-1)(j+1-n)}. \quad (3.44)
\end{aligned}$$

For the 3/2 model,

$$\begin{aligned}
\mathcal{B}_\theta u_{ij} + j\kappa\theta u_{ij} &= \frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{i(j-1)} + \rho(j-1)\partial_x u_{(i-1)(j-1)} + 2\rho\theta j\partial_x u_{(i-1)j} \\
&\quad + \rho\theta^2(j+1)\partial_x u_{(i-1)(j+1)} + \frac{1}{2}(j-1)(j-2)u_{(i-2)(j-1)} \\
&\quad + \frac{3}{2}\theta j(j-1)u_{(i-2)j} + \frac{3}{2}\theta^2 j(j+1)u_{(i-2)(j+1)}
\end{aligned}$$

$$+\frac{1}{2}\theta^3(j+1)(j+2)u_{(i-2)(j+2)} - (j-1)\kappa u_{i(j-1)}. \quad (3.45)$$

All equations should be solved with the initial condition (3.18), and the terms with negative indexes vanish by definition. For (3.40), (3.41) and (3.42), u_{00} satisfies the homogeneous Black−Scholes equation with an initial condition that can be solved by Lemma 2.1.2 and other terms satisfy the inhomogeneous Black−Scholes equation that can be solved by Lemma 2.1.3. □

As in the previous case, after the iterative expression for u_{ij} is given, the coefficients should be calculated in a particular order to ensure that when calculating u_{ij}, all the terms involved (Figure 3.3) have been calculated. For (η, v)-expansion, the order illustrated in

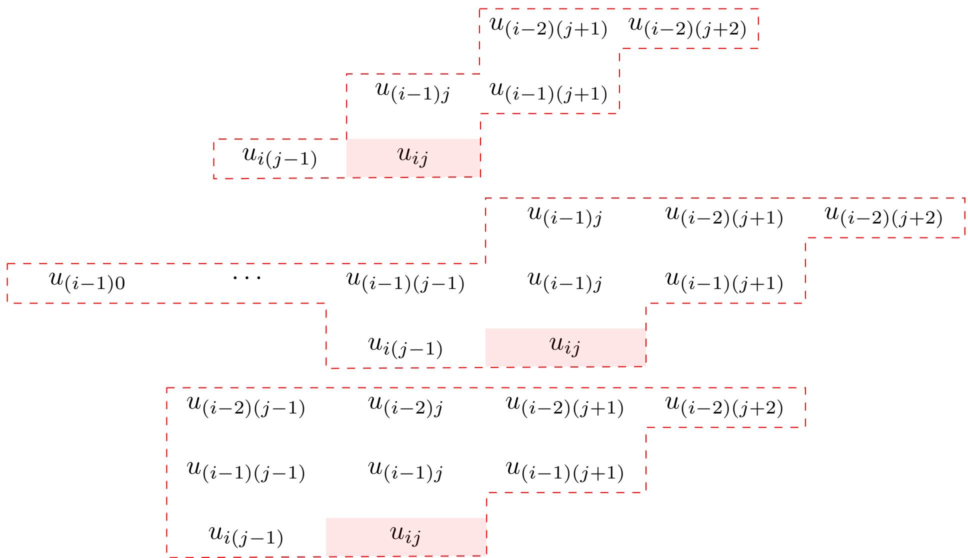

Figure 3.3: Terms involved in the calculation of u_{ij} under the Heston model (above), the GARCH model (middle) and the 3/2 model (below).

Figure 3.4 is recommended.

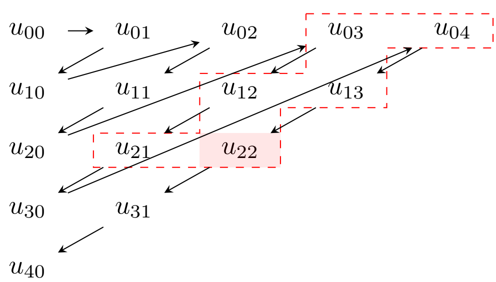

Figure 3.4: Order of calculation for (κ, η), (η, v) and (η, θ)-expansion.

The following corollary shows the general form of (κ, η)-expansion coefficients u_{ij}.

Corollary 3.2.5. *For solution* (3.32)*, the coefficients* u_{ij}*, except the leading term* u_{00}*, are BSDE functions,*

$$u_{ij} \in \Sigma_1', \quad \text{for } i + j > 0. \tag{3.46}$$

Proof. Define

$$w = \frac{1}{2\sqrt{2\pi\theta t}} \exp\left(-\frac{x^2}{2\theta t} + ax + bt\right), \tag{3.47a}$$

$$a = \frac{1}{2} - \frac{r}{\theta}, \quad b = -\frac{\theta}{2}\left(\frac{1}{2} + \frac{r}{\theta}\right)^2. \tag{3.47b}$$

For the Heston and GARCH models, the leading terms in Order 3.2.2 are

$$u_{O(1)} = u_{00} \notin \Sigma_1' \tag{3.48a}$$

$$u_{O(2)} = u_{01} = \frac{w\left(1 - e^{-\kappa t}\right)}{\kappa} \in \Sigma_1'. \tag{3.48b}$$

For the 3/2 model,

$$u_{O(1)} = u_{00} \notin \Sigma_1' \tag{3.49a}$$

$$u_{O(2)} = u_{01} = \frac{w\left(1 - e^{-\kappa\theta t}\right)}{\kappa\theta} \in \Sigma_1'. \tag{3.49b}$$

One can check that when $n > 2$, $u_{O(n)}$ does not involve u_{00} on the right-hand side of (3.40). Therefore, if

$$u_{O(k)} \in \Sigma_1', \quad 2 \le k \le n-1, \tag{3.50}$$

by Lemma 2.2.8 and 2.2.10, (3.43)–(3.45) give

$$u_{O(n)} \in \Sigma_1'. \tag{3.51}$$

By induction

$$u_{O(n)} \in \Sigma_1', \quad n \ge 2. \tag{3.52}$$

□

3.2.3 (η, θ)-expansion

The following proposition shows how to derive expansion terms in (3.7), and how it differs from the previous (η, v)-expansion.

Proposition 3.2.6. *The solution of* (3.2) *with initial condition* (3.3) *can be written as*

$$u(t, x) = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(t, x, v)\eta^i(\theta - v)^j, \tag{3.53}$$

with

$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2, \tag{3.54a}$$

$$u_{00}(t,x) = \frac{1}{2}e^x \operatorname{erfc}\left[-\frac{x+(1-a)vt}{\sqrt{2vt}}\right] - \frac{1}{2}e^{-rt}\operatorname{erfc}\left[-\frac{x-avt}{\sqrt{2vt}}\right] \tag{3.54b}$$

$$\begin{aligned} u_{ij}(s,y) = e^{ay+bs-j\kappa v^\alpha s} &\int_0^s \frac{\mathrm{d}t}{\sqrt{2\pi v(s-t)}} \int_{-\infty}^{\infty} \mathrm{d}x \exp\left[-\frac{(x-y)^2}{2v(s-t)} - ax - bt + j\kappa v^\alpha t\right] \\ &\times \Big[\rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{(i-1)j} - (j+1)\rho v^{\beta+\frac{1}{2}}\partial_x u_{(i-1)(j+1)} + \frac{1}{2}v^{2\beta}\partial_v^2 u_{(i-2)j} \\ &\quad -(j+1)v^{2\beta}\partial_v u_{(i-2)(j+1)} + \frac{1}{2}(j+1)(j+2)v^{2\beta}u_{(i-2)(j+2)} \\ &\quad +\kappa v^\alpha \partial_v u_{i(j-1)}\Big]. \end{aligned} \tag{3.54c}$$

Proof. Since when $\eta = 0$ and $v = \theta$, the Black−Scholes model is recovered, besides (3.37) in the Proposition 3.2.4, the option price can also be written as

$$u = \sum_{\substack{i=0\\j=0}}^{\infty} u_{ij}(v)\eta^i(\theta - v)^j, \tag{3.55a}$$

$$u_{00} = \frac{1}{2}e^x \operatorname{erfc}\left[-\frac{x+(1-a)vt}{\sqrt{2vt}}\right] - \frac{1}{2}e^{-rt}\operatorname{erfc}\left[-\frac{x-avt}{\sqrt{2vt}}\right], \tag{3.55b}$$

where the Black−Scholes model with variance v is used as a starting point. The difference with Proposition 3.2.4 is that θ is regarded as expansion parameter, and is used in all structural constants a, b and in the inverse Black−Scholes operator B_v^{-1}.

The v-derivatives act on both the coefficients $u_{ij}(v)$ and the power terms

$$\begin{aligned} \partial_v u &= \sum_{\substack{i=0\\j=0}}^{\infty} \partial_v u_{ij}\eta^i(\theta-v)^j - \sum_{\substack{i=0\\j=0}}^{\infty} j u_{ij}\eta^i(\theta-v)^{j-1} \\ &= \sum_{\substack{i=0\\j=0}}^{\infty} \partial_v u_{ij}\eta^i(\theta-v)^j - \sum_{\substack{i=0\\j=0}}^{\infty} (j+1)u_{i(j+1)}\eta^i(\theta-v)^j, \end{aligned} \tag{3.56a}$$

$$\begin{aligned} \partial_v^2 u &= \sum_{\substack{i=0\\j=0}}^{\infty} \partial_v^2 u_{ij}\eta^i(\theta-v)^j - 2\sum_{\substack{i=0\\j=0}}^{\infty} j\partial_v u_{ij}\eta^i(\theta-v)^{j-1} + \sum_{\substack{i=0\\j=0}}^{\infty} j(j-1)u_{ij}\eta^i(\theta-v)^{j-2} \\ &= \sum_{\substack{i=0\\j=0}}^{\infty} \partial_v^2 u_{ij}\eta^i(\theta-v)^j - 2\sum_{\substack{i=0\\j=0}}^{\infty} (j+1)\partial_v u_{i(j+1)}\eta^i(\theta-v)^j \\ &\quad + \sum_{\substack{i=0\\j=0}}^{\infty} (j+1)(j+2)u_{i(j+2)}\eta^i(\theta-v)^j, \end{aligned} \tag{3.56b}$$

$$\partial_z^n u = \sum_{\substack{i=0\\j=0}}^{\infty} \partial_z^n u_{ij}\eta^i(v-\theta)^j, \tag{3.56c}$$

where z denotes variables other than v. With ansatz (3.55) and derivatives (3.56), (3.2) becomes

$$\sum_{\substack{i=0\\j=0}}^{\infty} \mathcal{B}_v u_{ij}\eta^i(\theta-v)^j - \sum_{\substack{i=1\\j=0}}^{\infty} \rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{(i-1)j}\eta^i(\theta-v)^j$$
$$+\sum_{\substack{i=1\\j=0}}^{\infty}(j+1)\rho v^{\beta+\frac{1}{2}}\partial_x u_{(i-1)(j+1)}\eta^i(\theta-v)^j - \sum_{\substack{i=2\\j=0}}^{\infty}\frac{1}{2}v^{2\beta}\partial_v^2 u_{(i-2)j}\eta^i(\theta-v)^j$$
$$+\sum_{\substack{i=2\\j=0}}^{\infty}(j+1)v^{2\beta}\partial_v u_{(i-2)(j+1)}\eta^i(\theta-v)^j - \sum_{\substack{i=2\\j=0}}^{\infty}\frac{1}{2}(j+1)(j+2)v^{2\beta}u_{(i-2)(j+2)}\eta^i(\theta-v)^j$$
$$-\sum_{\substack{i=0\\j=1}}^{\infty}\kappa v^{\alpha}\partial_v u_{i(j-1)}\eta^i(\theta-v)^j + \sum_{\substack{i=0\\j=1}}^{\infty}\kappa v^{\alpha} j u_{ij}\eta^i(\theta-v)^j = 0, \quad (3.57)$$

with initial condition (3.18). The coefficients of $\eta^i(\theta-v)^j$ satisfy

$$\begin{aligned}\mathcal{B}_v u_{ij} + \kappa v^{\alpha} j u_{ij} = {} & \rho v^{\beta+\frac{1}{2}}\partial_x\partial_v u_{(i-1)j} - (j+1)\rho v^{\beta+\frac{1}{2}}\partial_x u_{(i-1)(j+1)} + \frac{1}{2}v^{2\beta}\partial_v^2 u_{(i-2)j}\\ & - (j+1)v^{2\beta}\partial_v u_{(i-2)(j+1)} + \frac{1}{2}(j+1)(j+2)v^{2\beta}u_{(i-2)(j+2)}\\ & + \kappa v^{\alpha}\partial_v u_{i(j-1)}. \end{aligned} \quad (3.58)$$

Similarly they can be solved by Lemma 2.1.2 and 2.1.3. □

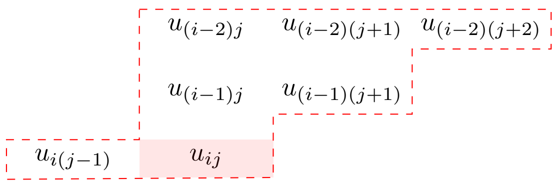

Figure 3.5: Terms involved in the calculation of u_{ij}.

The terms involved in every iteration of (η,θ)-expansion are shown in Figure 3.5. The order illustrated in Figure 3.6 (Order 3.2.2) can be applied. The following corollary

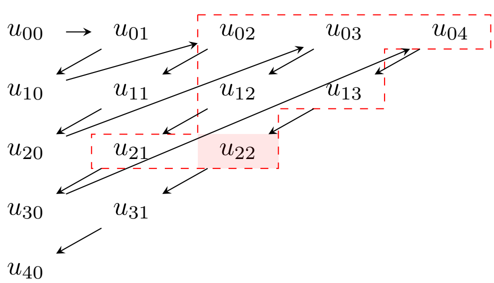

Figure 3.6: Order of calculation for (κ,η), (η,v) and (η,θ)-expansion.

ensures the form of the coefficients u_{ij}.

Corollary 3.2.7. *For solution* (3.53)*, the coefficients* u_{ij}*, except the leading term* u_{00}*, are BSDE functions,*

$$u_{ij} \in \Sigma_1', \quad \text{for } i+j>0. \tag{3.59}$$

Proof. Define

$$w = \frac{1}{2\sqrt{2\pi vt}} \exp\left(-\frac{x^2}{2vt} + ax + bt\right), \tag{3.60}$$

$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2. \tag{3.61}$$

For the Heston and GARCH models, the leading terms in Order 3.2.2 are

$$u_{O(1)} = u_{00} \notin \Sigma_1' \tag{3.62}$$

$$u_{O(2)} = u_{01} = \frac{w\left(1-e^{-\kappa t}\right)}{\kappa} \in \Sigma_1'. \tag{3.63}$$

For the 3/2 model,

$$u_{O(1)} = u_{00} \notin \Sigma_1' \tag{3.64}$$

$$u_{O(2)} = u_{01} = \frac{w\left(1-e^{-\kappa vt}\right)}{\kappa v} \in \Sigma_1'. \tag{3.65}$$

One can check that when $n>2$, $u_{O(n)}$ does not involve u_{00} on the right-hand side of (3.40). Therefore, if

$$u_{O(k)} \in \Sigma_1', \quad 2 \le k \le n-1, \tag{3.66}$$

by Lemma 2.2.8 and 2.2.10, (3.40) gives

$$u_{O(n)} \in \Sigma_1'. \tag{3.67}$$

By induction,

$$u_{O(n)} \in \Sigma_1', \quad n \ge 2. \tag{3.68}$$

□

3.2.4 (κ, v)-expansion

The next proposition shows how to derive expansion terms in (3.10).

Proposition 3.2.8. *The solution of* (3.2) *with initial condition* (3.3) *can be written as*

$$u(t,x) = \sum_{\substack{i=0\\j=0}}^{\infty} u_{ij}(t,x,\theta)\kappa^{-i}(v-\theta)^j, \tag{3.69}$$

with

$$a = \frac{1}{2} - \frac{r}{\theta}, \quad b = -\frac{\theta}{2}\left(\frac{1}{2} + \frac{r}{\theta}\right)^2, \tag{3.70a}$$

$$u_{00}(t,x) = \frac{1}{2}e^x \operatorname{erfc}\left[-\frac{x+(1-a)\theta t}{\sqrt{2\theta t}}\right] - \frac{1}{2}e^{-rt}\operatorname{erfc}\left[-\frac{x-a\theta t}{\sqrt{2\theta t}}\right]. \tag{3.70b}$$

For the Heston model,

$$u_{i0}(s,y) = e^{ay+bs}\int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left(-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt\right) \times \left(\rho\eta\theta\partial_x u_{i1} + \eta^2\theta u_{i2}\right), \tag{3.71a}$$

$$u_{i(j>0)}(t,x) = \frac{1}{j}\Big[-\mathcal{B}_\theta u_{(i-1)j} + \frac{1}{2}\left(\partial_x^2 - \partial_x\right)u_{(i-1)(j-1)} + \rho\eta j\partial_x u_{(i-1)j} + \rho\eta\theta(j+1)\partial_x u_{(i-1)(j+1)} + \frac{\eta^2}{2}j(j+1)u_{(i-1)(j+1)} + \frac{\eta^2\theta}{2}(j+1)(j+2)u_{(i-1)(j+2)}\Big]. \tag{3.71b}$$

For the GARCH model,

$$u_{i0}(s,y) = e^{ay+bs}\int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left(-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt\right) \times \left(\rho\eta\theta^{\frac{3}{2}}\partial_x u_{i1} + \eta^2\theta^2 u_{i2}\right), \tag{3.72a}$$

$$u_{i(j>0)}(t,x) = \frac{1}{j}\Big[-\mathcal{B}_\theta u_{(i-1)j} + \frac{1}{2}\left(\partial_x^2 - \partial_x\right)u_{(i-1)(j-1)} + \frac{\eta^2}{2}j(j-1)u_{(i-1)j} + \frac{\eta^2\theta}{2}j(j+1)u_{(i-1)(j+1)} + \frac{\eta^2\theta^2}{2}(j+1)(j+2)u_{(i-1)(j+2)} + \sum_{n=0}^{j+1}\frac{3\rho\eta(j+1-n)(2n-5)!!}{(2n)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{n-2}\partial_x u_{(i-1)(j+1-n)}\Big]. \tag{3.72b}$$

For the 3/2 model,

$$u_{i0}(s,y) = e^{ay+bs}\int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left(-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt\right) \times \left(\rho\eta\theta^2\partial_x u_{i1} + \eta^2\theta^3 u_{i2}\right), \tag{3.73a}$$

$$u_{i(j>0)}(t,x) = \frac{1}{j\theta}\Big[-\mathcal{B}_\theta u_{(i-1)j} + \frac{1}{2}\left(\partial_x^2 - \partial_x\right)u_{(i-1)(j-1)} + \rho\eta(j-1)\partial_x u_{(i-1)(j-1)} + 2\rho\eta\theta j\partial_x u_{(i-1)j} + \rho\eta\theta^2(j+1)\partial_x u_{(i-1)(j+1)} + \frac{1}{2}\eta^2(j-1)(j-2)u_{(i-1)(j-1)} + \frac{3}{2}\eta^2\theta j(j-1)u_{(i-1)j} + \frac{3}{2}\eta^2\theta^2 j(j+1)u_{(i-1)(j+1)} + \frac{1}{2}\eta^2\theta^3(j+1)(j+2)u_{(i-1)(j+2)} - (j-1)u_{i(j-1)}\Big]. \tag{3.73b}$$

Proof. From (3.1), it has been shown that when $\frac{1}{\kappa} \to 0$ ($\kappa \to \infty$), the variance process is constant at $v_t = \theta$. Therefore the option price can be expanded at $\frac{1}{\kappa} = 0$ and $v = \theta$

$$u = \sum_{\substack{i=0\\j=0}}^{\infty} u_{ij}(\theta)\kappa^{-i}(v-\theta)^j, \tag{3.74}$$

$$u_{00} = \frac{e^x}{2}\,\mathrm{erfc}\left[-\frac{x+(1-a)\theta t}{\sqrt{2\theta t}}\right] - \frac{e^{-rt}}{2}\,\mathrm{erfc}\left[-\frac{x-a\theta t}{\sqrt{2\theta t}}\right]. \tag{3.75}$$

As in (3.38), the derivatives involved in this expansion are

$$\partial_v u = \sum_{\substack{i=0\\j=0}}^{\infty} j u_{ij}\kappa^{-i}(v-\theta)^{j-1} = \sum_{\substack{i=0\\j=0}}^{\infty}(j+1)u_{i(j+1)}\kappa^{-i}(v-\theta)^j, \tag{3.76a}$$

$$\partial_v^2 u = \sum_{\substack{i=0\\j=0}}^{\infty} j(j-1)u_{ij}\kappa^{-i}(v-\theta)^{j-2} = \sum_{\substack{i=0\\j=0}}^{\infty}(j+1)(j+2)u_{i(j+2)}\kappa^{-i}(v-\theta)^j, \tag{3.76b}$$

$$\partial_z^n u = \sum_{\substack{i=0\\j=0}}^{\infty} \partial_z^n u_{ij}\kappa^{-i}(v-\theta)^j, \tag{3.76c}$$

where z indicates variables other than v. The power terms of v in (3.2) are expanded as in (3.39). Then for the Heston model, (3.2) becomes

$$\begin{aligned}\sum_{\substack{i=0\\j=0}}^{\infty}\mathcal{B}_\theta u_{ij}\kappa^{-i}(v-\theta)^j - \sum_{\substack{i=0\\j=1}}^{\infty}\frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{i(j-1)}\kappa^{-i}(v-\theta)^j - \sum_{\substack{i=0\\j=1}}^{\infty}\rho\eta j\partial_x u_{ij}\kappa^{-i}(v-\theta)^j \\ - \sum_{\substack{i=0\\j=0}}^{\infty}\rho\eta\theta(j+1)\partial_x u_{i(j+1)}\kappa^{-i}(v-\theta)^j - \sum_{\substack{i=0\\j=1}}^{\infty}\frac{\eta^2}{2}j(j+1)u_{i(j+1)}\kappa^{-i}(v-\theta)^j \\ - \sum_{\substack{i=0\\j=0}}^{\infty}\frac{\eta^2\theta}{2}(j+1)(j+2)u_{i(j+2)}\kappa^{-i}(v-\theta)^j + \sum_{\substack{i=-1\\j=1}}^{\infty} j u_{(i+1)j}\kappa^{-i}(v-\theta)^j = 0.\end{aligned} \tag{3.77}$$

For the GARCH model, (3.2) becomes

$$\begin{aligned}\sum_{\substack{i=0\\j=0}}^{\infty}\mathcal{B}_\theta u_{ij}\kappa^{-i}(v-\theta)^j - \sum_{\substack{i=0\\j=1}}^{\infty}\frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{i(j-1)}\kappa^{-i}(v-\theta)^j - \sum_{\substack{i=0\\j=2}}^{\infty}\frac{\eta^2}{2}j(j-1)u_{ij}\kappa^{-i}(v-\theta)^j \\ - \sum_{\substack{i=0\\j=1}}^{\infty}\frac{\theta\eta^2}{2}j(j+1)u_{i(j+1)}\kappa^{-i}(v-\theta)^j - \sum_{\substack{i=0\\j=0}}^{\infty}\frac{\theta^2\eta^2}{2}(j+1)(j+2)u_{i(j+2)}\kappa^{-i}(v-\theta)^j \\ - \sum_{\substack{i=0\\j=0}}^{\infty}\sum_{n=0}^{j+1}\frac{3\rho\eta(j+1-n)(2n-5)!!}{(2n)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{n-2}\partial_x u_{i(j+1-n)}\kappa^{-i}(v-\theta)^j \\ + \sum_{\substack{i=-1\\j=1}}^{\infty} j u_{(i+1)j}\kappa^{-i}(v-\theta)^j = 0.\end{aligned} \tag{3.78}$$

For the 3/2 model, (3.2) becomes

$$\sum_{\substack{i=0\\j=0}}^{\infty}\mathcal{B}_\theta u_{ij}\kappa^{-i}(v-\theta)^j - \sum_{\substack{i=0\\j=1}}^{\infty}\frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{i(j-1)}\kappa^{-i}(v-\theta)^j$$

$$
\begin{aligned}
&-\sum_{\substack{i=0\\j=2}}^{\infty}\rho\eta(j-1)\partial_x u_{i(j-1)}\kappa^{-i}(v-\theta)^j-\sum_{\substack{i=0\\j=1}}^{\infty}2\rho\eta\theta j\partial_x u_{ij}\kappa^{-i}(v-\theta)^j\\
&-\sum_{\substack{i=0\\j=0}}^{\infty}\rho\eta\theta^2(j+1)\partial_x u_{i(j+1)}\kappa^{-i}(v-\theta)^j-\sum_{\substack{i=0\\j=3}}^{\infty}\frac{\eta^2}{2}(j-1)(j-2)u_{i(j-1)}\kappa^{-i}(v-\theta)^j\\
&-\sum_{\substack{i=0\\j=2}}^{\infty}\frac{3\theta\eta^2}{2}j(j-1)u_{ij}\kappa^{-i}(v-\theta)^j-\sum_{\substack{i=0\\j=1}}^{\infty}\frac{3\theta^2\eta^2}{2}j(j+1)u_{i(j+1)}\kappa^{-i}(v-\theta)^j\\
&-\sum_{\substack{i=0\\j=0}}^{\infty}\frac{\theta^3\eta^2}{2}(j+1)(j+2)u_{i(j+2)}\kappa^{-i}(v-\theta)^j+\sum_{\substack{i=-1\\j=2}}^{\infty}(j-1)u_{(i+1)(j-1)}\kappa^{-i}(v-\theta)^j\\
&+\sum_{\substack{i=-1\\j=1}}^{\infty}\theta j u_{(i+1)j}\kappa^{-i}(v-\theta)^j=0. \qquad (3.79)
\end{aligned}
$$

The coefficients for $\kappa^{-1}(v-\theta)^j$ result in

$$\mathcal{B}_\theta u_{i0}=\rho\eta\theta\partial_x u_{i1}+\eta^2\theta u_{i2}, \qquad (3.80a)$$

$$
\begin{aligned}
ju_{i(j>0)}(t,x)=&-\mathcal{B}_\theta u_{(i-1)j}+\frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{(i-1)(j-1)}+\rho\eta j\partial_x u_{(i-1)j}\\
&+\rho\eta\theta(j+1)\partial_x u_{(i-1)(j+1)}+\frac{\eta^2}{2}j(j+1)u_{(i-1)(j+1)}\\
&+\frac{\eta^2\theta}{2}(j+1)(j+2)u_{(i-1)(j+2)} \qquad (3.80b)
\end{aligned}
$$

for the Heston model,

$$\mathcal{B}_\theta u_{i0}=\rho\eta\theta^{\frac{3}{2}}\partial_x u_{i1}+\eta^2\theta^2 u_{i2}, \qquad (3.81a)$$

$$
\begin{aligned}
ju_{i(j>0)}(t,x)=&-\mathcal{B}_\theta u_{(i-1)j}+\frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{(i-1)(j-1)}+\frac{\eta^2}{2}j(j-1)u_{(i-1)j}\\
&+\frac{\eta^2\theta}{2}j(j+1)u_{(i-1)(j+1)}+\frac{\eta^2\theta^2}{2}(j+1)(j+2)u_{(i-1)(j+2)}\\
&+\sum_{n=0}^{j+1}\frac{3\rho\eta(j+1-n)(2n-5)!!}{(2n)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{n-2}\partial_x u_{(i-1)(j+1-n)} \qquad (3.81b)
\end{aligned}
$$

for the GARCH model and

$$\mathcal{B}_\theta u_{i0}=\rho\eta\theta^2\partial_x u_{i1}+\eta^2\theta^3 u_{i2}, \qquad (3.82a)$$

$$
\begin{aligned}
j\theta u_{i(j>0)}(t,x)=&-\mathcal{B}_\theta u_{(i-1)j}+\frac{1}{2}\left(\partial_x^2-\partial_x\right)u_{(i-1)(j-1)}+\rho\eta(j-1)\partial_x u_{(i-1)(j-1)}\\
&+2\rho\eta\theta j\partial_x u_{(i-1)j}+\rho\eta\theta^2(j+1)\partial_x u_{(i-1)(j+1)}\\
&+\frac{1}{2}\eta^2(j-1)(j-2)u_{(i-1)(j-1)}+\frac{3}{2}\eta^2\theta j(j-1)u_{(i-1)j}\\
&+\frac{3}{2}\eta^2\theta^2 j(j+1)u_{(i-1)(j+1)}+\frac{1}{2}\eta^2\theta^3(j+1)(j+2)u_{(i-1)(j+2)}\\
&-(j-1)u_{i(j-1)} \qquad (3.82b)
\end{aligned}
$$

for the 3/2 model. Using Lemma 2.1.3, we can solve these equations for u_{i0}. □

As shown previously, when κ goes to infinity, stochastic volatility reduces to constant volatility θ, regardless of the initial volatility v. Therefore, the calculation order of (κ, v)-expansion is different than previous expansions, because the expansion terms exhibit a unique structure.

Corollary 3.2.9. *For all models in Proposition 3.2.8, the expansion coefficients are*

$$u_{0j} = 0, \quad \textit{for } j \geq 1. \tag{3.83}$$

Proof. When $\kappa = \infty$, the variance process is constant $v_t = \theta$ and the Black−Scholes model with variance θ is recovered. Therefore the solution is

$$u|_{\kappa=\infty} = \sum_{j=0}^{\infty} u_{0j}(v-\theta)^j = \frac{1}{2}e^x \operatorname{erfc}\left[-\frac{x+(1-a)\theta t}{\sqrt{2\theta t}}\right] - \frac{1}{2}e^{-rt} \operatorname{erfc}\left[-\frac{x-a\theta t}{\sqrt{2\theta t}}\right]. \tag{3.84}$$

Since the above solution (Black−Scholes solution with variance θ) does not depend on the initial variance v, the higher order coefficients are

$$u_{0j} = 0, \quad \text{for } j \geq 1. \tag{3.85}$$

□

The expansion coefficients for the Heston model look like a lower triangular matrix.

Corollary 3.2.10. *For the Heston model in Proposition 3.2.8, the expansion coefficients are*

$$u_{ij} = 0, \quad \textit{for } j > i. \tag{3.86}$$

Proof. Assume that

$$u_{(i-1)j} = 0, \quad \text{for } \forall i \text{ and } j \geq i. \tag{3.87}$$

Because for the Heston model (3.80), u_{ij} is linear in four terms:

$$u_{ij} = Au_{(i-1)(j-1)} + Bu_{(i-1)j} + Cu_{(i-1)(j+1)} + Du_{(i-1)(j+2)}, \quad \text{for } j \geq 1. \tag{3.88}$$

If $j \geq i+1$, then $j-1 \geq i$ and

$$u_{(i-1)(j-1)} = u_{(i-1)j} = u_{(i-1)(j+1)} = u_{(i-1)(j+2)} = 0, \tag{3.89}$$

consequently,

$$u_{i(j+1)} = 0, \quad \text{for } j \geq i. \tag{3.90}$$

By the previous corollary,

$$u_{0j} = 0, \quad \text{for } j \geq 1. \tag{3.91}$$

By induction,

$$u_{ij} = 0, \quad \text{for } j > i. \tag{3.92}$$

□

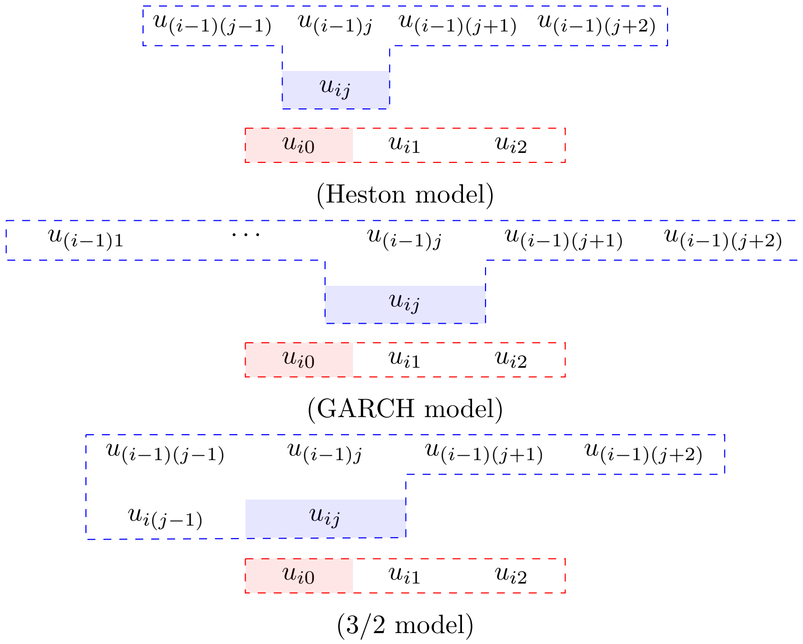

Figure 3.7: Terms involved in the calculation of u_{ij}.

For the (κ, v)-expansion, the terms involved in the calculation of u_{ij} are shown in Figure 3.7.

As shown in Figure 3.8, the expansion terms can be calculated as in Order 3.2.11 for the Heston model, and Order 3.2.12 for the GARCH and 3/2 models.

Order 3.2.11. *Let $i, j, n \in \mathbb{N}$ and $O(1) = (0,0)$. If $O(n) = (i,j)$, and then*

$$O(n+1) = \begin{cases} (i+1,1), & j = 0, \\ (i,j+1), & 0 < j < i, \\ (i,0), & j = i. \end{cases} \tag{3.93}$$

Order 3.2.12. *Let $i, j, n \in \mathbb{N}$ and $O(1) = (0,0)$. If $O(n) = (i,j)$, and then*

$$O(n+1) = \begin{cases} (1,2i+1), & j = 0, \\ (1,2i), & j = 1, \\ (i,0), & j = 2, \\ (i+1,j-2), & j \geq 3. \end{cases} \tag{3.94}$$

Like in the previous cases, we could show that the expansion terms are all BSE functions, except u_{00}.

Corollary 3.2.13. *In Proposition 3.2.8, the expansion coefficients*

$$u_{ij} \in \Sigma_1, \quad \textit{for } j \geq 1. \tag{3.95}$$

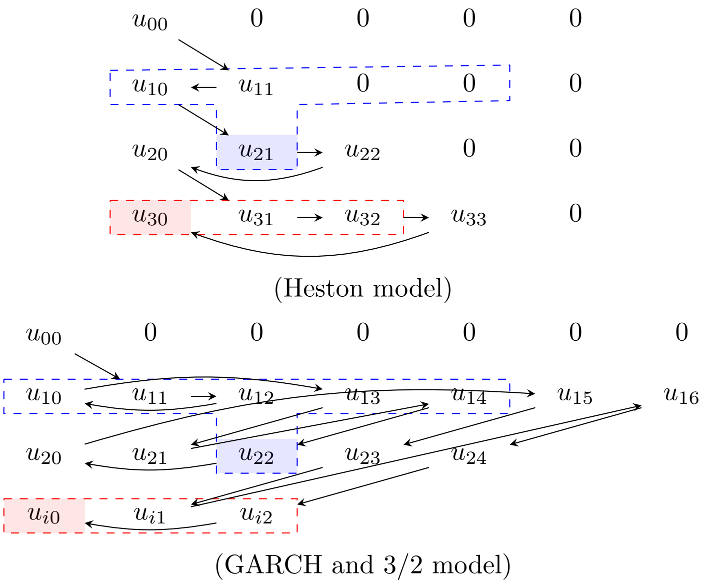

Figure 3.8: Order of calculation for (κ, v)-expansion.

Proof. For the Heston model,

$$u_{11} = \frac{1}{2}\left(\partial_x^2 - \partial_x\right) u_{00} = \frac{1}{2\sqrt{2\pi\theta t}} \exp\left(-\frac{x^2}{2\theta t} + ax + bt\right) \in \Sigma_1, \tag{3.96}$$

$$u_{10} = \mathcal{B}^{-1} \rho\eta\theta \partial_x u_{11} \in \Sigma_1, \tag{3.97}$$

$$u_{1j} = 0 \in \Sigma_1, \quad j > 1. \tag{3.98}$$

For the GARCH model,

$$u_{11} = \frac{1}{2}\left(\partial_x^2 - \partial_x\right) u_{00} = \frac{1}{2\sqrt{2\pi\theta t}} \exp\left(-\frac{x^2}{2\theta t} + ax + bt\right) \in \Sigma_1, \tag{3.99}$$

$$u_{10} = \mathcal{B}^{-1} \rho\eta\theta^{\frac{3}{2}} \partial_x u_{11} \in \Sigma_1, \tag{3.100}$$

$$u_{1j} = 0 \in \Sigma_1, \quad j > 1. \tag{3.101}$$

For the 3/2 model,

$$u_{11} = \frac{1}{2\theta}\left(\partial_x^2 - \partial_x\right) u_{00} = \frac{1}{2\theta\sqrt{2\pi\theta t}} \exp\left(-\frac{x^2}{2\theta t} + ax + bt\right) \in \Sigma_1, \tag{3.102}$$

$$u_{10} = \mathcal{B}^{-1} \rho\eta\theta^2 \partial_x u_{11} \in \Sigma_1, \tag{3.103}$$

$$u_{1j} = -\frac{j-1}{j\theta} u_{1(j-1)} \in \Sigma_1, \quad j > 1. \tag{3.104}$$

Therefore, for all models,

$$u_{1j} \in \Sigma_1, \quad j \geq 0. \tag{3.105}$$

Suppose

$$u_{ij} \in \Sigma_1, \quad j \geq 0. \tag{3.106}$$

Then,

$$u_{(i+1)j} \in \Sigma_1, \tag{3.107}$$

because the calculation of $u_{(i+1)j}$ only involves BSE functions

$$u_{ik}, \quad k = 0, 1, \ldots, j+2, \tag{3.108}$$

which are closed under $\mathcal{B}^{-1}$ and differential operators. By induction,

$$u_{ij} \in \Sigma_1, \quad \text{for } i > 1. \tag{3.109}$$

□

3.3 Bounded basis expansions

In the last section, we expanded the option prices with powers of parameters p^i. Conceptually, in the infinite-dimensional function space, all sets of basis vectors are equivalent. However, in practical calculations only a limited number of terms can be evaluated. Therefore, we have to cut off the polynomial to a certain degree:

$$u = \sum_{i,j=0}^{\infty} u_{ij} p^i q^j \approx \sum_{i,j=0}^{N} u_{ij} p^i q^j, \tag{3.110}$$

where p and q are expansion parameters and N is the largest power we calculate to. With a cut off limit of N, basis vector sets are no longer equivalent. If more appropriate basis vectors are chosen, the approximation error will be smaller.

For stochastic volatility models, the option price can be calculated using the Black−Scholes formula

$$C(\bar{v}) = \frac{1}{2} e^x \operatorname{erfc}\left[-\frac{x + \left(\frac{\bar{v}}{2} + r\right) t}{\sqrt{2\bar{v}t}}\right] - \frac{1}{2} e^{-rt} \operatorname{erfc}\left[-\frac{x - \left(\frac{\bar{v}}{2} - r\right) t}{\sqrt{2\bar{v}t}}\right], \tag{3.111}$$

with $\bar{v}$ being the average variance in the period $[0, t]$

$$\bar{v} = \frac{1}{t} \int_0^t v_s \, \mathrm{d}s, \tag{3.112}$$

and the SDE for variance process v_t is

$$\mathrm{d}v_t = \kappa(\theta - v_t) v_t^{\alpha} \, \mathrm{d}t + \eta v_t^{\beta} \, \mathrm{d}Z_t. \tag{3.113}$$

Because $C(\bar{v})$ is continuous on $0 \le \bar{v} < \infty$ and

$$C(0) = \left(e^x - e^{-rt}\right)^+, \quad C(\infty) = e^x, \tag{3.114}$$

$C(\bar{v})$ is bounded with respect to $\bar{v}$. Therefore, the option price obtained under stochastic volatility models is also bounded with respect to parameters (η, κ and v) of the SDE (3.113).

Figure 3.9 shows the asymptotic behavior of functions x^i and $\left(\frac{x}{1+x}\right)^i$. As $x \to \infty$,

$$\lim_{x\to\infty} x^i = \infty, \quad i \ge 1, \tag{3.115}$$

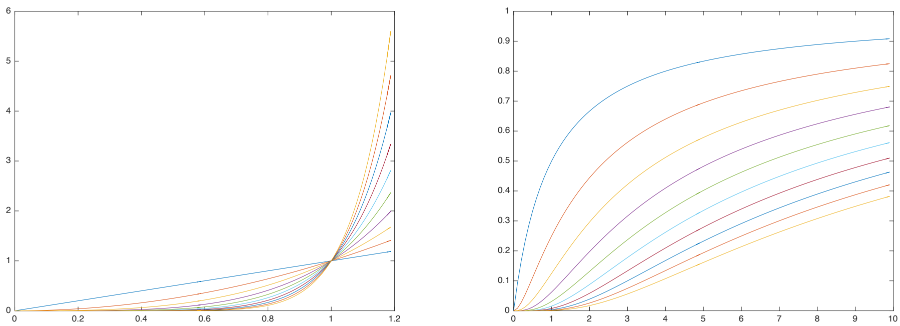

Figure 3.9: Asymptotic behavior of x^i (left) and $\left(\frac{x}{1+x}\right)^i$ (right).

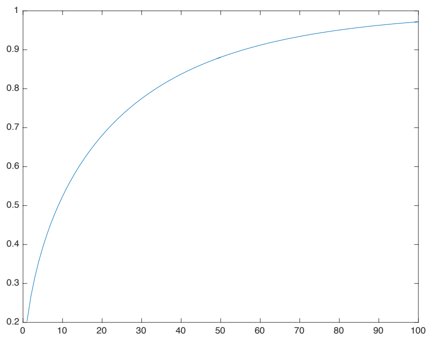

Figure 3.10: Asymptotic behavior of model parameter v.

$$\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^i = 1, \quad i \geq 1. \tag{3.116}$$

Figure 3.10 shows the asymptotic behavior of option prices with respect to a parameter of the variance process (3.113), which is the initial variance v. It confirms the boundedness of the option price and it looks very similar to Figure 3.9 of the bounded function

$$\left(\frac{x}{1+x}\right)^i, \quad i \geq 1. \tag{3.117}$$

Therefore, in the following propositions, we use (3.117) instead of x^i, as basis functions of expansion.

First, we have to determine how basis functions behave under differentiation and multiplication, because these operations are not as trivial as those for x^i.

Lemma 3.3.1. *For $i, k \in \mathrm{N}$, and $x > -1$,*

$$x\left(\frac{x}{1+x}\right)^i = \sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^{i+k+1}, \tag{3.118a}$$

$$x^2\left(\frac{x}{1+x}\right)^i = \sum_{k=0}^{\infty}\frac{2^{(k)}}{k!}\left(\frac{x}{1+x}\right)^{i+k+2}, \tag{3.118b}$$

$$\partial_x\left(\frac{x}{1+x}\right)^i = i\left(\frac{x}{1+x}\right)^{i-1} - 2i\left(\frac{x}{1+x}\right)^{i} + i\left(\frac{x}{1+x}\right)^{i+1}, \tag{3.118c}$$

$$x\partial_x\left(\frac{x}{1+x}\right)^i = i\left(\frac{x}{1+x}\right)^{i} - i\left(\frac{x}{1+x}\right)^{i+1}, \tag{3.118d}$$

$$x^2\partial_x\left(\frac{x}{1+x}\right)^i = i\left(\frac{x}{1+x}\right)^{i+1}, \tag{3.118e}$$

$$\begin{aligned}\partial_x^2\left(\frac{x}{1+x}\right)^i &= i(i-1)\left(\frac{x}{1+x}\right)^{i-2} + i(2-4i)\left(\frac{x}{1+x}\right)^{i-1} + 6i^i\left(\frac{x}{1+x}\right)^{i}\\ &\quad + i(-2-4i)\left(\frac{x}{1+x}\right)^{i+1} + i(i+1)\left(\frac{x}{1+x}\right)^{i+2},\end{aligned} \tag{3.118f}$$

$$\begin{aligned}x\partial_x^2\left(\frac{x}{1+x}\right)^i &= i(i-1)\left(\frac{x}{1+x}\right)^{i-1} - i(3i-1)\left(\frac{x}{1+x}\right)^{i}\\ &\quad + i(3i+1)\left(\frac{x}{1+x}\right)^{i+1} - i(i+1)\left(\frac{x}{1+x}\right)^{i+2},\end{aligned} \tag{3.118g}$$

$$x^2\partial_x^2\left(\frac{x}{1+x}\right)^i = i(i-1)\left(\frac{x}{1+x}\right)^{i} - 2i^2\left(\frac{x}{1+x}\right)^{i+1} + i(i+1)\left(\frac{x}{1+x}\right)^{i+2}, \tag{3.118h}$$

$$x^3\partial_x^2\left(\frac{x}{1+x}\right)^i = i(i-1)\left(\frac{x}{1+x}\right)^{i+1} - i(i+1)\left(\frac{x}{1+x}\right)^{i+2}. \tag{3.118i}$$

Proof. Suppose

$$\sum_{l=0}^{k}\frac{i^{(k-l)}}{(k-l)!} = \frac{(i+1)^{(k)}}{k!}, \tag{3.119}$$

then

$$\begin{aligned}\sum_{l=0}^{k+1}\frac{i^{(k+1-l)}}{(k+1-l)!} &= \sum_{l=0}^{k}\frac{i^{(k-l)}}{(k-l)!} + \frac{i^{(k+1)}}{(k+1)!} = \frac{(i+1)^{(k)}}{k!} + \frac{i^{(k+1)}}{(k+1)!}\\ &= \frac{(k+1)(i+1)^{(k)} + i(i+1)^{(k)}}{(k+1)!} = \frac{(i+k+1)(i+1)^{(k)}}{(k+1)!} = \frac{(i+1)^{(k+1)}}{(k+1)!}.\end{aligned} \tag{3.120}$$

Because (3.119) holds for $k = 0$, by induction, it holds for $k \in \mathrm{N}$. If

$$\frac{1}{(1-y)^i} = \sum_{k=0}^{\infty}\frac{i^{(k)}}{k!}y^k, \tag{3.121}$$

then

$$\frac{1}{(1-y)^{i+1}} = \frac{1}{(1-y)^i}\frac{1}{1-y} = \sum_{k=0}^{\infty}\frac{i^{(k)}}{k!}y^k\sum_{l=0}^{\infty}y^l$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{i^{(k-l)}}{(k-l)!} y^k = \sum_{k=0}^{\infty} \frac{(i+1)^{(k)}}{k!} y^k. \quad (3.122)$$

Because (3.121) holds for $i = 0$, by induction, it holds for $i \in \mathrm{N}$. Define

$$y = \frac{x}{1+x}. \quad (3.123)$$

Because when $0 < x < \infty$, $0 < y < 1$. Therefore, (3.121) can be used to prove the first two identities of this lemma. The rest can be proved by direct computation. □

In this section, we show the validity of bounded basis version of the (η, v)-expansion. However all four expansion methods introduced earlier in the last section can be modified in the same way.

3.3.1 $(\eta + 1, v + 1)$-expansion

Proposition 3.3.2. *The solution of* (3.2) *with initial condition* (3.3) *can be written as*

$$u(t, x) = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(t, x, \theta) X^i Y^j, \quad (3.124)$$

with

$$X = \frac{\eta}{1+\eta}, \quad Y = \frac{v-\theta}{1+v-\theta}, \quad a = \frac{1}{2} - \frac{r}{\theta}, \quad b = -\frac{\theta}{2}\left(\frac{1}{2} + \frac{r}{\theta}\right)^2, \quad (3.125)$$

and

$$u_{00}(t, x) = \frac{1}{2} e^x \operatorname{erfc}\left[-\frac{x + (1-a)\theta t}{\sqrt{2\theta t}}\right] - \frac{1}{2} e^{-rt} \operatorname{erfc}\left[-\frac{x - a\theta t}{\sqrt{2\theta t}}\right]. \quad (3.126)$$

For the Heston model,

$$\begin{aligned}
u_{ij}(s, y) &= e^{ay+bs-j\kappa s} \int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left[-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt + j\kappa t\right] \\
&\times \Bigg\{ \frac{1}{2}\left(\partial_x^2 - \partial_x\right) \sum_{k=0}^{j-1} u_{i(j-1-k)} + \rho \left[j\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} - (j-1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} \right] \\
&+ \rho\theta \left[(j+1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j+1)} - 2j\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} + (j-1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} \right] \\
&+ \frac{1}{2} \Bigg[j(j+1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+1)} - j(3j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} \\
&\quad + (j-1)(3j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} - (j-2)(j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \Bigg] \\
&+ \frac{\theta}{2} \Bigg[(j+1)(j+2) \sum_{k=0}^{i-2} (k+1) u_{(i-2)(j+2)} - (j+1)(4j+2) \sum_{k=0}^{i-1} (k+1) u_{(i-2-k)(j+1)}
\end{aligned}$$

$$+6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}-(j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}$$
$$\left.\left.+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\right]+\kappa(j-1)u_{i(j-1)}\right\}. \tag{3.127}$$

For the GARCH model,

$$u_{ij}(s,y)=e^{ay+bs-j\kappa s}\int_0^s \mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\frac{1}{\sqrt{2\pi\theta(s-t)}}\exp\left[-\frac{(x-y)^2}{2\theta(s-t)}-ax-bt+j\kappa t\right]$$
$$\times\left\{\frac{1}{2}\left(\partial_x^2-\partial_x\right)\sum_{k=0}^{j-1}u_{i(j-1-k)}\right.$$
$$+\rho\left[\sum_{k=0}^{j+1}\sum_{l=0}^{j-k+1}\sum_{m=0}^{i-1}\frac{3(j-k-l+1)(2k-5)!!}{(2k)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{k-2}\frac{k^{(l)}}{l!}\partial_x u_{(i-1-m)(j-k-l+1)}\right.$$
$$+\sum_{k=0}^{j}\sum_{l=0}^{j-k}\sum_{m=0}^{i-1}\frac{6(j-k-l)(2k-5)!!}{(2k)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{k-2}\frac{k^{(l)}}{l!}\partial_x u_{(i-1-m)(j-k-l)}$$
$$\left.+\sum_{k=0}^{j-1}\sum_{l=0}^{j-k-1}\sum_{m=0}^{i-1}\frac{3(j-k-l-1)(2k-5)!!}{(2k)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{k-2}\frac{k^{(l)}}{l!}\partial_x u_{(i-1-m)(j-k-l-1)}\right]$$
$$-\frac{1}{2}\left[j(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}-2(j-1)^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}\right.$$
$$\left.+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\right]+\theta\left[j(j+1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}\right.$$
$$-j(3j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}+(j-1)(3j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}$$
$$\left.-(j-2)(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\right]$$
$$+\frac{\theta^2}{2}\left[(j+1)(j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+2)}-(j+1)(4j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}\right.$$
$$+6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}-(j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}$$
$$\left.\left.+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\right]+\kappa(j-1)u_{i(j-1)}\right\}. \tag{3.128}$$

For the 3/2 model,

$$u_{ij}(s,y)=e^{ay+bs-j\kappa\theta s}\int_0^s \mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\frac{1}{\sqrt{2\pi\theta(s-t)}}\exp\left[-\frac{(x-y)^2}{2\theta(s-t)}-ax-bt+j\kappa\theta t\right]$$
$$\times\left\{\frac{1}{2}\left(\partial_x^2-\partial_x\right)\sum_{k=0}^{j-1}u_{i(j-1-k)}+\rho(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}\right.$$

$$
\begin{aligned}
&+ 2\rho\theta \left[j\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} - (j-1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} \right] \\
&+ \theta^2 \left[(j+1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j+1)} - 2j\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} + (j-1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} \right] \\
&+ \frac{1}{2} \left[(j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} - (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \right] \\
&+ \frac{3\theta}{2} \left[j(j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} - 2(j-1)^2 \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} \right. \\
&\quad \left. + (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \right] + \frac{3\theta^2}{2} \left[-j(j+1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+1)} \right. \\
&\quad - j(3j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} + (j-1)(3j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} \\
&\quad \left. - (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \right] \\
&+ \frac{\theta^3}{2} \left[(j+1)(j+2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+2)} - (j+1)(4j+2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+1)} \right. \\
&\quad + 6j^2 \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} - (j-1)(4j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} \\
&\quad \left. + (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \right] - \kappa(j-1) u_{i(j-1)} + \kappa\theta(j-1) u_{i(j-1)} \Bigg\}.
\end{aligned} \tag{3.129}
$$

Proof. From the model definition (3.1), the Black−Scholes model can be recovered by setting $\eta = 0$ and $v = \theta$. Thus, the solution can be written as

$$
u = \sum_{\substack{i=0 \\ j=0}}^{\infty} u_{ij}(\theta) \left(\frac{\eta}{1+\eta} \right)^i \left(\frac{v-\theta}{1+v-\theta} \right)^j, \tag{3.130a}
$$

$$
u_{00} = \frac{1}{2} e^x \operatorname{erfc}\left[-\frac{x + (1-a)\theta t}{\sqrt{2\theta t}} \right] - \frac{1}{2} e^{-rt} \operatorname{erfc}\left[-\frac{x - a\theta t}{\sqrt{2\theta t}} \right]. \tag{3.130b}
$$

In the above expansion, v-derivatives in (3.2) act only on the power terms

$$
Y^j = \left(\frac{v-\theta}{1+v-\theta} \right)^j, \tag{3.131}
$$

which can be obtained by replacing x with $v - \theta$ in Lemma 3.3.1.

After decomposing all v-terms to Y-terms in (3.2) and shifting the starting value of indexes i and j, the PDE (3.2) can be written in terms of $X^i Y^j$. For the Heston model, (3.2) becomes

$$
\begin{aligned}
&\sum_{\substack{i=0\\j=0}}^{\infty}\mathcal{B}_\theta u_{ij}X^iY^j-\sum_{\substack{i=0\\j=1}}^{\infty}\frac{1}{2}\left(\partial_x^2-\partial_x\right)\sum_{k=0}^{j-1}u_{i(j-1-k)}X^iY^j-\rho\left[\sum_{\substack{i=1\\j=1}}^{\infty}j\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)j}X^iY^j\right.\\
&\quad\left.-\sum_{\substack{i=1\\j=2}}^{\infty}(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}X^iY^j\right]-\rho\theta\left[\sum_{\substack{i=1\\j=0}}^{\infty}(j+1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j+1)}X^iY^j\right.\\
&\qquad\left.-\sum_{\substack{i=1\\j=1}}^{\infty}2j\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)j}X^iY^j+\sum_{\substack{i=1\\j=2}}^{\infty}(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}X^iY^j\right]\\
&-\frac{1}{2}\left[\sum_{\substack{i=2\\j=1}}^{\infty}j(j+1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}X^iY^j-\sum_{\substack{i=2\\j=1}}^{\infty}j(3j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}X^iY^j\right.\\
&\qquad+\sum_{\substack{i=2\\j=2}}^{\infty}(j-1)(3j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}X^iY^j\\
&\qquad\left.-\sum_{\substack{i=2\\j=3}}^{\infty}(j-2)(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}X^iY^j\right]\\
&\qquad-\frac{\theta}{2}\left[\sum_{\substack{i=2\\j=0}}^{\infty}(j+1)(j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2)(j+2)}X^iY^j\right.\\
&\quad-\sum_{\substack{i=2\\j=0}}^{\infty}(j+1)(4j+2)\sum_{k=0}^{i-1}(k+1)u_{(i-2-k)(j+1)}X^iY^j+\sum_{\substack{i=2\\j=1}}^{\infty}6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}X^iY^j\\
&\qquad-\sum_{\substack{i=2\\j=2}}^{\infty}(j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}X^iY^j\\
&\qquad\left.+\sum_{\substack{i=2\\j=3}}^{\infty}(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}X^iY^j\right]\\
&\qquad\qquad+\kappa\left[\sum_{\substack{i=0\\j=1}}^{\infty}ju_{ij}X^iY^j-\sum_{\substack{i=0\\j=2}}^{\infty}(j-1)u_{i(j-1)}X^iY^j\right]=0.
\end{aligned}
\tag{3.132}
$$

For the GARCH model, there is a particular term that needs special attention. Because

$$
Y=\frac{v-\theta}{1+v-\theta},
\tag{3.133}
$$

then

$$
v^{\frac{3}{2}}=\left(\frac{Y}{1-Y}+\theta\right)^{\frac{3}{2}}=\sum_{i=0}^{\infty}Y^i\mathcal{D}_Y^i\left(\frac{Y}{1-Y}+\theta\right)^{\frac{3}{2}}.
\tag{3.134}
$$

According to the previous lemma,

$$\partial_v \sum_{j=0}^{\infty} u_j Y^j = \sum_{j=0}^{\infty} j u_j \left(Y^{j-1} - 2Y^j + Y^{j+1}\right). \tag{3.135}$$

Therefore,

$$\begin{aligned}
v^{\frac{3}{2}} \partial_v \sum_{j=0}^{\infty} u_j Y^j &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} j u_j \mathcal{D}_Y^i \left(\frac{Y}{1-Y} + \theta\right)^{\frac{3}{2}} \left(Y^{j+i-1} - 2Y^{j+i} + Y^{j+i+1}\right) \\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{j} (j-i+1) u_{j-i+1} \mathcal{D}_Y^i \left(\frac{Y}{1-Y} + \theta\right)^{\frac{3}{2}} Y^j \\
&\quad - 2\sum_{j=1}^{\infty}\sum_{i=0}^{j-1} (j-i) u_{j-i} \mathcal{D}_Y^i \left(\frac{Y}{1-Y} + \theta\right)^{\frac{3}{2}} Y^j \\
&\quad + \sum_{j=2}^{\infty}\sum_{i=0}^{j-2} (j-i-1) u_{j-i-1} \mathcal{D}_Y^i \left(\frac{Y}{1-Y} + \theta\right)^{\frac{3}{2}} Y^j.
\end{aligned} \tag{3.136}$$

Then, (3.2) becomes

$$\begin{aligned}
&\sum_{\substack{i=0\\j=0}}^{\infty} \mathcal{B}_\theta u_{ij} X^i Y^j - \sum_{\substack{i=0\\j=1}}^{\infty} \frac{1}{2}\left(\partial_x^2 - \partial_x\right) \sum_{k=0}^{j-1} u_{i(j-1-k)} X^i Y^j \\
&\quad - \rho \Bigg[\sum_{\substack{i=1\\j=0}}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{i-1} (j-k+1)\partial_x u_{(i-1-l)(j-k+1)} \mathcal{D}_Y^k \left(\frac{Y}{1-Y} + \theta\right)^{\frac{3}{2}} X^i Y^j \\
&\qquad -2 \sum_{\substack{i=1\\j=1}}^{\infty}\sum_{k=0}^{j-1}\sum_{l=0}^{i-1} (j-k)\partial_x u_{(i-1-l)(j-k)} \mathcal{D}_Y^k \left(\frac{Y}{1-Y} + \theta\right)^{\frac{3}{2}} X^i Y^j \\
&\qquad + \sum_{\substack{i=1\\j=2}}^{\infty}\sum_{k=0}^{j-2}\sum_{l=0}^{i-1} (j-k-1)\partial_x u_{(i-1-l)(j-k-1)} \mathcal{D}_Y^k \left(\frac{Y}{1-Y} + \theta\right)^{\frac{3}{2}} X^i Y^j \Bigg] \\
&- \frac{1}{2} \Bigg[\sum_{\substack{i=2\\j=2}}^{\infty} j(j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} X^i Y^j - \sum_{\substack{i=2\\j=2}}^{\infty} 2(j-1)^2 \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} X^i Y^j \\
&\qquad + \sum_{\substack{i=2\\j=3}}^{\infty} (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} X^i Y^j \Bigg] \\
&- \theta \Bigg[\sum_{\substack{i=2\\j=1}}^{\infty} j(j+1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+1)} X^i Y^j - \sum_{\substack{i=2\\j=1}}^{\infty} j(3j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} X^i Y^j \\
&\qquad + \sum_{\substack{i=2\\j=2}}^{\infty} (j-1)(3j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} X^i Y^j
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{i=2\\j=3}}^{\infty} (j-2)(j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} X^i Y^j \Bigg] \\
&- \frac{\theta^2}{2} \Bigg[\sum_{\substack{i=2\\j=0}}^{\infty} (j+1)(j+2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+2)} X^i Y^j \\
&- \sum_{\substack{i=2\\j=0}}^{\infty} (j+1)(4j+2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+1)} X^i Y^j + \sum_{\substack{i=2\\j=1}}^{\infty} 6j^2 \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} X^i Y^j \\
&- \sum_{\substack{i=2\\j=2}}^{\infty} (j-1)(4j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} X^i Y^j \\
&+ \sum_{\substack{i=2\\j=3}}^{\infty} (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} X^i Y^j \Bigg] \\
&+ \kappa \Bigg[\sum_{\substack{i=0\\j=1}}^{\infty} j u_{ij} X^i Y^j - \sum_{\substack{i=0\\j=2}}^{\infty} (j-1) u_{i(j-1)} X^i Y^j \Bigg] = 0. \quad (3.137)
\end{aligned}
$$

For the 3/2 model, (3.2) becomes

$$
\begin{aligned}
&\sum_{\substack{i=0\\j=0}}^{\infty} \mathcal{B}_\theta u_{ij} X^i Y^j - \sum_{\substack{i=0\\j=1}}^{\infty} \frac{1}{2} \left(\partial_x^2 - \partial_x \right) \sum_{k=0}^{j-1} u_{i(j-1-k)} X^i Y^j \\
&- \rho \sum_{\substack{i=1\\j=2}}^{\infty} (j-1) \partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} X^i Y^j - 2\rho\theta \Bigg[\sum_{\substack{i=1\\j=1}}^{\infty} j \partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} X^i Y^j \\
&- \sum_{\substack{i=1\\j=2}}^{\infty} (j-1) \partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} X^i Y^j \Bigg] - \rho\theta^2 \Bigg[\sum_{\substack{i=1\\j=0}}^{\infty} (j+1) \partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j+1)} X^i Y^j \\
&- \sum_{\substack{i=1\\j=1}}^{\infty} 2j \partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} X^i Y^j + \sum_{\substack{i=1\\j=2}}^{\infty} (j-1) \partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} X^i Y^j \Bigg] \\
&- \frac{1}{2} \Bigg[\sum_{\substack{i=2\\j=3}}^{\infty} (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} X^i Y^j \\
&- \sum_{\substack{i=2\\j=3}}^{\infty} (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} X^i Y^j \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3\theta}{2}\left[\sum_{\substack{i=2\\j=2}}^{\infty} j(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}X^iY^j\right.\\
&\quad-\sum_{\substack{i=2\\j=2}}^{\infty} 2(j-1)^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}X^iY^j\\
&\quad\left.+\sum_{\substack{i=2\\j=3}}^{\infty}(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}X^iY^j\right]\\
&-\frac{3\theta^2}{2}\left[\sum_{\substack{i=2\\j=1}}^{\infty} j(j+1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}X^iY^j\right.\\
&\quad-\sum_{\substack{i=2\\j=1}}^{\infty} j(3j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}X^iY^j\\
&\quad+\sum_{\substack{i=2\\j=2}}^{\infty}(j-1)(3j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}X^iY^j\\
&\quad\left.-\sum_{\substack{i=2\\j=3}}^{\infty}(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}X^iY^j\right]\\
&-\frac{\theta^3}{2}\left[\sum_{\substack{i=2\\j=0}}^{\infty}(j+1)(j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+2)}X^iY^j\right.\\
&-\sum_{\substack{i=2\\j=0}}^{\infty}(j+1)(4j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}X^iY^j+\sum_{\substack{i=2\\j=1}}^{\infty}6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}X^iY^j\\
&\quad-\sum_{\substack{i=2\\j=2}}^{\infty}(j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}X^iY^j\\
&\quad\left.+\sum_{\substack{i=2\\j=3}}^{\infty}(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}X^iY^j\right]+\kappa\sum_{\substack{i=0\\j=2}}^{\infty}(j-1)u_{i(j-1)}X^iY^j\\
&\quad+\kappa\theta\left[\sum_{\substack{i=0\\j=1}}^{\infty} j u_{ij}X^iY^j-\sum_{\substack{i=0\\j=2}}^{\infty}(j-1)u_{i(j-1)}X^iY^j\right]=0. \qquad (3.138)
\end{aligned}
$$

The above equations hold if the coefficients of X^iY^j satisfy the following inhomogeneous Black−Scholes equations:

$$\begin{aligned}
\mathcal{B}_\theta u_{ij} + j\kappa u_{ij} = \frac{1}{2}\left(\partial_x^2 - \partial_x\right)\sum_{k=0}^{j-1} u_{i(j-1-k)} \\
+ \rho\left[j\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} - (j-1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)}\right] \\
+ \rho\theta\left[(j+1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j+1)} - 2j\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} + (j-1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)}\right] \\
+ \frac{1}{2}\Bigg[j(j+1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)} - j(3j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j} \\
+(j-1)(3j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)} - (j-2)(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg] \\
+ \frac{\theta}{2}\Bigg[(j+1)(j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2)(j+2)} - (j+1)(4j+2)\sum_{k=0}^{i-1}(k+1)u_{(i-2-k)(j+1)} \\
+6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j} - (j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)} \\
+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg] + \kappa(j-1)u_{i(j-1)} \qquad (3.139)
\end{aligned}$$

for the Heston model,

$$\begin{aligned}
\mathcal{B}_\theta u_{ij} + \kappa j u_{ij} = \frac{1}{2}\left(\partial_x^2 - \partial_x\right)\sum_{k=0}^{j-1} u_{i(j-1-k)} \\
+ \rho\Bigg[\sum_{k=0}^{j+1}\sum_{l=0}^{j-k+1}\sum_{m=0}^{i-1} \frac{3(j-k-l+1)(2k-5)!!}{(2k)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{k-2}\frac{k^{(l)}}{l!}\partial_x u_{(i-1-m)(j-k-l+1)} \\
+\sum_{k=0}^{j}\sum_{l=0}^{j-k}\sum_{m=0}^{i-1} \frac{6(j-k-l)(2k-5)!!}{(2k)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{k-2}\frac{k^{(l)}}{l!}\partial_x u_{(i-1-m)(j-k-l)} \\
+\sum_{k=0}^{j-1}\sum_{l=0}^{j-k-1}\sum_{m=0}^{i-1} \frac{3(j-k-l-1)(2k-5)!!}{(2k)!!\sqrt{\theta}}\left(-\frac{1}{\theta}\right)^{k-2}\frac{k^{(l)}}{l!}\partial_x u_{(i-1-m)(j-k-l-1)}\Bigg] \\
- \frac{1}{2}\Bigg[j(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j} - 2(j-1)^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)} \\
+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg] + \theta\Bigg[j(j+1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)} \\
-j(3j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j} + (j-1)(3j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)} \\
-(j-2)(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\theta^2}{2}\Bigg[(j+1)(j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+2)}-(j+1)(4j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}\\
&\qquad+6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}-(j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}\\
&\qquad\qquad+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]+\kappa(j-1)u_{i(j-1)} \qquad (3.140)
\end{aligned}
$$

for the GARCH model and

$$
\begin{aligned}
\mathcal{B}_\theta u_{ij}+\kappa\theta j u_{ij} &= \frac{1}{2}\left(\partial_x^2-\partial_x\right)\sum_{k=0}^{j-1}u_{i(j-1-k)}+\rho(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}\\
&\quad+2\rho\theta\left[j\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)j}-(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}\right]\\
&+\theta^2\left[(j+1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j+1)}-2j\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)j}+(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}\right]\\
&+\frac{1}{2}\left[(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}-(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\right]\\
&\quad+\frac{3\theta}{2}\Bigg[j(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}-2(j-1)^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}\\
&+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]+\frac{3\theta^2}{2}\Bigg[-j(j+1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}\\
&\quad-j(3j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}+(j-1)(3j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}\\
&\qquad-(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]\\
&+\frac{\theta^3}{2}\Bigg[(j+1)(j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+2)}-(j+1)(4j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}\\
&\qquad+6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}-(j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}\\
&+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]-\kappa(j-1)u_{i(j-1)}+\kappa\theta(j-1)u_{i(j-1)} \qquad (3.141)
\end{aligned}
$$

for the 3/2 model, with the initial condition (3.18) and the terms with negative indexes vanish by definition. For all models, u_{00} satisfies the homogeneous Black−Scholes equation which can be solved by Lemma 2.1.2. Higher order terms satisfy the inhomogeneous Black−Scholes equation which can be solved by Lemma 2.1.3. □

Because the method is an extension of the (η, v)-expansion, the calculation follows Order

3.2.2. The coefficients u_{ij} also have the same form as in Proposition 3.2.4, by which the following corollary can be proven.

Corollary 3.3.3. *In Proposition 3.3.2, the expansion coefficients*

$$u_{ij} \in \Sigma_1', \quad \text{for } i+j>0. \tag{3.142}$$

3.4 Symmetry breaking

Symmetry (invariance) is an important tool for analysing the structure of an object. The solution of PDE (3.2) exhibits scale invariance. For model definition (3.1), the option price is invariant under the following scaling.

Proposition 3.4.1. *The European option price under the Heston model is invariant under the scaling*

$$C(t,x,r,v,\theta,\rho,\eta,\kappa) = C\left(\frac{t}{\lambda}, x, \lambda r, \lambda v, \lambda\theta, \rho, \lambda\eta, \lambda\kappa\right), \quad \text{for } \lambda \in (0,\infty). \tag{3.143}$$

Proposition 3.4.2. *The European option price under the GARCH model is invariant under the scaling*

$$C(t,x,r,v,\theta,\rho,\eta,\kappa) = C\left(\frac{t}{\lambda}, x, \lambda r, \lambda v, \lambda\theta, \rho, \sqrt{\lambda}\eta, \lambda\kappa\right), \quad \text{for } \lambda \in (0,\infty). \tag{3.144}$$

Proposition 3.4.3. *The vanilla option price under the 3/2 model is invariant under the scaling*

$$C(t,x,r,v,\theta,\rho,\eta,\kappa) = C\left(\frac{t}{\lambda}, x, \lambda r, \lambda v, \lambda\theta, \rho, \eta, \kappa\right), \quad \text{for } \lambda \in (0,\infty). \tag{3.145}$$

The propositions can be proven by substituting the scaled parameters in the PDE (3.2) and the boundary conditions (3.3). Up to an overall constant, the PDE and boundary conditions stay the same.

For the four expansion methods introduced in Section 3.2, the expansion coefficients $u_{ij}\kappa^i\eta^j$ are scale-invariant. For example, in (κ,η)-expansion, as we have shown in (3.26a)

$$u_{10}\kappa = \frac{1}{2} w v^\alpha t^2 (\theta - v)\kappa, \tag{3.146a}$$

$$w = \frac{1}{2\sqrt{2\pi v t}} \exp\left(-\frac{x^2}{2vt} + ax + bt\right). \tag{3.146b}$$

Because vt and r/v are scale-invariant, so are a, bt and w consequently. For the Heston and GARCH models, $v^\alpha = 1 \to 1$ and $\kappa \to \lambda\kappa$. For the 3/2 model, $v^\alpha = v \to \lambda v$ and $\kappa \to \kappa$. Therefore, in both cases, $u_{10}\kappa$ and higher order terms are scale-invariant.

However, for the $(\eta+1, v+1)$-expansion

$$V = \sum_{i,j=0}^{\infty} u_{ij} \left(\frac{\eta}{1+\eta}\right)^i \left(\frac{v-\theta}{1+v-\theta}\right)^j, \tag{3.147}$$

the scale-invariance is broken, because the scaling effect cannot be expressed by an overall constant. For the Heston and GARCH models,

$$\frac{\lambda\eta}{1+\lambda\eta} \neq \lambda' \frac{\eta}{1+\eta}. \tag{3.148}$$

For all models,

$$\frac{\lambda v - \lambda\theta}{1+\lambda v - \lambda\theta} \neq \lambda' \frac{v-\theta}{1+v-\theta}. \tag{3.149}$$

The breaking of scale invariance gives us one more degree of freedom to fine-tune the convergence of the finite-term approximation

$$V = \sum_{i+j=0}^{N} u_{ij} \left(\frac{\eta}{1+\eta}\right)^i \left(\frac{v-\theta}{1+v-\theta}\right)^j \tag{3.150}$$

because the convergence is different when we choose different λ-value.

The fine-tuning, however, does not work for the terms

$$u_{i0} \left(\frac{\eta}{1+\eta}\right)^i \tag{3.151}$$

in the 3/2 model because η does not change under scaling. In this case, there is another degree of freedom we can use. Instead of scaling η, we scale '1' in the basis function

$$\frac{\eta}{1+\eta} \to \frac{\eta}{\gamma+\eta}. \tag{3.152}$$

Because γ is not a model parameter but a pure gauge choice, the option price stays the same as we change 1 to γ. Such symmetry is also called gauge invariance. Similarly, the symmetry is broken for a finite-term approximation and can be used for convergence fine-tuning.

3.4.1 $(\eta+\gamma, v+1)$-expansion

With the additional parameter, we can also expand the option price as follows.

Proposition 3.4.4. *The solution of* (3.2) *with initial condition* (3.3) *can be written as*

$$u(t,x) = \sum_{\substack{i=0\\ j=0}}^{\infty} u_{ij}(t,x,\theta) X^i Y^j, \tag{3.153}$$

with

$$X = \frac{\eta}{\gamma+\eta}, \quad Y = \frac{v-\theta}{1+v-\theta}, \quad a = \frac{1}{2} - \frac{r}{\theta}, \quad b = -\frac{\theta}{2}\left(\frac{1}{2}+\frac{r}{\theta}\right)^2, \tag{3.154}$$

$$u_{00}(t,x) = \frac{1}{2} e^x \operatorname{erfc}\left[-\frac{x+(1-a)\theta t}{\sqrt{2\theta t}}\right] - \frac{1}{2} e^{-rt} \operatorname{erfc}\left[-\frac{x - a\theta t}{\sqrt{2\theta t}}\right]. \tag{3.155}$$

For the Heston model,

$$u_{ij}(s,y) = e^{ay+bs-j\kappa s} \int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left[-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt + j\kappa t\right]$$

$$
\times \left\{ \frac{1}{2}\left(\partial_x^2 - \partial_x\right) \sum_{k=0}^{j-1} u_{i(j-1-k)} + \rho\gamma \left[j\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} - (j-1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} \right] \right.
$$
$$
+ \rho\theta\gamma \left[(j+1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j+1)} - 2j\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)j} + (j-1)\partial_x \sum_{k=0}^{i-1} u_{(i-1-k)(j-1)} \right]
$$
$$
+ \frac{\gamma^2}{2} \left[j(j+1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+1)} - j(3j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} \right.
$$
$$
\left. + (j-1)(3j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} - (j-2)(j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \right]
$$
$$
+ \frac{\theta\gamma^2}{2} \left[(j+1)(j+2) \sum_{k=0}^{i-2} (k+1) u_{(i-2)(j+2)} - (j+1)(4j+2) \sum_{k=0}^{i-1} (k+1) u_{(i-2-k)(j+1)} \right.
$$
$$
+ 6j^2 \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} - (j-1)(4j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)}
$$
$$
\left. \left. + (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \right] + \kappa(j-1) u_{i(j-1)} \right\}. \tag{3.156}
$$

For the GARCH model,

$$
u_{ij}(s,y) = e^{ay+bs-j\kappa s} \int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{\sqrt{2\pi\theta(s-t)}} \exp\left[-\frac{(x-y)^2}{2\theta(s-t)} - ax - bt + j\kappa t \right]
$$
$$
\times \left\{ \frac{1}{2}\left(\partial_x^2 - \partial_x\right) \sum_{k=0}^{j-1} u_{i(j-1-k)} \right.
$$
$$
+ \rho\gamma \left[\sum_{k=0}^{j+1} \sum_{l=0}^{j-k+1} \sum_{m=0}^{i-1} \frac{3(j-k-l+1)(2k-5)!!}{(2k)!!\sqrt{\theta}} \left(-\frac{1}{\theta}\right)^{k-2} \frac{k^{(l)}}{l!} \partial_x u_{(i-1-m)(j-k-l+1)} \right.
$$
$$
+ \sum_{k=0}^{j} \sum_{l=0}^{j-k} \sum_{m=0}^{i-1} \frac{6(j-k-l)(2k-5)!!}{(2k)!!\sqrt{\theta}} \left(-\frac{1}{\theta}\right)^{k-2} \frac{k^{(l)}}{l!} \partial_x u_{(i-1-m)(j-k-l)}
$$
$$
\left. + \sum_{k=0}^{j-1} \sum_{l=0}^{j-k-1} \sum_{m=0}^{i-1} \frac{3(j-k-l-1)(2k-5)!!}{(2k)!!\sqrt{\theta}} \left(-\frac{1}{\theta}\right)^{k-2} \frac{k^{(l)}}{l!} \partial_x u_{(i-1-m)(j-k-l-1)} \right]
$$
$$
- \frac{\gamma^2}{2} \left[j(j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} - 2(j-1)^2 \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)} \right.
$$
$$
\left. + (j-1)(j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \right] + \theta\gamma^2 \left[j(j+1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j+1)} \right.
$$
$$
- j(3j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)j} + (j-1)(3j-2) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-1)}
$$
$$
\left. - (j-2)(j-1) \sum_{k=0}^{i-2} (k+1) u_{(i-2-k)(j-2)} \right]
$$

$$
\begin{aligned}
&+\frac{\theta^2\gamma^2}{2}\Bigg[(j+1)(j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+2)}\\
&\quad-(j+1)(4j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}+6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}\\
&\quad-(j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]\\
&+\kappa(j-1)u_{i(j-1)}\Bigg\}. \qquad (3.157)
\end{aligned}
$$

For the 3/2 model,

$$
\begin{aligned}
&u_{ij}(s,y)=e^{ay+bs-j\kappa\theta s}\int_0^s \mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\frac{1}{\sqrt{2\pi\theta(s-t)}}\exp\left[-\frac{(x-y)^2}{2\theta(s-t)}-ax-bt+j\kappa\theta t\right]\\
&\times\Bigg\{\frac{1}{2}\left(\partial_x^2-\partial_x\right)\sum_{k=0}^{j-1}u_{i(j-1-k)}+\rho\gamma(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}+2\rho\theta\gamma\Bigg[j\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)j}\\
&\quad-(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}\Bigg]+\rho\theta^2\gamma\Bigg[(j+1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j+1)}\\
&\quad-2j\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)j}+(j-1)\partial_x\sum_{k=0}^{i-1}u_{(i-1-k)(j-1)}\Bigg]\\
&+\frac{\gamma^2}{2}\Bigg[(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}-(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]\\
&+\frac{3\theta\gamma^2}{2}\Bigg[j(j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}-2(j-1)^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}\\
&\quad+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]+\frac{3\theta^2\gamma^2}{2}\Bigg[-j(j+1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}\\
&\quad-j(3j-1)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}+(j-1)(3j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}\\
&\quad-(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]\\
&+\frac{\theta^3\gamma^2}{2}\Bigg[(j+1)(j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+2)}\\
&\quad-(j+1)(4j+2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j+1)}+6j^2\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)j}\\
&\quad-(j-1)(4j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-1)}+(j-1)(j-2)\sum_{k=0}^{i-2}(k+1)u_{(i-2-k)(j-2)}\Bigg]
\end{aligned}
$$

$$- \kappa(j-1)u_{i(j-1)} + \kappa\theta(j-1)u_{i(j-1)} \Bigg\}. \tag{3.158}$$

Proof. The proposition can be proven in the same way as in Proposition 3.3.2, with two identities:

$$\eta \left(\frac{\eta}{\gamma+\eta}\right)^i = \gamma \sum_{k=0}^{\infty} \left(\frac{\eta}{\gamma+\eta}\right)^{i+k+1}, \tag{3.159a}$$

$$\eta^2 \left(\frac{\eta}{\gamma+\eta}\right)^i = \gamma^2 \sum_{k=0}^{\infty}(k+1) \left(\frac{\eta}{\gamma+\eta}\right)^{i+k+2}. \tag{3.159b}$$

□

In this particular expansion, although the scaling of the parameter ($\eta \to \eta/\gamma$) and gauge transformation ($1 \to \gamma$) are equivalent

$$\frac{\eta}{\gamma+\eta} = \frac{\eta/\gamma}{1+\eta/\gamma}, \tag{3.160}$$

they are fundamentally different in nature. Scale invariance is a general property of the model and therefore is expansion-independent and is determined by model specifications, while gauge choice has no physical significance and is expansion-specific, with respect to $\eta/(\gamma+\eta)$. However, both of them can be used to achieve better convergence with finite-term approximation.

3.5 Numerical performance

3.5.1 Parameter and computational settings

In this section, option prices calculated using expansion methods are compared with those calculated using popular numerical methods. Due to its general applicability, prices obtained using Monte Carlo methods are the benchmark prices for every model. Options are priced by simulating 3×10^8 paths using Euler discretisation with daily time steps. The prices for the Heston and 3/2 models using the FFT are calculated as well [64, 65].

For the calculation in Table 3.3−3.5, we use the parameter estimates presented in Table 3.2. It should be noted that empirical estimates of these model parameters vary greatly in the extant literature, depending on the estimation method and data being used. The parameters are chosen in a reasonable range to demonstrate the validity of the expansion methods.

Model\Parameter	K	r	v	θ	κ	η	ρ
Heston model	1	0.04	0.05	0.04	6	0.2	−0.8
GARCH model	1	0.04	0.05	0.04	6	1	−0.8
3/2 model	1	0.04	0.05	0.04	60	2	−0.8

Table 3.2: Parameter values in numerical computations.

In each cell of Table 3.3−3.5, the option prices are calculated using the expansion methods introduced in this chapter, the Monte Carlo simulation and the FFT (only calculated

Time\Moneyness	0.4	0.6	0.8	1	1.2	1.4	1.6
1 week (κ,η)	0	0	0	0.0129	0.2008	0.4008	0.6008
(η,v)	0	0	0	0.0129	0.2008	0.4008	0.6008
(η,θ)	0	0	0	0.0129	0.2008	0.4008	0.6008
(κ,v)	0	0	-	-	-	0.4008	0.6008
$(\eta+1,v+1)$	0	0	0	0.0129	0.2008	0.4008	0.6008
MC	0	0	0	0.0129	0.2008	0.4008	0.6008
FFT	0	0	0	0.0129	0.2008	0.4008	0.6008
IV $(\eta+1,v+1)$	-	-	-	0.2223	0.2496	-	-
IV MC	-	-	-	0.2226	-	-	-
1 month (κ,η)	0	0	0	0.0268	0.2034	0.4033	0.6033
(η,v)	0	0	0	0.0268	0.2034	0.4033	0.6033
(η,θ)	0	0	0	0.0268	0.2034	0.4033	0.6033
(κ,v)	0	0	-	0.0289	0.1753	0.4272	0.6036
$(\eta+1,v+1)$	0	0	0	0.0268	0.2034	0.4033	0.6033
MC	0	0	0	0.0268	0.2034	0.4033	0.6033
FFT	0	0	0	0.0268	0.2034	0.4033	0.6033
IV $(\eta+1,v+1)$	-	-	0.1807	0.2186	0.2453	0.2644	0.2602
IV MC	-	-	0.1824	0.2188	0.2434	-	-
3 months (κ,η)	0	0	0.0003	0.0473	0.2126	0.4101	0.6100
(η,v)	0	0	0.0003	0.0473	0.2126	0.4101	0.6100
(η,θ)	0	0	0.0002	0.0472	0.2126	0.4100	0.6099
(κ,v)	0	0	0.0001	0.0474	0.2139	0.4104	0.6099
$(\eta+1,v+1)$	0	0	0.0003	0.0473	0.2126	0.4101	0.6100
MC	0	0	0.0003	0.0473	0.2126	0.4101	0.6100
FFT	0	0	0.0003	0.0473	0.2126	0.4101	0.6100
IV $(\eta+1,v+1)$	-	-	0.1835	0.2123	0.2336	0.2500	0.2634
IV MC	-	-	0.1836	0.2123	0.2336	0.2512	0.2785
1 year (κ,η)	0.0484	-	0.0062	0.0499	0.5251	0.4578	0.3955
(η,v)	0	0.0002	0.0153	0.1009	0.2569	0.4440	0.6405
(η,θ)	0	-	0.0156	0.1009	0.2576	0.4440	0.6401
(κ,v)	0	0.0001	0.0148	0.1009	0.2574	0.4443	0.6407
$(\eta+1,v+1)$	0	0.0002	0.0153	0.1009	0.2569	0.4440	0.6405
MC	0	0.0002	0.0153	0.1009	0.2569	0.4440	0.6405
FFT	0	0.0002	0.0153	0.1009	0.2569	0.4440	0.6405
IV $(\eta+1,v+1)$	0.1747	0.1762	0.1922	0.2044	0.2141	0.2222	0.2290
IV MC	0.1515	0.1762	0.1922	0.2044	0.2141	0.2221	0.2288

Table 3.3: Prices for the Heston model. The table shows the calculated call option prices and implied volatilities (IV) in terms of moneyness and time to maturity. The model parameters are shown in Table 3.2. '-' indicates a divergent result and '0' indicates a number with a magnitude of less than 1×10^{-4}.

for the Heston and 3/2 models) and implied volatilities are inverted from the prices of the $(\eta+1,v+1)$-expansion. Theoretically, more terms generate more accurate results. However, practically, the number of terms is limited by computational resources. In the

Time\Moneyness	0.4	0.6	0.8	1	1.2	1.4	1.6
1 week (κ,η)	0	0	0	0.0129	0.2008	0.4008	0.6008
(η,v)	0	0	0	0.0129	0.2008	0.4008	0.6008
(η,θ)	0	0	0	0.0129	0.2008	0.4008	0.6008
(κ,v)	0	0	-	-	-	0.4008	0.6008
$(\eta+1,v+1)$	0	0	0	0.0129	0.2008	0.4008	0.6008
MC	0	0	0	0.0129	0.2008	0.4008	0.6008
IV $(\eta+1,v+1)$	-	-	-	0.2221	0.2463	-	-
IV MC	-	-	-	0.2225	0.3294	0.5897	0.8120
1 month (κ,η)	0	0	0	0.0268	0.2035	0.4033	0.6033
(η,v)	0	0	0	0.0268	0.2035	0.4033	0.6033
(η,θ)	0	0	0	0.0268	0.2034	0.4033	0.6033
(κ,v)	0	-	-	-	-	-	0.6044
$(\eta+1,v+1)$	0	0	0	0.0268	0.2035	0.4033	0.6033
MC	0	0	0	0.0268	0.2034	0.4033	0.6033
IV $(\eta+1,v+1)$	-	-	0.1837	0.2182	0.2489	0.2605	0.2514
IV MC	-	-	0.1823	0.2184	0.2485	-	-
3 months (κ,η)	0	0	0.0003	0.0471	0.2127	0.4101	0.6100
(η,v)	0	0	0.0003	0.0471	0.2127	0.4101	0.6100
(η,θ)	0	0	0.0001	0.0473	0.2127	0.4100	0.6099
(κ,v)	0	0.0061	-	-	-	-	-
$(\eta+1,v+1)$	0	0	0.0003	0.0471	0.2127	0.4101	0.6100
MC	0	0	0.0003	0.0471	0.2127	0.4101	0.6100
IV $(\eta+1,v+1)$	-	-	0.1837	0.2115	0.2358	0.2553	0.2658
IV MC	-	-	0.1841	0.2115	0.2358	0.2558	0.2489
1 year (κ,η)	0.3717	-0.1084	0.0131	0.0597	0.4795	0.3930	0.4709
(η,v)	0	0.0002	0.0152	0.1007	0.2570	0.4441	0.6406
(η,θ)	-	-	0.0157	0.1014	0.2578	0.4440	0.6401
(κ,v)	0	-	0.0147	0.1019	0.2577	0.4438	0.6407
$(\eta+1,v+1)$	0	0.0002	0.0152	0.1007	0.2570	0.4441	0.6406
MC	0	0.0002	0.0152	0.1007	0.2569	0.4441	0.6406
IV $(\eta+1,v+1)$	-	0.1780	0.1920	0.2038	0.2142	0.2234	0.2315
IV MC	0.1628	0.1782	0.1919	0.2037	0.2141	0.2234	0.2317

Table 3.4: Prices for the GARCH model. The table shows the calculated call option prices and implied volatilities (IV) in terms of moneyness and time to maturity. The model parameters are shown in Table 3.2. '-' indicates a divergent result and '0' indicates a number with a magnitude of less than 1×10^{-4}.

calculations, the series is truncated up to 5th order terms

$$u = \sum_{i+j=0}^{5} u_{ij} X^i Y^j, \tag{3.161}$$

where X and Y are respective expansion parameters.

Time\Moneyness	0.4	0.6	0.8	1	1.2	1.4	1.6
1 week (κ, η)	0	0	0	0.0129	0.2008	0.4008	0.6008
(η, v)	0	0	0	0.0129	0.2008	0.4008	0.6008
(η, θ)	0	0	0	0.0129	0.2008	0.4008	0.6008
(κ, v)	0	0	0	0	0	0.4008	0.6008
$(\eta+1, v+1)$	0	0	0	0.0129	0.2008	0.4008	0.6008
MC	0	0	0	0.0129	0.2008	0.4008	0.6008
FFT	0	0	0	0.0129	0.2008	0.4008	0.6008
IV $(\eta+1, v+1)$	-	-	0.2491	0.2229	0.2349	-	-
IV MC	-	-	-	0.2230	-	-	-
1 month (κ, η)	0	0	0	0.0271	0.2034	0.4033	0.6033
(η, v)	0	0	0	0.0271	0.2034	0.4033	0.6033
(η, θ)	0	0	0	0.0271	0.2034	0.4033	0.6033
(κ, v)	0	0	0.0483	0.292	0.0997	0.4758	0.6041
$(\eta+1, v+1)$	0	0	0	0.0271	0.2034	0.4033	0.6033
MC	0	0	0	0.0271	0.2034	0.4033	0.6033
FFT	0	0	0	0.0271	0.2034	0.4033	0.6033
IV $(\eta+1, v+1)$	0.2072	0.2101	0.2033	0.2209	0.2344	0.2389	0.2348
IV MC	-	-	0.2041	0.2209	0.2344	-	-
3 months (κ, η)	0	0	0.0006	0.0481	0.2124	0.4100	0.6100
(η, v)	0	0	0.0006	0.0481	0.2124	0.4100	0.6100
(η, θ)	0	0	0.0005	0.0482	0.2124	0.4100	0.6100
(κ, v)	0	0.0001	0.0003	0.0503	0.2147	0.4105	0.6097
$(\eta+1, v+1)$	0	0	0.0006	0.0481	0.2124	0.4100	0.6100
MC	0	0	0.0006	0.0481	0.2124	0.4100	0.6099
FFT	0	0	0.0006	0.0481	0.2124	0.4100	0.6099
IV $(\eta+1, v+1)$	-	0.1858	0.2024	0.2167	0.2287	0.2374	0.2418
IV MC	-	0.1815	0.2019	0.2166	0.2293	0.2400	-
1 year (κ, η)	-0.0772	-0.2796	-0.2790	-0.0869	0.1732	0.4196	0.6338
(η, v)	0	0.0003	0.0166	0.1019	0.2570	0.4438	0.6404
(η, θ)	0	0.0001	0.0167	0.1021	0.2573	0.4438	0.6402
(κ, v)	0	0.0001	0.0152	0.1015	0.2576	0.4444	0.6407
$(\eta+1, v+1)$	0	0.0003	0.0168	0.1021	0.2570	0.4438	0.6403
MC	0	0.0003	0.0166	0.1019	0.2570	0.4438	0.6404
FFT	0	0.0003	0.0166	0.1019	0.2570	0.4438	0.6404
IV $(\eta+1, v+1)$	0.1813	0.1870	0.1989	0.2074	0.2144	0.2203	0.2252
IV MC	0.1708	0.1869	0.1980	0.2069	0.2144	0.2208	0.2270

Table 3.5: Prices for the 3/2 model. The table shows the calculated call option prices and implied volatilities (IV) in terms of moneyness and time to maturity. The model parameters are shown in Table 3.2. '-' indicates a divergent result and '0' indicates a number with a magnitude of less than 1×10^{-4}.

3.5.2 Accuracy of expansion methods

Unlike the model parameters of the Heston model, which are bounded by Feller's condition,

$$2\kappa\theta \leq \eta^2, \tag{3.162}$$

the parameters of the GARCH and 3/2 models can be arbitrarily large. As the convergence of the expansion methods is unproven, it is no surprise that some of the expansion methods fail to produce accurate results in some cases.

- (κ, η)-expansion

  As is evident for all three models, (κ, η)-expansion is extremely accurate for short-term options. The shorter the maturity, the more accurate the price is. However, when the maturity becomes longer than three months, the series fails to be convergent, possibly because (κ, η)-expansion starts with the Black−Scholes solution u_{00} with an initial variance v, which is very close to the average of stochastic variance process $\{v_t; t \geq 0\}$. Therefore only small corrections to u_{00} (i.e. a few higher-order terms) are needed. However, the difference between initial variance v and effective variance

  $$\bar{v} = \frac{1}{t} \int_0^t v_s \, \mathrm{d}s \tag{3.163}$$

  becomes bigger as the maturity increases. Therefore, v is no longer a good estimate for $\bar{v}$.

- (κ, v)-expansion

  Contrary to the previous case, this method works well with long-term options. With either large κ or large t, the method produces convergent results, though not as accurately as the other methods. The reason for large κ is obvious, because of the κ^{-i} factor in the expansion form. The convergence for long-term options can also be explained by effective variance. Due to mean-reversion, the variance process fluctuates around long-term variance θ. In a long enough time frame, the fluctuations cancel each other, making the effective variance stable and close to θ, which is the starting point of this expansion.

- (η, v) and (η, θ)-expansions

  These two expansions seem to work for both long-term options and short-term options, in all three models. The convergence rate of the series is generically dependent on the magnitude of η and $v - \theta$. For the parameter values in Table 3.2, the methods produce accurate results for all three models

- $(\eta + 1, v + 1)$ expansion

  This bounded basis expansion outperforms other expansion methods dominantly. For both long-term and short-term options with either large or small expansion parameters, this method produces results very close to the ones given by Monte Carlo methods. Detailed comparison with its unbounded version, (η, v)-expansion, will be discussed later.

In terms of implied volatility, because the prices produced by expansion and Monte Carlo methods are approximate values, they can be outside of the theoretical price bounds, which makes the implied volatility not applicable in those cases. However, in those cases when calculable, the implied volatilities are not constant across moneyness and maturity.

In general, $(\eta + 1, v + 1)$-expansion is preferable since it works in most cases. Especially for multi-year options, which are most challenging to obtain using Monte Carlo methods, very few terms are needed to obtain a convergent result, given that the parameters are reasonably large. For ultra short-term options, (κ, η)-expansion is more effective.

3.5.3 Numerical convergence

$i\backslash j$	0	1	2	3	4	5
0	9.925(−2)	1.585(−3)	−1.598(−5)	3.173(−7)	−7.888(−9)	2.194(−10)
1	4.240(−4)	−2.297(−5)	2.794(−7)	−6.886(−10)	−2.612(−10)	
2	−3.032(−4)	1.510(−6)	1.291(−7)	−1.099(−8)		
3	2.412(−7)	1.763(−7)	−3.223(−9)			
4	3.250(−6)	−1.070(−7)				
5	−9.217(−8)					

Table 3.6: Magnitude of $u_{ij}\eta^i(v-\theta)^j$ in (η, v)-expansion. Numbers in the table are denoted by $a(b) = a \times 10^b$.

To give a general picture of a convergent result, Table 3.6 shows the convergence of the (η, v)-expansion for the Heston model with $x = 0$ and $t = 1$. The higher order terms converge well:

$$|u_{ij}| < 1 \times 10^{-6}, \quad i + j = 6. \tag{3.164}$$

The small magnitude of higher order terms ensures that the sum is a good approximation for practical purposes. In Table 3.3, we can see that the value does agree with the reference prices produced using the Monte Carlo and the FFT.

Although results in some methods converge well for extreme parameters, such as in $(\eta + 1, v + 1)$-expansion, convergence is not proven theoretically. Therefore, expansion methods should be used with caution: it is always beneficial to check the solution matrix $u_{ij}X^iY^j$, such as Table 3.6, if higher order terms are small enough.

3.5.4 Unbounded vs. bounded series expansion

$\eta\backslash v$	0.04	0.24	0.44	0.64	0.84	method
0	0.0993	0.1265	0.1597	0.2738	0.6822	(η, v)
	0.0993	0.1263	0.1479	0.1671	0.1850	$(\eta+1, v+1)$
	0.0993	0.1263	0.1476	0.1658	0.1819	FFT
0.5	0.0985	0.1248	0.1531	0.2517	0.6241	(η, v)
	0.0986	0.1249	0.1459	0.1641	0.1808	$(\eta+1, v+1)$
	0.0985	0.1249	0.1458	0.1637	0.1794	FFT
1	0.0956	0.1197	0.1367	0.2009	0.4998	(η, v)
	0.0955	0.1216	0.1422	0.1599	0.1759	$(\eta+1, v+1)$
	0.0954	0.1212	0.1419	0.1596	0.1752	FFT
1.5	0.0936	0.1117	0.1078	0.1157	0.3001	(η, v)
	0.0918	0.1177	0.1382	0.1556	0.1712	$(\eta+1, v+1)$
	0.0915	0.1166	0.1370	0.1545	0.1700	FFT
2	0.0967	0.1003	0.0607	-0.0150	0.0084	(η, v)
	0.0881	0.1139	0.1344	0.1517	0.1670	$(\eta+1, v+1)$
	0.0876	0.1118	0.1318	0.1490	0.1643	FFT

Table 3.7: Prices of the Heston model calculated using (η, v), $(\eta+1, v+1)$ and FFT for extreme η and v values.

Table 3.7 compares (η, v)-expansion with its bounded counterpart, $(\eta + 1, v + 1)$. Both are truncated to 6th-order terms

$$\sum_{i+j=0}^{6} u_{ij} X^i Y^j, \tag{3.165}$$

where in (η, v)

$$X = \eta, \quad Y = v - \theta, \tag{3.166}$$

and in $(\eta + 1, v + 1)$

$$X = \frac{\eta}{1+\eta}, \quad Y = \frac{v-\theta}{1+v-\theta}. \tag{3.167}$$

Parameters η and v range from 0 to extremely large values. The other parameters are

$$x = 0, \quad t = 1, \quad r = 0.04, \quad \theta = 0.04, \quad \kappa = 6, \quad \rho = -0.8. \tag{3.168}$$

Though values like $\eta = 2$ make no financial sense because they violate Feller's condition, they are calculated to test the approximability of the methods from a mathematical point of view. The numbers show that $(\eta + 1, v + 1)$ produces much more accurate prices than (η, v). Even in the worst case of $\eta = 2$ and $v - \theta = 0.8$, $(\eta + 1, v + 1)$ produces a reasonable (about 3% error) approximation of the real value, while (η, v) produces completely unusable results. The reference prices are produced by the FFT, because for the Heston model, they are generally considered accurate. The result corroborates our hypothesis that bounded basis functions capture asymptotic behavior better than power series, which explode at infinity.

Method\\λ	0.001	0.01	0.1	1	10	100
(η, v)	0.124754	0.124754	0.124754	0.124754	0.124754	0.124754
$(\eta + 1, v + 1)$	0.124755	0.12476	0.124799	0.12492	0.123746	0.106067

Table 3.8: Breaking of scale invariance.

Table 3.8 shows the breaking of scale invariance for the Heston model numerically, with the expansion terms specified in (3.165), (3.166) and (3.167). The parameters used are

$$\begin{aligned} &x = 0, \quad t = \frac{1}{\lambda}, \quad r = 0.04\lambda, \quad v = 0.24\lambda, \\ &\theta = 0.04\lambda, \quad \eta = 0.5\lambda, \quad \kappa = 6\lambda, \quad \rho = -0.8. \end{aligned} \tag{3.169}$$

Because (η, v) is scale-invariant, the first row does not change over λ. However, the second row changes as λ scales, as described in (3.148) and (3.149). With the additional degree of freedom, the convergence can be fine-tuned. For example, when $\lambda = 100$, the series is not convergent, because it is far from the 'real' value. In this case, we can scale the parameter set to $\lambda = 0.001$. In the region of $\lambda < 1$, the price is stable as λ gets smaller. This signals that the series may have been convergent already. The same rule applies to the gauge parameter γ in (3.160).

3.6 Summary

In this chapter, we showed how expansion methods can be applied to European options under most popular stochastic volatility models. After the solution is written as a power series of parameters or variables, the pricing PDE is transformed to a set of solvable (in)homogeneous Black−Scholes equations, that relate neighbouring terms of the expansion terms. Given a specific order, the two-dimensional expansion matrix can be calculated up to an arbitrary order of terms.

With bounded basis functions, the expansion methods are able to approximate prices of extreme parameter values. Furthermore, scale invariance is broken in these expansion methods, which gives us one more tool with which to fine-tune the convergence of the result.

The numerical results show the validity of all the expansion methods with the affine Heston model and the non-affine GARCH and 3/2 models. However, $(\eta + 1, v + 1)$-expansion is shown to result in better convergence than the other expansion methods and is therefore recommended for practical use (code available online [59]).

4 American options under the Black–Scholes model

The holder of an American option has the right to exercise it at any time before expiration. However, such flexibility makes American option much harder to price than its European counterpart. Although nowadays most exchange-traded equity options are American, only a few computationally intensive numerical methods (e.g. the least-squares Monte Carlo method, tree methods and finite-difference methods) can be used to calculate their prices. The problem is even more difficult to overcome for advanced models (stochastic volatility and/or jumps), as none of the above methods produce satisfying results.

In this chapter, with the same series expansion logic demonstrated in the previous chapter, the American option will be examined from an analytical viewpoint.

4.1 Problem setting

Under the Black−Scholes model, price dynamics are governed by the SDE:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sqrt{v}S_t\,\mathrm{d}W_t, \tag{4.1}$$

where r is the interest rate and v is the variance.

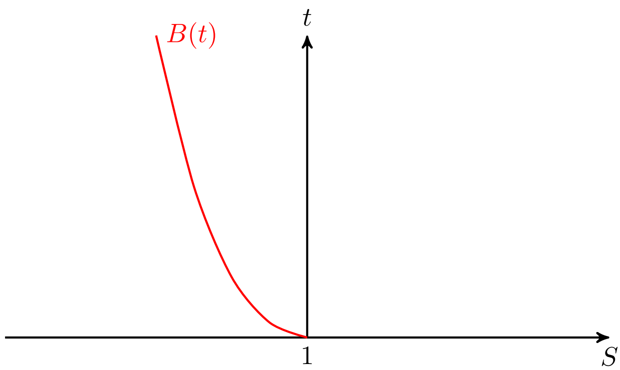

Figure 4.1: Moving exercise boundary of American put option.

Without dividends, an American call option should always be held rather than exercised, since the continuation value of an American call is greater than or equal to the immediate value of exercising it:

$$e^{-r(T-t)}\mathbb{E}\left[(S_T-K)^+\right] \geq e^{-r(T-t)}\left(\mathbb{E}[S_T]-K\right)^+$$
$$= \left(e^{-r(T-t)}\mathbb{E}\left[S_T\right]-e^{-r(T-t)}K\right)^+ = \left(S_t-e^{-r(T-t)}K\right)^+ \geq (S_t-K)^+. \quad (4.2)$$

Because x^+ is a convex function, the first inequality is held by Jensen's inequality. Since it is never optimal to exercise an American call, it has the same value as its European counterpart. However, with dividends, American call options should still be solved with early exercise boundary [66].

However, the last inequality of (4.2) does not hold for American put options. Depending on the current price, the holder of an American put option may exercise it at any time to sell the equity at the strike price. In order to account for the hold-exercise criterion, we introduce the exercise boundary $B(t)$ (Figure 4.1) below which the put option should be exercised. In the region above the boundary $B(t)$, using the same hedging argument as for European put options, we obtain the same PDE as that of European put options for $t > 0$ and $S > B(t)$

$$\left(\partial_t - \frac{\sigma^2}{2}S^2\partial_S^2 - rS\partial_S + r\right)V = 0, \quad (4.3)$$

with

$$V(0,S) = 0, \quad (4.4a)$$
$$V(t,B(t)) = 1 - B(t), \quad (4.4b)$$
$$\partial_S V(t,B(t)) = -1. \quad (4.4c)$$

Hereafter, the strike is set to 1 for simplicity, and the option prices of arbitrary strike K can be obtained by scaling

$$P_K(t,S) = KP_1(t,S/K). \quad (4.5)$$

The boundary conditions (4.4b) and (4.4c) at the exercise boundary $S = B(t)$ are called the *smooth pasting* conditions of American put options (Figure 4.2). The conditions reflect the fact that regardless of execution costs, there is no difference between exercise and hold when the price is at the exercise boundary exactly.

What makes the American option difficult to price is the fact that there are two coupled unknown functions $V(t,S)$ and $B(t)$ that must be solved simultaneously.

In order to reduce the irregular domain $(t,\infty)\times(B(t),\infty)$ to a regular one $(0,\infty)\times(0,\infty)$, the *front-fixing* technique [61] is required. If we change the independent variables (t,S) to (t,x) with

$$x = \ln S - d(t), \quad d(t) = \ln B(t), \quad (4.6)$$

then the derivatives in t and S change accordingly

$$\partial_t V \to \partial_t V - d'(t)\partial_x V, \quad (4.7a)$$
$$\partial_S V \to e^{-x}\partial_x V, \quad (4.7b)$$
$$\partial_S^2 V \to e^{-2x}\left(\partial_x^2 - \partial_x\right)V. \quad (4.7c)$$

Now, the PDE becomes, for $t > 0$ and $x > 0$,

$$\mathcal{B}V := \left[\partial_t - \frac{v}{2}(\partial_x^2 - \partial_x) - r(\partial_x - 1)\right]V = d(t)'\partial_x V, \quad (4.8)$$

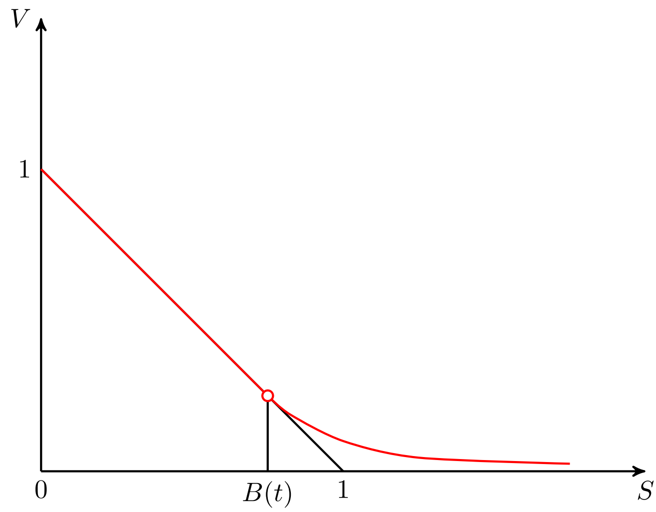

Figure 4.2: Smooth pasting around moving boundary.

with

$$V(0,x) = 0, \tag{4.9a}$$

$$V(t,0) = 1 - e^{d(t)}, \tag{4.9b}$$

$$\partial_x V(t,0) = -e^{d(t)}. \tag{4.9c}$$

The PDE (4.8) and boundary conditions (4.9) is the problem we are trying to solve.

4.2 Series expansion

In [57], Zhu pioneered the method of expanding the price V and moving boundary B of an American put option as an infinite series

$$V = \sum_{i=0}^{\infty} V_i, \quad B = \sum_{i=0}^{\infty} B_i. \tag{4.10}$$

Though he succeeded in obtaining the formula that relates the unknown terms V_i and B_i to previously calculated terms V_j and B_j, $j < i$, the formula (Equation (23) in [57]) involves double integrals that can only be evaluated numerically. Therefore, this method is hard to apply to high order terms.

Fortunately, there are many ways to perform the expansion (4.10). In order to ensure that V_i and B_i can be written explicitly, the leading terms V_0 and B_0 and iterative step

$$V_i = \mathcal{B}^{-1} f(V_{i-1}, V_{i-1}, \ldots, V_0) \tag{4.11}$$

should be carefully chosen so that every step is analytically computable.

The set of BSCE functions (Σ_2) is closed under many operations (see Section 2.2). Therefore it is an ideal basis for the expansion. $\mathcal{I}^{-1}$, L_i and $\mathcal{D}_p^i$ are defined in Notation 2.2.15, 2.2.16 and 2.2.17.

Proposition 4.2.1 (ABS-I)**.** *The solution of* (4.8) *with the initial and boundary conditions* (4.9) *can be written as*

$$V = \sum_{i=0}^{\infty} V_i(t,x), \quad d = \sum_{i=0}^{\infty} d_i(t), \tag{4.12}$$

with

$$V_0(t,x) = -xe^{ax+bt}\operatorname{erfc}\left(\frac{x}{\sqrt{2vt}}\right), \quad d_0 = bt, \quad R_0 = 0,$$
$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2, \tag{4.13}$$

then for $i \geq 1$*,*

$$V_i(t,x) = f_i + u_i + w_i, \tag{4.14a}$$
$$f_i(t,x) = e^{ax+bt}\operatorname{erfc}\left(\frac{x}{\sqrt{2vt}}\right)\mathcal{D}_p^{i+1}\left[(1-e^{xp})\,e^{-(ax+bt)p}\right], \tag{4.14b}$$
$$u_i(t,x) = -f_{i-1} + \mathcal{B}^{-1}\left(\sum_{j=0}^{i-1} d_j'\partial_x V_{i-1-j}\right), \tag{4.14c}$$
$$w_i(t,x) = \mathcal{I}^{-1}\left[-u_i(t,0) + \partial_x u_i(t,0) + (R_i - R_{i-1})e^{bt}\right], \tag{4.14d}$$
$$R_i(t,x) = L_i\left\{e^{-bt}[-u_i(t,0) + \partial_x u_i(t,0)] - R_{i-1}\right\}, \tag{4.14e}$$
$$d_i(t) = -e^{-bt}\partial_x V_i(t,0) - \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} d_j p^j\right). \tag{4.14f}$$

Proof. Define functions

$$\bar{V} := \sum_{i=0}^{\infty} p^i V_i = \sum_{i=0}^{\infty} p^i\left(f_i + u_i + w_i\right), \quad \bar{d} := \sum_{i=0}^{\infty} p^i d_i. \tag{4.15}$$

For $x > 0$,

$$\lim_{t\to 0^+}\operatorname{erfc}\left(\frac{x}{\sqrt{2vt}}\right) = 0, \tag{4.16}$$

therefore, for $i \geq 0$,

$$f_i(0,x) = 0. \tag{4.17}$$

By Lemma 2.2.10 and 2.1.4,

$$u_i(0,x) = 0, \quad w_i(0,x) = 0, \tag{4.18}$$

therefore,

$$\bar{V}(0,x) = \sum_{i=0}^{\infty} p^i\left(f_i + u_i + w_i\right) = 0. \tag{4.19}$$

And by Lemma 2.1.4,

$$w_i(t,0) - \partial_x w_i(t,0) = -u_i(t,0) + \partial_x u_i(t,0) + (R_i - R_{i-1}\mathbf{1}_{i>0})e^{bt}, \tag{4.20a}$$

$$f_i(t,0) - \partial_x f_i(t,0) = e^{bt}\frac{(-bt)^i}{i!}. \tag{4.20b}$$

therefore,

$$\begin{aligned}\bar{V}(t,0) - \partial_x \bar{V}(t,0) &= e^{bt}\sum_{i=0}^{\infty} p^i \frac{(-bt)^i}{i!} + e^{bt}\left[R_0 + \sum_{i=1}^{\infty} p^i (R_i - R_{i-1})\right] \\ &= e^{bt(1-p)} + e^{bt}(1-p)\sum_{i=0}^{\infty} p^i R_i. \end{aligned}\tag{4.21}$$

By Lemma 2.2.19,

$$d_i + \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right) = \mathcal{D}_p^i \exp\left(\sum_{j=1}^{\infty} p^j d_j\right), \tag{4.22}$$

therefore,

$$\partial_x \bar{V}(t,0) = -e^{bt}\sum_{i=0}^{\infty} p^i \mathcal{D}_p^i \exp\left(\sum_{j=1}^{\infty} p^j d_j\right) = -\exp\left(\sum_{j=0}^{\infty} p^j d_j\right) = -e^{\bar{d}}. \tag{4.23}$$

By definition,

$$\mathcal{B}u_i = \left(-\mathcal{B}f_{i-1} + \sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j}\right)\mathbf{1}_{i>0}, \tag{4.24a}$$

$$\mathcal{B}w_i = 0, \tag{4.24b}$$

therefore,

$$\begin{aligned}\mathcal{B}\bar{V} = \sum_{i=0}^{\infty} p^i \mathcal{B}(f_i + u_i + w_i) &= \sum_{i=0}^{\infty} p^i \mathcal{B}f_i + \sum_{i=1}^{\infty} p^i \left(-\mathcal{B}f_{i-1} + \sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j}\right) \\ &= (1-p)\sum_{i=0}^{\infty} p^i \mathcal{B}f_i + p\sum_{i=1}^{\infty} p^{i-1}\sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j} \\ &= (1-p)\sum_{i=0}^{\infty} p^i \mathcal{B}f_i + p\left(\sum_{i=0}^{\infty} p^i d'_i\right)\left(\sum_{j=0}^{\infty} p^j \partial_x V_j\right) \\ &= (1-p)\sum_{i=0}^{\infty} p^i \mathcal{B}f_i + p\bar{d}'\partial_x \bar{V}. \end{aligned}\tag{4.25}$$

According to (4.25), (4.19), (4.21) and (4.23), $\bar{V}$ and $\bar{d}$ are the solutions of

$$\mathcal{B}\bar{V} = (1-p)\sum_{i=0}^{\infty} p^i \mathcal{B}f_i + p\bar{d}'\partial_x \bar{V}, \tag{4.26}$$

with

$$\bar{V}(0,x) = 0, \tag{4.27a}$$

$$\bar{V}(0,x) - \partial_x \bar{V}(0,x) = e^{bt(1-p)} + e^{bt}(1-p)\sum_{i=0}^{\infty} p^i R_i, \tag{4.27b}$$

$$\partial_x \bar{V}(0,x) = -e^{\bar{d}}. \tag{4.27c}$$

By setting $p = 1$,

$$V = \sum_{i=0}^{\infty} V_i, \quad d = \sum_{i=0}^{\infty} d_i \tag{4.28}$$

are the solutions of

$$\mathcal{B}V = d' \partial_x V, \tag{4.29}$$

with

$$V(0,x) = 0, \tag{4.30a}$$

$$V(t,0) - \partial_x V(t,0) = 1, \tag{4.30b}$$

$$\partial_x V(t,0) = -e^{d}. \tag{4.30c}$$

□

As in the last chapter, the above proposition states that if it is convergent, the sum of expansion series $\sum_i V_i$ and $\sum_i d_i$ serves as the option price and log-boundary for American put options respectively. This expansion is computable because of the following corollary, which ensures that the expansion terms can be expressed in terms of BSCE functions (Definition 2.2.3) and BST functions (Definition 2.2.4):

Corollary 4.2.2. *For $i \geq 0$,*

$$V_i \in \Sigma_2, \quad d_i \in \Sigma_3. \tag{4.31}$$

Proof. According to the definitions,

$$V_0 = -xe^{ax+bt} \operatorname{erfc}\left(\frac{x}{\sqrt{2vt}}\right) \in \Sigma_2, \quad d_0 = bt \in \Sigma_3. \tag{4.32}$$

Consider V_i and d_i. Assume that for $0 \leq j \leq i-1$,

$$V_j \in \Sigma_2, \quad d_j \in \Sigma_3. \tag{4.33}$$

Because $d_i' \in \Sigma_3$ and $\partial_x V_i \in \Sigma_2$,

$$\sum_{j=0}^{i-1} d_j' \partial_x V_{i-1-j} \in \Sigma_2. \tag{4.34}$$

By Lemma 2.2.10,

$$u_i = -f_{i-1} + \mathcal{B}^{-1}\left(\sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j}\right) \in \Sigma_2. \tag{4.35}$$

Since $e^{-bt}[-u_i(t,0) + \partial_x u_i(t,0)] \in \Sigma_3$, according to Theorem 2.2.14

$$w_i \in \Sigma_2. \tag{4.36}$$

Therefore,

$$V_i = f_i + u_i + w_i \in \Sigma_2. \tag{4.37}$$

Since $e^{-bt}\partial_x V_i(t,0) \in \Sigma_3$ and by Lemma 2.2.19,

$$d_i = -e^{-bt}\partial_x V_i(t,0) - \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right) \in \Sigma_3 \tag{4.38}$$

By induction for $i \geq 0$,

$$V_i \in \Sigma_2, \quad d_i \in \Sigma_3. \tag{4.39}$$

□

One key difference between the expansion methods in this chapter and the last chapter is the expansion basis p, which serves as a purely formal structure to decompose the option price. In other words, the expansion coefficients V_i and d_i are not the derivatives of the option price with respect to any model parameter. As a result, the method of decomposition is not unique.

Proposition 4.2.3 (ABS-II)**.** *The solution of* (4.8) *with initial and boundary conditions* (4.9) *can be written as*

$$V = \sum_{i=0}^{\infty} V_i(t,x), \quad d = \sum_{i=0}^{\infty} d_i(t), \tag{4.40}$$

with

$$V_0(t,x) = 0, \quad d_0 = bt, \quad R_0 = -1, \quad a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2, \tag{4.41}$$

then for $i \geq 1$,

$$V_i(t,x) = u_i + w_i, \tag{4.42a}$$

$$u_i(t,x) = \mathcal{B}^{-1}\left(\sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j}\right), \tag{4.42b}$$

$$w_i(t,x) = \mathcal{I}^{-1}\left[e^{bt}\frac{(-bt)^i}{i!} - u_i(t,0) + \partial_x u_i(t,0) + (R_i - R_{i-1})e^{bt}\right], \tag{4.42c}$$

$$R_i(t,x) = L_i\left[\frac{(-bt)^i}{i!} + e^{-bt}[-u_i(t,0) + \partial_x u_i(t,0)] - R_{i-1}\right], \tag{4.42d}$$

$$d_i(t) = \frac{(-bt)^i}{i!} - e^{-bt}V_i(t,0) - \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} d_j p^j\right). \tag{4.42e}$$

Proof. Define functions

$$\bar{V} := \sum_{i=0}^{\infty} p^i V_i = \sum_{i=0}^{\infty} p^i \left(u_i + w_i\right), \quad \bar{d} := \sum_{i=0}^{\infty} p^i d_i. \tag{4.43}$$

By Lemma 2.2.10 and 2.1.4,

$$u_i(0,x) = 0, \quad w_i(0,x) = 0, \tag{4.44}$$

therefore,

$$\bar{V}(0,x) = \sum_{i=0}^{\infty} p^i \left(u_i + w_i\right) = 0. \tag{4.45}$$

And by Lemma 2.1.4,

$$w_i(t,0) - \partial_x w_i(t,0) = e^{bt}\frac{(-bt)^i}{i!} - u_i(t,0) + \partial_x u_i(t,0) + (R_i - R_{i-1}\mathbf{1}_{i>0})e^{bt}, \tag{4.46}$$

therefore,

$$\begin{aligned}\bar{V}(t,0) - \partial_x \bar{V}(t,0) &= e^{bt}\sum_{i=0}^{\infty} p^i \frac{(-bt)^i}{i!} + e^{bt}R_0 + e^{bt}\sum_{i=1}^{\infty} p^i \left(R_i - R_{i-1}\right) \\ &= e^{bt(1-p)} + e^{bt}(1-p)\sum_{i=0}^{\infty} p^i R_i. \end{aligned}\tag{4.47}$$

By Lemma 2.2.19,

$$d_i + \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right) = \mathcal{D}_p^i \exp\left(\sum_{j=1}^{\infty} p^j d_j\right), \tag{4.48}$$

therefore,

$$\begin{aligned}\bar{V}(t,0) &= e^{bt}\sum_{i=0}^{\infty}\frac{(-bt)^i}{i!}p^i - e^{bt}\sum_{i=0}^{\infty} p^i \mathcal{D}_p^i \exp\left(\sum_{j=1}^{\infty} p^j d_j\right) \\ &= e^{bt(1-p)} - e^{bt}\exp\left(\sum_{j=1}^{\infty} p^j d_j\right) \\ &= e^{bt(1-p)} - e^{\bar{d}}. \end{aligned}\tag{4.49}$$

By definition,

$$\mathcal{B}u_i = \left(\sum_{j=0}^{i-1} d_j' \partial_x V_{i-1-j}\right)\mathbf{1}_{i>0}, \tag{4.50a}$$

$$\mathcal{B}w_i = 0, \tag{4.50b}$$

therefore,

$$\mathcal{B}\bar{V} = \sum_{i=0}^{\infty} p^i \mathcal{B}(u_i + w_i) = \sum_{i=1}^{\infty} p^i \left(\sum_{j=0}^{i-1} d_j' \partial_x V_{i-1-j}\right)$$

$$= p \sum_{i=1}^{\infty} p^{i-1} \sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j}$$

$$= p \left(\sum_{i=0}^{\infty} p^i d'_i \right) \left(\sum_{j=0}^{\infty} p^j \partial_x V_j \right)$$

$$= p\bar{d}' \partial_x \bar{V}. \tag{4.51}$$

By (4.51), (4.45), (4.47) and (4.49), $\bar{V}$ and $\bar{d}$ are the solutions of

$$\mathcal{B}\bar{V} = p\bar{d}' \partial_x \bar{V}, \tag{4.52}$$

with

$$\bar{V}(0, x) = 0, \tag{4.53a}$$

$$\bar{V}(0, x) - \partial_x \bar{V}(0, x) = e^{bt(1-p)} + e^{bt}(1-p) \sum_{i=0}^{\infty} p^i R_i, \tag{4.53b}$$

$$\bar{V}(0, x) = e^{bt(1-p)} - e^{\bar{d}}. \tag{4.53c}$$

By setting $p = 1$,

$$V = \sum_{i=0}^{\infty} V_i, \quad d = \sum_{i=0}^{\infty} d_i \tag{4.54}$$

are the solutions of

$$\mathcal{B}V = d' \partial_x V, \tag{4.55}$$

with

$$V(0, x) = 0, \tag{4.56a}$$

$$V(t, 0) - \partial_x V(t, 0) = 1, \tag{4.56b}$$

$$\partial_x V(t, 0) = 1 - e^d. \tag{4.56c}$$

□

Similar to the previous expansion, the expansion terms can be expressed as BSCE functions.

Corollary 4.2.4. *For $i \geq 0$,*

$$V_i \in \Sigma_2, \quad d_i \in \Sigma_3. \tag{4.57}$$

Proof. According to the definitions,

$$V_0 = 0 \in \Sigma_2, \quad d_0 = bt \in \Sigma_3. \tag{4.58}$$

Consider V_i and d_i. Assume that for $0 \leq j \leq i - 1$,

$$V_j \in \Sigma_2, \quad d_j \in \Sigma_3. \tag{4.59}$$

Because $d'_i \in \Sigma_3$ and $\partial_x V_i \in \Sigma_2$,

$$\sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j} \in \Sigma_2. \tag{4.60}$$

By Lemma 2.2.10,

$$u_i = \mathcal{B}^{-1}\left(\sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j}\right) \in \Sigma_2. \tag{4.61}$$

Since $e^{-bt}[-u_i(t,0) + \partial_x u_i(t,0)] \in \Sigma_3$, according to Theorem 2.2.14

$$w_i \in \Sigma_2. \tag{4.62}$$

Therefore,

$$V_i = u_i + w_i \in \Sigma_2. \tag{4.63}$$

Since $e^{-bt}V_i(t,0) \in \Sigma_3$ and by Lemma 2.2.19

$$d_i = \frac{(-bt)^i}{i!} - e^{-bt}V_i(t,0) - \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right) \in \Sigma_3. \tag{4.64}$$

By induction for $i \geq 0$

$$V_i \in \Sigma_2, \quad d_i \in \Sigma_3. \tag{4.65}$$

□

Two methods of decomposing the American put options are presented above. However, the numerical results in the next section indicate that the methods fail in cases with high interest rates and low volatility. Fortunately, the expansion methods are very flexible. The Black–Scholes operator $\mathcal{B}_v$ determines the structure of the expansion terms. Therefore, it can be decomposed as

$$\begin{aligned}\mathcal{B}_v &= \partial_t - \frac{v}{2}\left(\partial_x^2 - \partial_x\right) - r\left(\partial_x - 1\right) \\ &= \partial_t - \frac{\theta}{2}\left(\partial_x^2 - \partial_x\right) - r\left(\partial_x - 1\right) - \frac{v-\theta}{2}\left(\partial_x^2 - \partial_x\right) = \mathcal{B}_\theta - \frac{v-\theta}{2}\left(\partial_x^2 - \partial_x\right).\end{aligned} \tag{4.66}$$

Consequently, the PDE (4.8) is transformed to

$$\mathcal{B}_\theta V := \left[\partial_t - \frac{\theta}{2}\left(\partial_x^2 - \partial_x\right) - r\left(\partial_x - 1\right)\right] V = d'\partial_x V + \frac{v-\theta}{2}\left(\partial_x^2 - \partial_x\right) V, \tag{4.67}$$

with unchanged boundary conditions. The 'effective' volatility has been replaced by an arbitrary constant, θ, which will be set to interest rate r. We applied this technique in the last chapter and set θ to long-term volatility. In this case, volatility can be increased to achieve better convergence.

Proposition 4.2.5 (ABS-III)**.** *The solution of* (4.8) *with initial and boundary conditions* (4.9) *can be written as*

$$V = \sum_{i=0}^{\infty} V_i(t,x), \quad d = \sum_{i=0}^{\infty} d_i(t), \tag{4.68}$$

with

$$V_0(t,x) = 0, \quad d_0 = bt, \quad R_0 = -1, \quad a = \frac{1}{2} - \frac{r}{\theta}, \quad b = -\frac{\theta}{2}\left(\frac{1}{2} + \frac{r}{\theta}\right)^2. \tag{4.69}$$

then for $i \geq 1$*,*

$$V_i(t,x) = u_i + w_i, \tag{4.70a}$$

$$u_i(t,x) = \mathcal{B}_\theta^{-1}\left[\sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j} + \frac{v-\theta}{2}\left(\partial_x^2 - \partial_x\right) V_{i-1}\right], \tag{4.70b}$$

$$w_i(t,x) = \mathcal{I}^{-1}\left[e^{bt}\frac{(-bt)^i}{i!} - u_i(t,0) + \partial_x u_i(t,0) + (R_i - R_{i-1})e^{bt}\right], \tag{4.70c}$$

$$R_i(t,x) = L_i\left[\frac{(-bt)^i}{i!} + e^{-bt}[-u_i(t,0) + \partial_x u_i(t,0)] - R_{i-1}\right], \tag{4.70d}$$

$$d_i(t) = \frac{(-bt)^i}{i!} - e^{-bt}V_i(t,0) - \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} d_j p^j\right). \tag{4.70e}$$

Proof. The PDE (4.8) is decomposed as

$$\mathcal{B}_\theta V_i = \sum_{j=0}^{i-1} d'_j \partial_x V_{i-1-j} + \frac{v-\theta}{2}\left(\partial_x^2 - \partial_x\right) V_{i-1}. \tag{4.71}$$

The rest of the proof follows Proposition 4.2.3. □

In our numerical calculation using this method, we ensure that the volatility is larger than or equal to the interest rate:

$$\theta = \begin{cases} v, & v \geq r, \\ r, & v < r. \end{cases} \tag{4.72}$$

According to this definition, the ratio r/θ is bounded from above.

Obviously, the extra operator

$$\frac{v-\theta}{2}\left(\partial_x^2 - \partial_x\right) \tag{4.73}$$

does not change the form of a BSCE function. Therefore, the expansion terms have the same form as in previous propositions.

Corollary 4.2.6. *For* $i \geq 0$*,*

$$V_i \in \Sigma_2, \quad d_i \in \Sigma_3. \tag{4.74}$$

The code for Proposition 4.2.5 can be found at [59].

After deriving the formula for V_i and d_i, the option price V can be calculated as

$$y = \sum_{i=0}^{N} d_i(t), \tag{4.75a}$$

$$x = \ln S - y, \tag{4.75b}$$

$$V = \begin{cases} \sum_{i=0}^{N} V_i(t,x), & x \geq 0, \\ \sum_{i=0}^{N} V_i(t,0) + e^y - e^{x+y}, & x < 0. \end{cases} \tag{4.75c}$$

Although this definition ensures that the price is continuous at $x = 0$, the price may not equal the execution price below the moving boundary ($x < 0$). Alternatively, we can define

$$V = \begin{cases} \sum_{i=0}^{N} V_i(t,x), & x \geq 0, \\ \sum_{i=0}^{N} 1 - e^y, & x < 0. \end{cases} \tag{4.76}$$

This definition guarantees the execution price, but the price may be discontinuous at $x = 0$.

4.3 Numerical results

In this section, as reference, the prices obtained using the binomial tree method (the average of 1000 and 1001 steps) are presented as the 'true' prices [36].

In Table 4.1, the parameter values used to calculate using Proposition 4.2.1 are

$$v = 0.09, \quad r = 0.1, \quad t = 1, \quad x = 0. \tag{4.77}$$

The table shows the convergence of the expansion terms in this proposition. Both V_i and d_i are decreasing in magnitude.

i	0	1	2	3	4	5
d_i	-0.116806	-0.0500598	-0.0615439	-0.00681467	-0.0205322	-0.00531739
V_i	0	0.14847	0.059714	0.0035771	0.0163554	0.00353447
	6	7	8	9	10	11
	-0.00639416	-0.0038259	-0.00182383	-0.0020473	-0.000450435	-0.000738586
	0.00438134	0.00277584	0.000943652	0.00144181	0.0000988823	0.000442853

Table 4.1: Convergence of Proposition 4.2.1.

For Propositions 4.2.1, 4.2.3 and 4.2.5, Tables 4.2, 4.4 and 4.6 show the boundaries and prices in terms of moneyness and times to maturity. The prices obtained using the tree method are extremely close to those produced using expansion methods. The relative error at the moving boundary is less than 1% and grows bigger as the stock price moves away from the boundary. This occurs because the option price is expanded as powers of the log-distance to the boundary $x = \ln S - d(t)$. Option prices below the boundary at which the put option should be exercised, differ from the options' intrinsic value, because the definition (4.75c) is employed.

Tables 4.3, 4.5 and 4.7 show the boundary and prices for various volatilities and interest rates for the propositions. In Tables 4.3 and 4.5, the straightforward expansion of Propositions 4.2.1 and 4.2.3 fail to produce convergent results in case with low volatility and high interest rates. This may be due to the fact the structural parameters

$$a = \frac{1}{2} - \frac{r}{v}, \quad b = -\frac{v}{2}\left(\frac{1}{2} + \frac{r}{v}\right)^2 \tag{4.78}$$

play an important role in those propositions and the expansion terms are expressed in powers of a and b. Therefore, the ratio r/v is important for achieving convergence of the series, because

$$a \to \infty, \quad b \to -\infty, \quad \text{if} \quad \frac{r}{v} \to \infty. \tag{4.79}$$

Table 4.7 confirms the validity of the volatility breakdown $v = \theta + (v - \theta)$ in Proposition 4.2.5. Therefore, it shows that the option prices with low volatility and high interest rates, which the previous propositions are unable to deal with, can be priced with a high degree of accuracy.

4.4 American options beyond the Black–Scholes model

All the expansion methods we proposed for European options under stochastic volatility models and American options under the Black−Scholes model are inspired by the notion that one can decompose a price to a series of coefficients and find computable relations among them. Therefore, it is appropriate to synthesise the methods to explore American options under advanced models.

Suppose the PDE of a European option beyond the Black−Scholes model can be written as

$$\mathcal{B}V = \mathcal{O}V, \tag{4.80}$$

where $\mathcal{O}$ represents the advanced dynamics. As discovered earlier, the PDE of an American put option under the Black−Scholes model is

$$\mathcal{B}V = d'\partial_x V, \tag{4.81}$$

with boundary conditions

$$V(0, x) = 0, \tag{4.82a}$$

$$V(t, 0) = 1 - e^{d(t)}, \tag{4.82b}$$

$$\partial_x V(t, 0) = -e^{d(t)}. \tag{4.82c}$$

An American option beyond the Black−Scholes model can be regarded as the combination of the two problems

$$\mathcal{B}V = \mathcal{O}V + d'\partial_x V, \tag{4.83}$$

Time\Distance	−0.2	−0.1	0	0.1	0.2
0.01	78.172	86.393	95.479	105.52	116.61
Tree	21.828	13.607	4.5519	0.0414	0.0000
Expansion	21.839	13.617	4.5312	0.0136	0.0000
0.03	76.801	83.773	92.584	102.32	1.1308
Tree	24.199	16.227	7.4473	1.0484	0.0140
Expansion	24.224	16.252	7.4410	0.7727	0.0036
0.1	71.879	79.439	87.793	97.027	107.23
Tree	28.121	20.561	12.223	4.9207	1.1108
Expansion	28.170	20.610	12.256	4.6009	0.8297
0.3	67.146	74.208	82.013	90.638	1.0017
Tree	32.854	25.792	17.987	10.644	5.2976
Expansion	32.904	25.842	18.037	10.512	4.9747
1	61.874	68.382	75.574	83.522	92.306
Tree	38.126	31.618	24.426	17.427	11.838
Expansion	38.109	31.602	24.410	17.331	11.439
3	59.037	65.246	72.108	79.692	88.073
Tree	40.963	34.754	27.892	21.459	16.281
Expansion	41.072	34.863	28.001	21.142	15.097

Table 4.2: American put option prices under the Black–Scholes model by Proposition 4.2.1. The first column indicates time to maturity in years. The first row indicates log-distance between the stock price and the moving boundary $\ln S - d(t)$. In each cell, the top number is the nominal price of the stock, the middle number is the price calculated using the tree method, and the bottom number is the price calculated using the proposition. The other parameters are $r = 0.1$, $v = 0.3^2$ and $K = 100$.

with the same boundary conditions (4.82). We will show that the extra $\mathcal{O}V$ term does not change the general structure and computability of the problem.

To provide a concrete example, we consider the CEV model, the pricing PDE of which is

$$\partial_t V - \frac{v}{2} S^{2-\alpha} \partial_S^2 V - rS\partial_S V + rV = 0, \tag{4.84}$$

where t denotes the time to maturity and $0 < \alpha < 2$. Written in log-price x, the operator

$$\mathcal{O} = \frac{v}{2}\left(e^{-\alpha x} - 1\right)\left(\partial_x^2 - \partial_x\right). \tag{4.85}$$

Thus, the PDE for a CEV American option is, for $t > 0$ and $x > 0$,

$$\mathcal{B}V = d(t)'\partial_x V + \frac{v}{2}\left(e^{-\alpha x} - 1\right)\left(\partial_x^2 - \partial_x\right) V, \tag{4.86}$$

with

$$V(0, x) = 0, \tag{4.87a}$$

$$V(t, 0) = 1 - e^{d(t)}, \tag{4.87b}$$

$$\partial_x V(t, 0) = -e^{d(t)}. \tag{4.87c}$$

The boundary conditions are the same as in the Black–Scholes model. The expansion can be performed as in Proposition 4.2.3.

Volatility\Interest	0.02	0.04	0.06	0.08	0.1
0.1	88.869	90.180	–	–	–
Tree	11.131	9.8204	–	–	–
Expansion	11.149	9.3152	–	–	–
0.2	75.971	79.153	81.889	84.227	86.024
Tree	24.058	20.847	18.111	15.773	13.976
Expansion	24.225	20.899	18.111	15.767	13.945
0.3	64.775	67.912	70.728	73.273	75.574
Tree	35.313	32.096	29.272	26.727	24.426
Expansion	35.755	32.341	29.363	26.738	24.410
0.4	55.150	58.105	60.840	63.377	65.733
Tree	44.997	41.922	39.160	36.623	34.267
Expansion	45.805	42.457	39.456	36.748	34.292
0.5	46.911	49.632	52.207	54.642	56.946
Tree	53.288	50.419	47.800	45.358	43.054
Expansion	54.517	51.305	48.371	45.679	43.201

Table 4.3: American put option prices under the Black–Scholes model by Proposition 4.2.1. The first column indicates volatility and the first row indicates interest rates. In each cell, the top number is the nominal price of the stock, the middle number is the price calculated using the tree method, and the bottom number is the price calculated using the proposition. The other parameters are $x = 0$ (at the moving boundary), $t = 1$ and $K = 100$.

Proposition 4.4.1. *The solution of* (4.86) *with initial and boundary conditions* (4.87) *can be written as*

$$V = \sum_{i=0}^{\infty} V_i(t, x), \quad d = \sum_{i=0}^{\infty} d_i(t), \tag{4.88}$$

with

$$V_0(t, x) = 0, \quad d_0 = bt, \quad R_0 = -1, \tag{4.89}$$

then for $i \geq 1$,

$$V_i(t, x) = u_i + w_i, \tag{4.90}$$

$$u_i(t, x) = \mathcal{B}^{-1}\left[\sum_{j=1}^{i} d'_{j-1}\partial_x V_{i-j} + \frac{v}{2}\sum_{j=1}^{i}\frac{(-\alpha x)^j}{j!}\left(\partial_x^2 - \partial_x\right)V_{i-j}\right], \tag{4.91}$$

$$w_i(t, x) = \mathcal{I}^{-1}\left[e^{bt}\frac{(-bt)^i}{i!} - u_i(t, 0) + \partial_x u_i(t, 0) + (R_i - R_{i-1})e^{bt}\right], \tag{4.92}$$

$$R_i(t, x) = L_i\left[e^{bt}\frac{(-bt)^i}{i!} + e^{-bt}[\partial_x u_i(t, 0) - u_i(t, 0)] - R_{i-1}\right], \tag{4.93}$$

$$d_i(t) = \frac{(-bt)^i}{i!} - e^{-bt}V_i(t, 0) - \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} d_j p^j\right). \tag{4.94}$$

Time\Distance	−0.2	−0.1	0	0.1	0.2
0.01	78.330	86.568	95.673	105.73	116.85
Tree	21.670	13.432	4.3693	0.0348	0.0000
Expansion	21.682	13.444	4.3394	0.0134	0.0000
0.03	76.054	84.053	92.892	102.66	1.1346
Tree	23.946	15.947	7.1547	0.9488	0.0115
Expansion	23.974	15.975	7.1352	0.7035	0.0036
0.1	72.249	79.848	88.245	97.526	107.78
Tree	27.751	20.152	11.788	4.6323	1.0091
Expansion	27.807	20.208	11.811	4.2898	0.7568
0.3	67.560	74.666	82.519	91.197	1.0079
Tree	32.440	25.334	17.488	10.253	5.0416
Expansion	32.517	25.412	17.559	10.116	4.7630
1	62.104	68.636	75.854	83.832	92.649
Tree	37.896	31.364	24.146	17.195	11.657
Expansion	37.924	31.392	24.174	17.269	11.717
3	58.726	64.902	71.728	79.271	87.608
Tree	41.274	35.098	28.272	21.765	16.526
Expansion	41.285	35.109	28.283	21.818	16.612

Table 4.4: American put option prices under the Black–Scholes model by Proposition 4.2.3. The first column indicates time to maturity in years. The first row indicates log-distance between the stock price and the moving boundary $\ln S - d(t)$. In each cell, the top number is the nominal price of the stock, the middle number is the price calculated using the tree method, and the bottom number is the price calculated using the proposition. The other parameters are $r = 0.1$, $v = 0.3^2$ and $K = 100$.

Proof. Define functions

$$\bar{V} := \sum_{i=0}^{\infty} p^i V_i = \sum_{i=0}^{\infty} p^i \left(u_i + w_i\right), \quad \bar{d} := \sum_{i=0}^{\infty} p^i d_i. \tag{4.95}$$

By definition,

$$\mathcal{B}u_i = \left[\sum_{j=1}^{i} d'_{j-1}\partial_x V_{i-j} + \frac{v}{2}\sum_{j=1}^{i}\frac{(-\alpha x)^j}{j!}\left(\partial_x^2 - \partial_x\right)V_{i-j}\right]\mathbf{1}_{i>0}, \tag{4.96}$$

$$\mathcal{B}w_i = 0, \tag{4.97}$$

therefore,

$$\mathcal{B}\bar{V} = \sum_{i=0}^{\infty} p^i \mathcal{B}(u_i + w_i) = \sum_{i=1}^{\infty} p^i \left[\sum_{j=1}^{i} d'_{j-1}\partial_x V_{i-j} + \frac{v}{2}\sum_{j=1}^{i}\frac{(-\alpha x)^j}{j!}\left(\partial_x^2 - \partial_x\right)V_{i-j}\right]. \tag{4.98}$$

Because

$$p\bar{d}'\partial_x\bar{V} + \frac{v}{2}\left(e^{-p\alpha x} - 1\right)\left(\partial_x^2 - \partial_x\right)\bar{V}$$

Volatility\Interest	0.02	0.04	0.06	0.08	0.1
0.1	89.050	91.608	85.671	–	–
Tree	10.952	8.3919	14.329	–	–
Expansion	10.975	8.4145	12.161	–	–
0.2	76.956	79.718	82.086	84.199	86.120
Tree	23.115	20.287	17.914	15.801	13.880
Expansion	23.270	20.376	17.938	15.794	13.920
0.3	66.470	69.162	71.612	73.835	75.854
Tree	33.715	30.883	28.394	26.165	24.146
Expansion	34.098	31.177	28.574	26.253	24.174
0.4	57.541	60.035	62.378	64.572	66.619
Tree	42.772	40.079	37.657	35.436	33.381
Expansion	43.447	40.648	38.077	35.716	33.551
0.5	49.974	52.239	54.410	56.481	58.452
Tree	50.469	47.959	45.673	43.550	41.557
Expansion	51.464	48.839	46.382	44.086	41.941

Table 4.5: American put option prices under the Black–Scholes model by Proposition 4.2.3. The first column indicates volatility and the first row indicates interest rates. In each cell, the top number is the nominal price of the stock, the middle number is the price calculated using the tree method, and the bottom number is the price calculated using the proposition. The other parameters are $x = 0$ (at the moving boundary), $t = 1$ and $K = 100$.

$$= p\left(\sum_{i=0}^{\infty} p^i d_i'\right)\left(\sum_{j=0}^{\infty} p^j \partial_x V_j\right) + \frac{v}{2}\left[\sum_{i=0}^{\infty} \frac{(-p\alpha x)^i}{i!} - 1\right]\left[\sum_{j=0}^{\infty} p^j \left(\partial_x^2 - \partial_x\right) V_j\right]$$

$$= \sum_{i=1}^{\infty} p^i \left[\sum_{j=1}^{i} d_{j-1}'\partial_x V_{i-j} + \frac{v}{2}\sum_{j=1}^{i} \frac{(-\alpha x)^j}{j!}\left(\partial_x^2 - \partial_x\right) V_{i-j}\right], \tag{4.99}$$

$\bar{V}$ and $\bar{d}$ satisfy the PDE:

$$\mathcal{B}\bar{V} = p\bar{d}'\partial_x\bar{V} + \frac{v}{2}\left(e^{-p\alpha x} - 1\right)\left(\partial_x^2 - \partial_x\right)\bar{V}. \tag{4.100}$$

By setting $p = 1$, V and d satisfy PDE

$$\mathcal{B}V = d(t)'\partial_x V + \frac{v}{2}\left(e^{-\alpha x} - 1\right)\left(\partial_x^2 - \partial_x\right) V. \tag{4.101}$$

The boundary conditions are proven in the same way as in Proposition 4.2.3. □

We can also show that the expansion terms are within the same function family as those in the Black–Scholes model.

Corollary 4.4.2. *For $i \geq 0$, in Proposition 4.4.1*

$$V_i \in \Sigma_2, \quad d_i \in \Sigma_3. \tag{4.102}$$

Proof. Define

$$V_0 = 0 \in \Sigma_2, \quad d_0 = bt \in \Sigma_3 \tag{4.103}$$

Time\Distance	−0.2	−0.1	0	0.1	0.2
0.01	78.439	86.689	95.806	105.88	117.02
Tree	21.561	13.311	4.2446	0.0348	0.0000
Expansion	21.569	13.319	4.2020	0.0309	0.0000
0.03	76.186	84.199	93.054	102.84	113.66
Tree	23.814	15.801	7.0030	0.8998	0.0104
Expansion	23.836	15.823	6.9681	0.6481	0.0031
0.1	72.386	79.999	88.413	97.711	107.99
Tree	27.614	20.001	11.629	4.5283	0.9734
Expansion	27.671	20.058	11.645	4.1525	0.7162
0.3	67.679	74.797	82.663	91.357	100.97
Tree	32.321	25.203	17.377	10.144	4.9680
Expansion	32.409	25.291	17.425	9.9769	4.6563
1	62.135	68.669	75.891	83.873	92.694
Tree	37.865	31.331	24.109	17.164	11.633
Expansion	37.901	31.367	24.145	17.236	11.681
3	58.559	64.718	71.524	79.047	87.360
Tree	41.441	35.282	28.476	21.930	16.659
Expansion	41.429	35.270	28.464	22.004	16.780

Table 4.6: American put option prices under the Black–Scholes model by Proposition 4.2.5. The first column indicates time to maturity in years. The first row indicates log-distance between the stock price and the moving boundary $\ln S - d(t)$. In each cell, the top number is the nominal price of the stock, the middle number is the price calculated using the tree method, and the bottom number is the price calculated using the proposition. The other parameters are $r = 0.1$, $v = 0.3^2$ and $K = 100$.

Consider V_i and d_i. Assume that for $0 \le j \le i-1$,

$$V_j \in \Sigma_2, \quad d_j \in \Sigma_3. \tag{4.104}$$

Because $d_i' \in \Sigma_3$, $\partial_x V_i \in \Sigma_2$ and $\partial_x^2 V_i \in \Sigma_2$,

$$\sum_{j=1}^{i} d_{j-1}' \partial_x V_{i-j} + \frac{v}{2} \sum_{j=1}^{i} \frac{(-\alpha x)^j}{j!} \left(\partial_x^2 - \partial_x\right) V_{i-j} \in \Sigma_2. \tag{4.105}$$

By Lemma 2.2.10,

$$u_i = \mathcal{B}^{-1} \left[\sum_{j=1}^{i} d_{j-1}' \partial_x V_{i-j} + \frac{v}{2} \sum_{j=1}^{i} \frac{(-\alpha x)^j}{j!} \left(\partial_x^2 - \partial_x\right) V_{i-j} \right] \in \Sigma_2. \tag{4.106}$$

Since $e^{-bt}[-u_i(t,0) + \partial_x u_i(t,0)] \in \Sigma_3$, according to Theorem 2.2.14

$$w_i \in \Sigma_2 \tag{4.107}$$

Therefore

$$V_i = u_i + w_i \in \Sigma_2. \tag{4.108}$$

Volatility\Interest	0.02	0.04	0.06	0.08	0.1
0.1	89.051	91.241	93.029	94.524	95.798
Tree	10.951	8.7590	6.9710	5.4760	4.2110
Expansion	10.968	8.7643	7.0301	5.6246	4.4560
0.2	77.296	79.887	82.093	84.070	85.844
Tree	22.794	20.121	17.907	15.930	14.156
Expansion	22.949	20.222	17.929	15.924	14.161
0.3	67.003	69.527	71.842	73.966	75.891
Tree	33.221	30.535	28.169	26.034	24.109
Expansion	33.598	30.851	28.375	26.141	24.145
0.4	58.213	60.555	62.765	64.845	66.799
Tree	42.158	39.593	37.287	35.168	33.201
Expansion	42.825	40.186	37.747	35.492	33.408
0.5	50.740	52.871	54.919	56.880	58.754
Tree	49.779	47.378	45.194	43.166	41.270
Expansion	50.766	48.284	45.952	43.762	41.706

Table 4.7: American put option prices under the Black−Scholes model by Proposition 4.2.5. The first column indicates volatility and the first row indicates interest rates. In each cell, the top number is the nominal price of the stock, the middle number is the price calculated using the tree method, and the bottom number is the price calculated using the proposition. The other parameters are $x = 0$ (at the moving boundary), $t = 1$ and $K = 100$.

Since $V_i(t, 0) \in \Sigma_3$ and by Lemma 2.2.19

$$d_i = \frac{(-bt)^i}{i!} - e^{-bt}V_i(t,0) - \mathcal{D}_p^i \exp\left(\sum_{j=1}^{i-1} p^j d_j\right) \in \Sigma_3. \tag{4.109}$$

By induction for $i \geq 0$

$$V_i \in \Sigma_2, \quad d_i \in \Sigma_3. \tag{4.110}$$

□

Although the results above apply to the CEV model, we can generalise them to other advanced models as well. The expansion can be performed, as long as the operator $\mathcal{O}$ only involves differential operators, as BSCE functions are closed under differentiation.

4.5 Summary

In this chapter, we showed how to apply expansion methods to American options under the Black−Scholes model and beyond. Because the Black−Scholes formula is not a good leading term for expanding American option, it is crucial to use BSCE functions as expansion terms. With BSCE functions, the expansion terms can be calculated analytically up to any order. The decomposition of the boundary condition at $x = 0$ (Theorem 2.2.14) is also important for ensuring the computability of BSCE functions.

The form of the expansion is highly flexible, as there are a number of ways to rewrite the PDE and every boundary condition in terms of the dummy variable p, which will be set to 1 in the end. The inaccuracy caused by low volatility and high interest rates can be avoided by regarding the Black−Scholes model as an advanced model with an extra operator and artificial volatility θ, which is bounded from below. Other advanced models, such as the CEV model, can be solved similarly, as long as the extra operator does not break the closedness of BSCE functions.

The numerical results show the validity of three expansion methods of the Black-Scholes model. The code for Proposition 4.2.5, which works in most cases, is available at [59].

5 Discussion

The current literature on expansion methods for option pricing [51, 52, 54, 57] has identified the formal relations between expansion terms in terms of the double integral

$$V = \sum_{i=0}^{\infty} V_i p^i, \quad V_i = \int_0^s \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x f(V_{i-1}, \ldots, V_0). \tag{5.1}$$

The authors show the validity of the methods by numerically evaluating those integrals in a few special cases. However, the limitations of the methods are not mentioned in the literature. In order to improve the pricing performance and our theoretical understanding of the methods, we derived the explicit formula for the leading terms (code available at [59])

$$V = \sum_{i=0}^{\infty} V_i p^i, \quad V_i = f_i(t, x). \tag{5.2}$$

We proved that the terms can be calculated up to arbitrary order i, as claimed in Chapter 1.

We also numerically demonstrated that although all the expansion methods work asymptotically, it is critical to choose appropriate basis functions when the expansion parameters change from infinitesimally small to moderately large. In the case of European options under stochastic volatility models, without the restriction of the HAM framework which dominates the current literature, the option price can be expanded in four different ways

$$V = \sum_{i,j=0}^{\infty} V_{ij} P^i Q^j, \tag{5.3}$$

where $(P, Q) = (\kappa, \eta)$, $(\eta, v - \theta)$, $(\eta, \theta - v)$ or $(1/\kappa, v)$. Like all expansion methods, these methods all converge with very few terms when the respective parameters go to zero

$$P \to 0, \quad Q \to 0. \tag{5.4}$$

Unfortunately, this fast convergence does not occur when P or Q is large enough to cover the domain calibrated by real market data. If we consider the behaviour of large parameters of option price, the expansion parameters can be promoted to their bounded version:

$$P \to \frac{P}{1+P}, \quad Q \to \frac{Q}{1+Q}. \tag{5.5}$$

The numerical results confirm that the modified expansion method indeed provides a very accurate approximation of the option price, even with some unrealistically large parameters (with a less than 3% error; see Table 3.7). The accuracy of Proposition 3.3.2 makes it usable for most practical applications.

For American options under the Black−Scholes model, the non-linear term $d'\partial_x V$ in the pricing PDE

$$\mathcal{B}_v V = d'\partial_x V \tag{5.6}$$

and the boundary conditions

$$V(t,0) = 1 - e^d, \quad \partial_x V(t,0) = -e^d, \tag{5.7}$$

arising from the front-fixing technique are taken care of by the choice of BSCE functions, as the expansion coefficients V_i and BST functions as d_i in

$$V = \sum_{i=0}^{\infty} V_i(t,x)p^i, \quad d = \sum_{i=0}^{\infty} d_i(t)p^i. \tag{5.8}$$

The two methods are proven and their validity is numerically demonstrated, but the series does not converge in cases with high interest rates and low volatility. Thus, a modified expansion to the pricing PDE

$$\mathcal{B}_\theta V = d'\partial_x V + \frac{v-\theta}{2}\left(\partial_x^2 - \partial_x\right) V \tag{5.9}$$

is proposed. The transformation of (5.6) to (5.9) has been demonstrated (Table 4.7) to overcome the aforementioned limitations, making Proposition 4.2.5 the best method for pricing American options.

Expansion methods have many advantages over other numerical methods. First of all, they are quick to evaluate, because the formula series contains much more information about the 'real' solution than the discretisation steps used by conventional methods like the Monte Carlo and finite-difference methods. Our numerical experiments also confirm that expansion methods with explicit formulae produce results within a second[1], while Monte Carlo methods may take minutes, or even hours, to produce results for multi-year options, with the same accuracy and complexity.

Secondly, expansion methods are general. Monte Carlo methods should be used with care when dealing with path-dependent options (e.g. American options [27]), finite-difference methods are inefficient for solving high-dimensional problems [40−42], and Fourier transform methods cannot deal with non-affine models without characteristic functions [11, 12]. Expansion methods may be applicable to any problem with a PDE representation.

Expansion methods are set apart from other methods because they serve as the first step towards an exact solution, which is the ultimate goal for a problem. In this thesis, we show that all the terms in the series can be derived, once all lower order terms are known:

$$V_i = g_i(t,x,\ldots), \quad i = 0,1,2,\ldots \tag{5.10}$$

Once V_i is expressed in terms of i

$$V_i = g(i,t,x,\ldots), \quad i = 0,1,2,\ldots, \tag{5.11}$$

[1] The expansion prices are evaluated in MATLAB, with a 2.6 GHz Intel Core i5 processor and 8 GB memory. With code [59], a single price takes about 0.2 second to evaluate.

a closed-form solution can be achieved.

Now, we can affirmatively answer the questions we asked in Chapter 1.

- *How well can expansion methods be applied to option pricing problems beyond European options under the Black – Scholes model?*
  Expansion methods can be applied to European options under stochastic volatility models and to American options under the Black – Scholes model with a high degree of accuracy, with $(\eta + 1, v + 1)$ and ABS-III-expansion being the respective best choice. American options under advanced models are outlined but not numerically illustrated.

- *Expansion methods usually work well as expansion parameters go to zero. However, do the methods work for reasonably large parameters, that are realistic in the actual market?*
  There are many ways to expand an option price. When one carefully chooses basis functions (e.g. bounded functions in $(\eta + 1, v + 1)$-expansion), the method can be used for extremely large parameter values (Table 3.7) that are calibrated by market data.

- *Are the series solutions convergent? If not, how efficient and accurate are the finite-term approximations?*
  Convergence is not always guaranteed, but when one choose the appropriate expansion form, pricing errors can be arbitrarily small, even for finite-term approximations.

Though expansion methods can be efficient in many cases, they may not be appropriate when dealing with models without a PDE representation (e.g. discrete-time GARCH models). In such cases, the discrete nature of Monte Carlo methods becomes an advantage, because the simulation's time step can be chosen to match the model's time step. Models with jumps are also difficult to apply to expansion methods, as their pricing equation, partial integro-differential equation (PIDE), involves integration, which destroys the closedness of expansion terms much more often than differentiation. Though they are not entirely impossible to solve, jump models require extra care, compared to models without jumps.

The flexibility of expansion methods is a double-edged sword. On one hand, it provides plenty of options to achieve better convergence and fewer terms. On the other hand, for researchers, it can be difficult when facing a new option, because the obvious method of expansion is usually not optimal in terms of the domain of applicability. In our experience, dozens of methods have to be tried before we can obtain a satisfying result.

Regarding the structure of expansion methods, the expansion terms are very specific to the option type we are investigating. As we demonstrated in the previous chapters, European and American options must be treated in completely different ways even though they differ in only one feature, when they can be exercised. We did not consider exotic options in this thesis, but we expect that they would have to be treated differently as well. Thus, the expansion formulae for different option types under different models should be derived individually.

However, this inconvenience should not disencourage researchers from exploring expansion methods because they provide better speed and accuracy.

6 Conclusion

In this thesis, expansion methods are applied to various types of options under various models that have no analytical solutions. Our numerical results confirm the validity of the methods. Due to their general applicability, the expansion methods have the potential to overcome some of the drawbacks of popular numerical methods.

For the sake of argument, several mathematical notions are introduced in Chapter 2. In addition to the solutions to the Black−Scholes equation with different boundary conditions, we introduce Black−Scholes special functions, which are closed under many operations. Finally, we demonstrate the similarity between the solutions of simple ODEs and PDEs, which serves as a general introduction to expansion methods.

Through numerical examples, we demonstrate that the expansion methods can be applied to European options under stochastic volatility models. The four expansion methods ((κ, η), (η, v), (η, θ), and (κ, v)) are proposed because the stochastic volatility models can be extended from the Black−Scholes model in various ways. They all work for small parameter values, although convergence is not guaranteed. A modified version of (η, v)-expansion, $(\eta + 1, v + 1)$-expansion, is proposed to improve convergence. This method uses bounded basis functions, instead of unbounded power series, to approximate the bounded target function, option price. The bounded basis functions can be also applied to the other three methods. The numerical results confirm that the error of $(\eta + 1, v + 1)$-expansion is indeed bounded, as the parameters go to infinity.

Scale invariance is a form of internal symmetry of the stochastic volatility models. For the four original unbounded power series expansions, the symmetry is preserved with a finite-term approximation. However, for the bounded series expansions (e.g. $(\eta+1, v+1)$), the symmetry is broken due to the form of the expansion basis functions. Thus, by scaling λ, we could obtain different values for the expansion series, which we could use as tools to fine-tune the convergence of $(\eta + 1, v + 1)$-expansion. It should be noted that the fine-tuning does not work for parameters that do not change through scaling, such as η in the 3/2 model. Furthermore, the fine-tuning cannot replace information that is contained in the form of a truncated series. Including more terms in a expansion is always beneficial, even when scaling can be performed.

American options are particularly difficult to price because the coupled boundary/price pair must be solved simultaneously and there is a non-linear term in the pricing PDE. In Chapter 4, we prove that the prices of American put options without dividends under the Black−Scholes model can be represented by BSCE functions introduced in Chapter 2. Though the numerical results confirm the general validity of the expansion methods (ABS-I and ABS-II), the truncated series cannot be used in cases of low volatility and high interest rates. An improved version, ABS-III, is proposed to deal with this difficulty. It further expands volatility to a modestly high level relative to the interest rate. With the decomposition, this method is able to deal with reasonable volatility, interest rate, moneyness and maturity.

American options under many popular advanced models can be treated similarly as long as the extra operators preserve the closedness of Black−Scholes special functions. ABS-III can be regarded as an advanced model with a fixed volatility of θ and an additional operator $\frac{v-\theta}{2}(\partial_x^2 - \partial_x)$. The pricing PDE is formally similar to that of an advanced model, however, it is a Black−Scholes PDE dynamically. The greatest advantage of expansion methods is that American options under advanced models are not significantly more difficult to price than those under the Black−Scholes model, unlike other numerical methods (e.g. tree methods and finite-difference methods).

In comparison to numerical methods, such as the Monte Carlo and integral transform methods, expansion methods are more efficient when computing a family of option prices with the same model parameters, since we can simplify the formula by substituting the common parameters. For example, if we calculate multiple option prices with different levels of moneyness, we can write the solution as a single variable function of moneyness $u(x, \text{parameters}) \to \bar{u}(x)$, which greatly reduces computation time. However, for the integral transform method (FFT), every numerical integration is independent and simplification is not possible. Thus, the computation time grows linearly with respect to the number of option prices.

Furthermore, expansion methods deal with option types and model dynamics in a single framework. Model dynamics correspond to the additional operators in the pricing PDE, and option type corresponds to the boundary conditions with which the PDE should be solved. When used in combination, they cover a great number of the options we are interested in. This thesis is an initial attempt to price complicated options with expansion methods. More follow-up research is required to deal with exotic options and advanced models.

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