Pricing of Contracts:
-Numerical Solutions of Backward SDEs-
-Modeling Volatility Risk Premium-

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Outline

1. Numerical Solution of Backward SDEs: Regression Later Algorithm, *SIAM Journal of Mathematical Finance (Submission)*.
Simple Example

\[(Market)\] \[\begin{align*}
    dS^0_t &= rS^0_t dt; \quad S^0_0 = 1 \quad \text{(Riskless Asset)} \\
    dS_t &= S_t \mu dt + S_t \sigma dW_t, \quad S^0_0 = x \quad \text{(Risky Asset)}
\end{align*}\]

At \(t \in [0, T]\), we build the self-financing portfolio \(Y\)

\[Y_t = \Delta_t S_t + \beta_t S^0_t\]

By the self-financing condition, \(dY_t = \Delta_t dS_t + \beta_t dS^0_t\)

\[dY_t = S_t \Delta_t (\mu dt + \sigma dW_t) + \beta_t dS^0_t\]

With \(\theta = [\mu - r]\sigma^{-1}\) and \(Z_t = \sigma \Delta_t S_t\), the couple \((Y_t, Z_t)\) solves:

\[-dY_t = f(t, S_t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \phi(S_T)\]

\[
\text{Call} \begin{cases} 
\phi(x) = (x - K)^+ \\
\phi(t, x, y, z) = -(ry + \theta z)
\end{cases}
\]
We consider the system (S)

\[
\begin{aligned}
X_t &= x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_r, \\
Y_t &= \Phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.
\end{aligned}
\]

'\mathcal{L}' is the differential generator defined by

\[
\forall \phi \in C^{1,2}([0, T] \times \mathbb{R}^d) \quad \mathcal{L}\phi =: b.\nabla \phi + \frac{1}{2} \text{Trace}(A.\nabla^2 \phi)
\]

\[A = \sigma \sigma^*\]
Let us consider $u \in C^{1,2}([0, T] \times \mathbb{R}^d)$ a solution of (1). Suppose that there exist two positive constants $C$ and $q$ such that:

$$
|u(t, x)| + |(\nabla_x u)(t, x)| \leq C(1 + |x|^q)
$$

\(\left\{
\begin{aligned}
\frac{\partial u}{\partial t}(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla u\sigma(t, x)) &= 0 \\
u(T, x) &= \Phi(x), \quad \text{with} \quad (t, x) \in [0, T] \times \mathbb{R}^d
\end{aligned}\right.\) (1)

**Theorem (Pardoux and Peng)**

$$
\forall t \in [0, T], \quad Y_t = u(t, X_t) \quad \text{and} \quad Z_t = \nabla_x u(t, X_t)\sigma(t, X_t).
$$
Linear BSDEs

\( (\varphi_t)_{0 \leq t \leq T} \in \mathcal{H}^2(\mathbb{R}), \ (\beta_t)_{0 \leq t \leq T} \in \mathbb{R}, \ (\gamma_t)_{0 \leq t \leq T} \in \mathbb{R}^d \) m.p.
and uniformly bounded

\[
\begin{aligned}
-dY_t &= (\varphi_t + \beta_t Y_t + \gamma_t Z_t)dt - Z_t dW_t. \\
Y_T &= \xi
\end{aligned}
\] (1)

\[\exists \! (Y_t, Z_t) \in S^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\]

\[H_t Y_t = \mathbb{E}[H_T \xi + \int_t^T H_s \varphi_s ds | \mathcal{F}_t].\]

\[
\begin{aligned}
dH_t &= H_t(\beta_t dt + \gamma_t dW_t), \\
H_0 &= 1
\end{aligned}
\] (2)
Euler Maruyama Scheme

Consider the subdivision $\pi$ of $[0, T]$, $0 = t_0 < t_1 < \ldots < t_N = T$,

- $\Delta_i := t_{i+1} - t_i$,
  
  $\Delta W_{t_i} := W_{t_{i+1}} - W_{t_i}$, $|\pi| = \max\{\Delta_i; 0 \leq i \leq N - 1\}$.

- $(X^\pi, Y^\pi, Z^\pi) \approx (X, Y, Z)$

\[
Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(s, X_s, Y_s, Z_s) ds - \int_{t_i}^{t_{i+1}} Z_s dW_s
\]

The Euler approximation of the previous stochastic integral is:

\[
Y_{t_i}^\pi = Y_{t_{i+1}}^\pi + f(t_i, X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi) \Delta_i - Z_{t_i}^\pi \Delta W_{t_i}
\]
Euler-Maruyama Scheme (Gobet et al.)

For $0 \leq i \leq N$

\[
\begin{aligned}
SDE \quad \left\{ 
\begin{align*}
X_0^\pi &= x \\
X_{t_{i+1}}^\pi &= X_{t_i}^\pi + \Delta_i b(t_i, X_{t_i}^\pi) + \sigma(t_i, X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i})
\end{align*}
\right.
\end{aligned}
\]

\[
\begin{aligned}
BSDE \quad \left\{ 
\begin{align*}
Y_T^\pi &= \phi(X_T^\pi) \\
Z_{t_i}^\pi &= \frac{1}{\Delta_i} \mathbb{E}[Y_{t_{i+1}}^\pi \Delta W_{t_i} | \mathcal{F}_{t_i}] = \frac{1}{\Delta_i} \mathbb{E}[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}] \\
Y_{t_i}^\pi &= \mathbb{E}[Y_{t_{i+1}}^\pi | \mathcal{F}_{t_i}] + \Delta_i f(t_i, X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi)
\end{align*}
\right.
\end{aligned}
\]
Error Estimation

Theorem (Gobet E., J.P. Lemor and X. Warin)

\[
\begin{cases}
1) & x \mapsto b(t,x), \sigma(t,x), \text{ uniformly Lipschitz, } \phi \text{ Lipschitz } \\
2) & f \text{ Lipschitz in } y, z \text{ and } \frac{1}{2} - \text{ Hölderian in } t
\end{cases}
\]

then with \( \Delta Y_{t_i} = Y_{t_i} - Y_{t_i}^\pi \), \( \Delta Z_t^\pi = Z_t - Z_{t_i}^\pi \)

\[
\max_{0 \leq i < N} \mathbb{E} \left[ |\Delta Y_{t_i}^\pi|^2 \right] + \mathbb{E} \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_i+1} |\Delta Z_t^\pi|^2 \, dt \right] \\
\leq C \left[ (1 + |x|^2) \Delta_i + \mathbb{E}[\phi(X_T) - \phi(X_N^\pi)]^2 \right]
\]
Hypothesis (H)

\[
\begin{align*}
(H1) \quad & (t, x) \mapsto b(t, x), \sigma(t, x) \text{ are Lipschitz in } x \\
& \text{ uniformly in } t \text{ and: } |b(t, x)| + |\sigma(t, x)| \leq K(1 + |x|) \\
(H2) \quad & \text{There exists a positive constant } K > 0, \text{ s.t. } \\
& |f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|), \forall (x_i, y_i, z_i)_{i=1,2} \\
(H3) \quad & \text{There exists } k_\sigma, K_\sigma > 0 \text{ such that } \forall t \in [0, T], \forall x, \zeta \in \mathbb{R}^m \\
& k_\sigma |\zeta|^2 \leq \sum_{i,j} [\sigma \sigma^*]_{i,j}(t, x) \zeta_i \zeta_j \leq K_\sigma |\zeta|^2
\end{align*}
\]
Hypothesis (G)

\[
\begin{cases}
(G1) \quad \sup_{t \in [0, T]} |f(t, 0, 0, 0)| \leq K \\
(G2) \quad x \mapsto \phi(x) \in C^2(\mathbb{R}^m, \mathbb{R}) \quad \text{and Lipshitz in } x \\
(G3) \quad f : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \text{ is continuously differentiable in } (x, y, z) \text{ with uniformly bounded derivatives} \\
(G4) \quad b \in C_b^{0,2}([0, T] \times \mathbb{R}^m, \mathbb{R}^m), \sigma \in C_b^{0,2}([0, T] \times \mathbb{R}^m, \mathbb{R}^{m \times d})
\end{cases}
\]
We built the Dynamic Programing Problem (DPP)

\[ \begin{align*}
Y^\pi_N &= \phi(X^\pi_T), \\
Z^\pi_N &= \sigma(T, X^\pi_T)(\nabla_x \phi)(X^\pi_T), \\
Y^\pi_{t_i} &= \mathbb{E}[Y^\pi_{t_{i+1}} | \mathcal{F}_{t_i}] + \Delta_i \mathbb{E}[f(t_{i+1}, X^\pi_{t_{i+1}}, Y^\pi_{t_{i+1}}, Z^\pi_{t_{i+1}}) | \mathcal{F}_{t_i}], \quad 0 \leq i \leq N - 1 \\
Z^\pi_{t_i} &= \sigma(t_i, X^\pi_{t_i}) \nabla_x Y^\pi_{t_i} \text{ with } \Delta_i := t_{i+1} - t_i, \quad 0 \leq i \leq N - 1.
\end{align*} \]
Convergence Results

Proposition

Under the hypotheses \((G) + (H)\), there exist two positive constant \(C_1, C_2\) independent of the partition \(\pi\) such that:

\[
\max_{0 \leq i < N} \mathbb{E} \left| Y_{t_i} - Y_{t_i}^{\pi} \right|^2 + \mathbb{E} \sum_{i=0}^{N-1} \int_{t_i}^{t_i+1} \left| Z_s - Z_{t_i}^{\pi} \right|^2 ds \leq C_1 (1 + |x|^2) |\pi| \\
+ C_1 \mathbb{E} \left| \phi(X_T) - \phi(X_T^{\pi}) \right|^2
\]
**Theorem (J. Zhang’s $L^2$-regularity)**

Let $\pi$ be any partition of $[0, T]$. Under some hypotheses and assume that $(Z_s)_{0 \leq s \leq T}$ is a càdlàg process, there is a positive constant $C_{T,K} > 0$ independent of $\pi$ such that:

$$\sum_{i=0}^{N-1} \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_s - Z_{t_i}|^2 + |Z_s - Z_{t_{i+1}}|^2 ds \leq C_{T,K}(1 + |x|^2)$$
Remark

For $|\pi|$ small enough, the function

$$x \mapsto u_{t_i}^{\pi}(x)$$

is Lipschitz

$$Y_{t_i}^{\pi} = u_{t_i}^{\pi}(X_{t_i}^{\pi}), \quad t_i \in \pi$$

The Euler approximation

$$Y_{t_i}^{\pi} = \mathbb{E} \left( Y_{t_{i+1}}^{\pi} + \Delta_i f(t_{i+1}, X_{t_{i+1}}^{\pi}, Y_{t_{i+1}}^{\pi}, Z_{t_{i+1}}^{\pi}) | \mathcal{F}_{t_i} \right).$$

$$X_s^{\pi} = X_{t_i}^{\pi} + \int_{t_i}^s b(t_i, X_{t_i}^{\pi}) du + \int_{t_i}^s \sigma(t_i, X_{t_i}^{\pi}) dW_u, \quad \text{with } s \in [t_i, t_{i+1}]. \quad (3)$$

By the Martingale Representation Theorem

$$Y_t^{\pi} = Y_{t_{i+1}}^{\pi} + \int_t^{t_{i+1}} f(t_{i+1}, X_{t_{i+1}}^{\pi}, Y_{t_{i+1}}^{\pi}, Z_{t_{i+1}}^{\pi}) ds - \int_t^{t_{i+1}} \tilde{Z}_s^{\pi} dW_s,$$
Lipschitz By backward induction

- Forward $X^{\pi,i}$ with initial conditions $x_i$, $(i = 1, 2)$
- $(Y^{\pi,x_i}, \bar{Z}^{\pi,x_i}), i = 1, 2$ (where $Y^\pi_t, x_i = u^\pi_t (X^\pi_t, i)$)

By Itô,

$$
\mathbb{E}\left| Y^\pi_t, x_1 - Y^\pi_t, x_2 \right|^2 + \mathbb{E} \int_t^{t+1} |\bar{Z}^\pi_s, x_1 - \bar{Z}^\pi_s, x_2|^2 \, ds = \mathbb{E}\left| Y^\pi_{t+1,1} - Y^\pi_{t+1,2} \right|^2 \\
+ 2\mathbb{E} \int_t^{t+1} (Y^\pi_s, x_1 - Y^\pi_s, x_2) \delta f^\pi_{t+1,1,2} \, ds.
$$

(4)

From $ab \leq \frac{1}{2\alpha} a^2 + \frac{1}{2} \alpha b^2$, $\alpha > 0$

$$
\mathbb{E}\left| \Delta Y^1,2_t \right|^2 + \mathbb{E} \int_t^{t+1} |\bar{Z}^\pi_s, x_1 - \bar{Z}^\pi_s, x_2|^2 \, ds \leq (1 + K \Delta_i \alpha) \mathbb{E}\left| \Delta Y^1,2_{t+1} \right|^2 \\
+ 2K \alpha \int_t^{t+1} \mathbb{E}\left| \Delta Y^1,2_s \right|^2 \, ds + \alpha K \Delta_i \mathbb{E}\left| Z^\pi_{t+1,1} - Z^\pi_{t+1,2} \right|^2.
$$

(5)
Lipschitz By backward induction

There exists $c_i^1$ and $c_i^2 > 0$ such that

$$
\mathbb{E} \int_{t_i}^{t_{i+1}} |Z_{t_{i+1}}^{\pi,x_1} - Z_{t_{i+1}}^{\pi,x_2}|^2 ds \leq c_i^1 (1 + |x_1|^2) |\pi|^2 + 3 \int_{t_i}^{t_{i+1}} \mathbb{E} |\bar{Z}_{s}^{\pi,x_1} - \bar{Z}_{s}^{\pi,x_2}|^2 ds \\
+ c_i^2 (1 + |x_2|^2) |\pi|^2. 
$$

(6)

For $\alpha = \frac{1}{6K}$, $|\pi|$ small enough

$$
\mathbb{E} |\Delta Y_{t,i+1}^{1,2}|^2 + \frac{1}{2} \mathbb{E} \int_{t}^{t_{i+1}} |\bar{Z}_{s}^{\pi,x_1} - \bar{Z}_{s}^{\pi,x_2}|^2 ds \leq (1 + \frac{1}{6} \Delta_i) \mathbb{E} |\Delta Y_{t,i+1}^{1,2}|^2 \\
+ 12K^2 \int_{t}^{t_{i+1}} \mathbb{E} |\Delta Y_{s}^{1,2}|^2 ds. 
$$

(7)

In particular

$$
\mathbb{E} |\Delta Y_{t}^{1,2}|^2 \leq (1 + \frac{1}{6} \Delta_i) \mathbb{E} |\Delta Y_{t_i+1}^{1,2}|^2 + 12K^2 \int_{t}^{t_{i+1}} \mathbb{E} |\Delta Y_{s}^{1,2}|^2 ds. 
$$

(8)
Lipschitz By backward induction

\( u_{t_{i+1}}^\pi \) is Lipschitz with \( C_{i+1} \) its Lipschitz constant. We know

\[
\mathbb{E}|X_{t_{i+1}}^{\pi,x_1} - X_{t_{i+1}}^{\pi,x_2}|^2 \leq (1 + C \Delta_i)|x_1 - x_2|^2.
\]

Gronwall inequality to \( t \in [t_i, t_{i+1}) \mapsto \mathbb{E} |\Delta Y_t^{1,2}|^2 \)

\[
\mathbb{E} |\Delta Y_t^{1,2}|^2 \leq C_i^2 |x_1 - x_2|^2.
\]

where

\[
C_i^2 = (1 + \frac{1}{6} \Delta_i)(1 + C \Delta_i)C_{i+1}^2 \exp(12K^2 \Delta_i)
\]
Non Explosion

Lemma (Gronwall Inequality)

Consider the partition \( \pi : 0 = t_0 < ... < t_N = T \) of the interval \([0, T]\) and \( \Delta_i \) its mesh. Consider the two families \((a_k)_{0 \leq k \leq N}, (b_k)_{0 \leq k \leq N} \geq 0\), \( \gamma > 0 \) where \( a_{k-1} \leq (1 + \gamma \Delta_i) a_k + b_k \), \( k = 1, \ldots, N \). Then,

\[
\max_{0 \leq i \leq N} \ a_i \leq e^{\gamma T} (a_N + \sum_{i=1}^{N} b_i).
\]

By Gronwall inequality, we have

\[
\max_{0 \leq i \leq N} \ C_i^2 \leq e^{CT} (C_\phi^2 + CT) \Rightarrow x \mapsto u_{t_i}^\pi(x)
\]

is Lipschitz where \( C_\phi \) is the Lipschitz constant of the function \( Y_T \).
With $\theta = [\mu - r]\sigma^{-1}$ and $Z_t = \sigma \Delta_t S_t$, the couple $(Y_t, Z_t)$ solves:

$$-dY_t = f(t, S_t, Y_t, Z_t)dt - Z_t dW_t,$$

$$Y_T = \phi(S_T)$$

Call $\left\{ \begin{align*} 
\phi(x) &= (x - K)^+ \\
 f(t, x, y, z) &= -(ry + \theta z)
\end{align*} \right.$$

$T = 1$, $r = 1\%$, $x = 100$, $K = 100$, $\mu = 1\%$, $\sigma = 2\%$
Pricing Vanilla Options: Log-Error

\[ k = 4, \ M = 10^4 \quad (Y_{t_0}, Z_{t_0}) = (1.3886, 1.39). \]

**Figure**: Log-Error curve to estimate \((Y_0, Z_0)\). European Call case
Log-Error Curve

\[ k = 4, \ M = 10^4 \quad (Y_0, Z_0) = (0.39, -0.60) \]

Figure: Log-Error curve to estimate \((Y_0, Z_0)\). European Put
Brownian Functional Case

\[
\begin{align*}
- dY_t &= f(s, W_s, Y_s, Z_s) dt - Z_t dW_t, \quad 0 \leq t < T, \\
Y_T &= \phi(W_T), \\
\phi(x) &= x \arctan(x) - \ln(\sqrt{1 + x^2}) \\
f(t, W_t, Y_t, Z_t) &= -\frac{1}{2(1 + \tan^2(Z_t))}.
\end{align*}
\]

\[
(Y_t, Z_t) = (-\frac{1}{2} \ln(1 + W_t^2) + W_t \arctan(W_t), \arctan(W_t)) \quad a.s.
\]

- \(x \mapsto \ln(x)\) satisfies the linear growth condition
- \(x \mapsto \arctan(x)\) is bounded,

\[
(Y_t, Z_t)_{0 \leq t \leq T} \in S^2(\mathbb{R}) \times H^2(\mathbb{R}) \\
(Y_0, Z_0) = (0, 0).
\]
Log-Error Curve

\( k = 4, \ M = 10^4 \)

Figure: Log-Error curve to estimate \((Y_0, Z_0) = (0, 0)\)
Advantages
- Simple Algorithm,
- High dimension (Basket Options, Some Insurance contracts )
- Reflected BSDEs

Challenges & Perspectives
- Multi Level Monte Carlo
- Propagation Error
Modeling Volatility Risk Premium

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Outline

1. Introduction

2. Backward Representation

3. Application 1: Affine Case

4. Application 2: Non-Affine Case

5. Conclusion
The expression of the variance risk premium ($VRP_t^T$)

$$VRP_t^T = \mathbb{E}(RV_{t,T} | \mathcal{F}_t) - \mathbb{E}^*(RV_{t,T} | \mathcal{F}_t), \quad 0 \leq t \leq T$$

- $RV_{t,T}$: Realized variance of an asset $S$
- $T$: Fixed time horizon
- $V$: is the variance process $S$
Variance Swap and VRP

Variance Swap = Forward contract on realized variance (RV). With the maturity time \( T \), the payoff is

\[
(RV_{t,T} - K)N, \tag{1}
\]

where:
- the integer \( N \) denotes the notional
- \( K \) is called the fixed leg of the variance swap contract
- \( RV_{t,T} \) is the realized annualized variance of the underlying asset over the period \([t, T]\).

\[
RV_{t,T}^d = \frac{C_t}{m-1} \sum_{i=0}^{m-1} \left( \ln \left( \frac{S_{t_{i+1}}}{S_{t_i}} \right) \right)^2, \tag{2}
\]
Estimation of the VRP

The fair conditional variance strike $K_{t,T}$ is

$$K_{t,T} = \mathbb{E}^*(RV_{t,T} | \mathcal{F}_t) = \frac{1}{T-t} \mathbb{E}^*\left( \int_t^T V_s ds | \mathcal{F}_t \right)$$ (3)

The continuous time version of $RV = \text{the quadratic variation of } S_t$


$$K_{t,T} = \frac{2}{T-t} \int_0^\infty \frac{O_t(\kappa, T)}{\kappa^2 B_{t,T}^{1.\$}} d\kappa + \epsilon_t^T$$ (4)

- $B_{t,T}^{1.\$}$ : Time $-t$ price of a Bond, $\epsilon_t^T$ is a Jump term
- $O_t(\kappa, T)$ : Time $t-$ price of OTM vanilla options prices with characteristics $(\kappa, T)$
In continuous time framework, the variance risk premium ($VRP^T_t$) satisfies the equation

$$
\begin{cases}
    dY_t = \frac{1}{T-t} Y_t dt - Z_t dB_t, & 0 \leq t < T, \\
    Y_T = VRP^T_T = 0
\end{cases}
$$

where $Z_t = \frac{1}{T-t} \begin{pmatrix} \gamma_t \\ \gamma^*_t \end{pmatrix}$ and $B_t = \begin{pmatrix} W_t \\ W^*_t \end{pmatrix}$. 
Vix, Variance Swap and VRP

Figure: Vix, Swap Rates, RV
The Model

$(V_t)_{0 \leq t \leq T}$ the adapted variance process of $S$

$$
\begin{cases}
    dV_t = \beta(V_t) dt + \sigma(V_t) dW_t, & 0 < t \leq T \\
    V_0 = v_0 > 0, \quad \beta(x) = b(x) - ax
\end{cases}
$$ (6)

$$
\begin{cases}
    dV_t = \beta^*(V_t) dt + \sigma^*(V_t) dW^*_t; & 0 < t \leq T \\
    V_0 = v_0 > 0, \quad \beta^*(x) = b^*(x) - a^*x
\end{cases}
$$ (7)

$(H) \left\{ b, b^* \sigma, \sigma^* \text{ are continuously differentiable with linear growth condition,} \right.$

$V$ and $V^*$ admit a unique Malliavin derivative.
Introduction

Backward Representation

Application 1: Affine Case

Application 2: Non-Affine Case

Conclusion

Model Based Representation

Proposition (Backward Representation)

Under the condition of the hypothesis \((H)\),

\[
VRP_T^T = \frac{1}{a^a(T - t)} \left[ \int_t^T a^* \mathbb{E}_s \left( D_s [\mathbb{E}_t (V_T)] \right) dW_s - \int_t^T a^* \mathbb{E}_s^* \left( D_s^* [\mathbb{E}_t^* (V_T)] \right) dW_s^* \right] \\
+ \frac{1}{a^*(T - t)} \int_t^T \sigma^* (V_s) dW_s^* - \frac{1}{a(T - t)} \int_t^T \sigma (V_s) dW_s - B_{T,t} + B^*_{T,t}
\]

\[
VRP_T^T = 0 \quad \text{where,}
\]

\[
a(T - t)B_{T,t} = \int_t^T \mathbb{E}_s \left( \int_s^T D_s b(V_u) du \right) dW_s
\]

\[
a^*(T - t)B^*_{T,t} = \int_t^T \mathbb{E}_s^* \left( \int_s^T D_s^* b^*(V_u) du \right) dW_s^*.
\]
Corollary (Affine Case)

We assume that \( \beta(x) = \mu - ax \) and \( \beta^*(x) = \mu^* - a^* x \).

\[
VRP_t^T = \frac{1}{a(T-t)} \left( \int_t^T e^{-a(T-t)} E_s (D_s V_t) \, dW_s - \int_t^T \sigma(V_s) \, dW_s \right) - \frac{1}{a^*(T-t)} \left( \int_t^T e^{-a^*(T-t)} E^*_s (D^*_s V_t) \, dW^*_s - \int_t^T \sigma^*(V_s) \, dW^*_s \right) 
\]

\( VRP_T^T = 0 \)
Heston Case [6]

Under the objective probability $\mathbb{P}$, for $0 \leq t \leq T$

$$S_t^x = x + \mu \int_0^t S_s^x ds + \int_0^t \sqrt{V_s} S_s^x dW_s^1$$

$$V_t = v_0 + \int_0^t \kappa (\theta - V_s) ds + \int_0^t \sigma \sqrt{V_s} dW_s^2.$$

Under the risk neutral probability $\mathbb{P}^*$,

$$S_t^x = x + \int_0^t rS_u^x du + \int_0^t \sqrt{V_t} S_u^x dW_u^{1*}, \quad 0 \leq t \leq T$$

$$V_t = v_0 + \int_0^t \kappa^* (\theta^* - V_u) du + \int_0^t \sigma \sqrt{V_u} dW_u^{2*}, \quad 0 \leq t \leq T$$

where

$$\kappa^* = \kappa + \lambda, \quad \theta^* = \frac{\kappa \theta}{\kappa + \lambda}.$$
VRP Modeling, Heston Case

Lemma

In one-dimensional setting and under the Feller condition $2\kappa \theta \geq \sigma^2$,

i) $\mathbb{P}\left( \inf\{t \geq 0, V_t = 0\} = \infty \right) = 1$, for $v_0 > 0$

ii) $\mathbb{E}(D_s V_t | \mathcal{F}_s) = \sigma e^{-\kappa(t-s)} \sqrt{V_s}, \quad s \leq t \leq T.$

Proposition

If $2\kappa \theta \geq \sigma^2$, the variance risk premium satisfies the backward representation

$$
\begin{cases}
VRP^T_T = \frac{\sigma}{\kappa(T-t)} \int_t^T f_T^\kappa(s) \sqrt{V_s} dW^2_s - \frac{\sigma}{\kappa^*(T-t)} \int_t^T f_T^{\kappa^*}(s) \sqrt{V_s} dW^{2^*}_s, \\
VRP^T_T = 0, \quad f_T^\rho(t) = e^{-\rho(T-t)} - 1, \quad 0 \leq t < T.
\end{cases}
$$
VRP Modeling, Heston Case

Lemma (Classical Result)

\[ VRP_T^t = \frac{1}{T-t} F(\kappa, \theta, T-t, V_t) - \frac{1}{T-t} F(\kappa^*, \theta^*, T-t, V_t) \]  \hspace{1cm} (9)

with \( VRP_T^T = 0 \) and where the function \( F \) is defined by

\[ F(x, y, s, v) = \frac{1}{x} (e^{-xs} - 1)(v - y) + ys. \]
The variance process of the 3/2 Model is described by

\[ V_t = v_0 + \int_0^t \eta V_s - b_\epsilon(V_s) ds + \int_0^t \sigma V_s^{3/2} dW_s^2, \quad \text{under} \quad \mathbb{P} \]

In Carr et al. [1] or in Itkin et al. [7], a risk neutral dynamic of the variance process of the 3/2 model is derived under some plausible assumptions.

\[ V_t = v_0 + \int_0^t \eta^* V_s - b_\epsilon^*(V_s) ds + \int_0^t \sigma^* V_s^{3/2} dW_s^{2*}, \quad \text{under} \quad \mathbb{P}^* \]

\[ v_0, \sigma, \sigma^* > 0 \quad \text{and} \quad b_\epsilon(x) = \epsilon x^2; \quad b_\epsilon^*(x) = \epsilon^* x^2. \]
 Proposition (Non-Affine Case)

If \( \frac{\epsilon}{\sigma^2} \geq \frac{3}{2} \), the variance risk premium \((\text{VRP}_t^T)\) satisfies for \( t \in [0, T] \),

\[
\text{VRP}_t^T = \frac{1}{\eta\eta^*(T-t)} \left[ (\eta - \eta^*)v_T + \eta^*E_t(v_T) + \eta E_t^*(v_T) \right] - \frac{\sigma^*}{\eta^*(T-t)} \int_t^T V_s^{3/2} dW_s^{2*} + \frac{\sigma}{\eta(T-t)} \int_t^T V_s^{3/2} dW_s^2 \\
- B_{T,t}^{\sigma,\eta,\epsilon} + B_{T,T}^{\sigma^*,\eta^*,\epsilon^*},
\]

where \( \eta(t - T)B_{T,t}^{\sigma,\eta,\epsilon} = \int_t^T H_s^T(\sigma, \eta, \epsilon) dW_s^2 \)

\( \eta^*(t - T)B_{T,t}^{\sigma^*,\eta^*,\epsilon^*} = \int_{s=T}^{s=t} H_s^T(\sigma^*, \eta^*, \epsilon^*) dW_s^{2*} \)
VRP Modeling in the 3/2 Model

\[
H_s^T(x, y, z) = \int_s^T e^{y(s-u)} F_{x,y,z}^u(l_u) du,
\]

\[
C_{x,y}(t) = \frac{2y}{x^2(1 - e^{-yt})} \mathbb{1}_{\{x > 0, y \in \mathbb{R}^*\}}
\]

\[
F_{x,y,z}^u(v) = \sqrt{v} C_{x,y}(u) \int_0^1 \theta^2 \left(1 - \theta\right)^{\frac{2z}{x^2} - 3} \exp \left\{ - \theta \exp(-yu) C_{x,y}(u) v \theta \right\} d\theta.
\]
Proof

By the Chain rule (See [9]), we have

\[
\mathbb{E}_s \left( \int_s^T D_s b_\epsilon(V_t) dt \right) = \mathbb{E}_s \left( \int_s^T b'_\epsilon(V_t) D_s V_t dt \right)
\]

\[D_s V_t = D_s \left( \frac{1}{I_t} \right) = -V_t^2 D_s I_t \tag{10}\]

\[
\mathbb{E}_s \left( \int_s^T D_s b_\epsilon(V_t) dt \right) = 2\epsilon \mathbb{E}_s \left( \int_s^T V_t^3 D_s I_t dt \right)
\]

The derivative term \(D_s I_t\) is known from (cf. [10]). By Fubini’s theorem

\[
\mathbb{E}_s \left( \int_s^T D_s b_\epsilon(V_t) dt \right) = 2\epsilon \sigma \int_s^T \mathbb{E}_s \left( \frac{V_t^3 Y_t}{Y_s} \right) dt \tag{11}\]

where

\[
\frac{Y_t}{Y_s} = \exp\{-\eta(t - s)\} \frac{M_t}{M_s}, \quad M_t = \exp \left( \frac{\sigma}{2} \int_0^t \sqrt{V_u} dW_u \right) - \frac{\sigma^2}{8} \int_0^t V_u du.
\]
Proof

Let us consider the probability $\mathbb{P}_M$ defined by the process $M$. The corresponding Radon-Nikodym densities is given by:

$$M_t = \frac{d\mathbb{P}_M}{d\mathbb{P}} \bigg|_{\mathcal{F}_t}, \quad M_t = \exp \left( \frac{\sigma}{2} \int_0^t \sqrt{V_u} dW_u^2 - \frac{\sigma^2}{8} \int_0^t V_u du \right).$$

$$\mathbb{E}_s \left( \int_s^T D_s b_{\epsilon}(V_t) dt \right) = 2\epsilon \sigma \int_s^T \mathbb{E}_s^M (V^3_t) dt, \quad (12)$$
Proof

\[ l_t = \frac{1}{v_0} + \int_0^t (\epsilon + \sigma^2 - \eta l_s) ds - \int_0^t \sigma l_s^{1/2} dW_s^2, \quad 0 \leq t \leq T. \]

Under the probability measure \( \mathbb{P}_M \), \( l_t \) follows then

\[ l_t = \frac{1}{v_0} + \int_0^t (\epsilon + \frac{1}{2} \sigma^2 - \eta l_s) ds - \int_0^t \sigma l_s^{1/2} dB_s^M, \quad 0 \leq t \leq T. \]

where \( B^M \) is a Brownian motion under \( \mathbb{P}_M \). The process \( B^M \) is defined by \( dB_t^M = dW_t^2 - \frac{1}{2} \sigma dt \). From the Lemma 1.1.5 in Diop et al. [5], we have

\[ \mathbb{E}_s^M (V_t^3) = \frac{1}{2 \sqrt{l_s}} F_{\sigma, \eta, \epsilon}^t (l_s) \]

\[ F_{x,y,z}^u (v) = \sqrt{v} \ C_{x,y} (u) \int_0^1 \theta^2 \left( 1 - \theta \right) \frac{2z}{\sqrt{\theta}} - 3 \ \exp \left\{ - \theta \exp(-yu) C_{x,y} (u) v \theta \right\} d\theta. \]
From Proposition 1, we know that

\[ \eta(t - T)B^{\sigma, \eta, \epsilon}_{T,t} = \int_t^T E_s \left( \int_s^T D_s b_\epsilon(V_u) du \right) dW_s^2 = \int_t^T H_s^T(\sigma, \eta, \epsilon) dW_s^2 \]

where

\[ H_s^T(\sigma, \eta, \epsilon) = E_s \left( \int_s^T D_s b_\epsilon(V_u) du \right) = 2\epsilon\sigma \int_s^T E_s^M(V_t^3) dt \]

We obtain the same structure result for the risk neutral part \( B^{\sigma^*, \eta^*, \epsilon^*}_{T,t} \). The Proposition 1 and the previous equality conclude the proof.
Summary & Perspectives

- **Advantages**
  - Illiquid Markets
  - Long dated portfolios (Insurance and Finance)

- **Challenges**
  - Uniqueness of EMM, Calibration of SVM

- **Perspectives**
  - Jumps Extension
  - Non Markovian Modeling
Thank you for your attention
References

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Fourier-Hermite Expansions Algorithm for Backward SDEs

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15 March 2016

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Outline

1. Introduction
2. Hermite Polynomials and BSDE
3. Uniqueness, Convergence
4. Applications
5. Conclusion
We consider

\[
\begin{cases}
-dY_t = g(t, W_t, Y_t, Z_t)dt - Z_t.dW_t, & 0 \leq t < T, \\
Y_T = \phi(W_T).
\end{cases}
\]  

(H2) \quad |g(t_1, x_1, y_1, z_1) - g(t_2, x_2, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|),

(H3) the function \( \phi \) is Lipschitz.

If \( g = 0 \),

\[
Y_t = \Phi(X_T) - \int_t^T Z_s dW_s, \quad \text{and} \quad Y_t = \mathbb{E}(\Phi(W_T)|\mathcal{F}_t).
\]
From Pardoux and Peng,

\[ \forall t \in [0, T], \quad Y_t = u(t, W_t) \quad \text{and} \quad Z_t = (\nabla_x u)(t, W_t), \quad (2) \]

where

\[
\begin{aligned}
\left\{
\begin{array}{c}
\frac{\partial u}{\partial t}(t, x) + \frac{1}{2} \triangle u(t, x) + g(t, x, u, \nabla u) = 0 \\
u(T, x) = \phi(x), \quad \text{with} \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\end{array}
\right.
\end{aligned}
\quad (3)
\]

\[ \triangle u(t, x) =: \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(t, x). \]
Hermite Polynomials

The system of the probabilist’s Hermite polynomials \((H_n(x))_{n \in \mathbb{N}}\) can be defined, for \(x \in \mathbb{R}\)

\[
H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad \text{with} \quad n \in \mathbb{N}^* \quad \text{and} \quad H_0(x) = 1.
\]

\((H_n(x))_{n \in \mathbb{N}}\) are orthogonal with respect to

\[
\phi(x) = \exp\left\{-\frac{1}{2} x^2\right\}/\sqrt{2\pi}, \quad x \in \mathbb{R}.
\]

Hence for \((n, m) \in \mathbb{N}^2\), any pair \((H_n(x), H_m(x))\) satisfies the orthogonality relationship

\[
\int_{-\infty}^{\infty} H_n(x) H_m(x) \phi(x) \, dx = n! \delta_{nm}, \quad \delta_{nm} = \mathbb{1}_{\{n=m\}}, \quad (4)
\]
Hermite Polynomials

We will introduce the "generalized" Hermite polynomial

\[ H_n^{[\theta]}(x) := \theta^{\frac{n}{2}} H_n \left( \frac{x}{\sqrt{\theta}} \right), \quad \theta > 0 \quad H_n^{[0]}(x) = x^n. \]  

We have the addition formula

\[ H_n^{[\theta]}(x + y) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} H_k^{[\theta]}(x). \]  

\[ \bar{H}_n^{[\theta]}(x) := \frac{1}{\sqrt{\theta^n n!}} H_n^{[\theta]}(x), \quad \theta > 0. \]
We introduce
\[
\tilde{H}^{[\theta]}_n(x) := \frac{1}{\sqrt{\theta^n n!}} H^{[\theta]}_n(x), \quad \theta > 0.
\]

This system of the generalized and normalized Hermite polynomials satisfies
\[
\int_{-\infty}^{\infty} \tilde{H}^{[\theta]}_n(x) \tilde{H}^{[\theta]}_m(x) \frac{e^{-\frac{1}{2} \frac{x^2}{\theta}}}{\sqrt{2\pi \theta}} \, dx = \delta_{nm}, \quad m, n \in \mathbb{N}, \quad \theta > 0,
\]  
(7)
Hermite Polynomials Basis

This system satisfies

\[
\int_{-\infty}^{\infty} \bar{H}^{[\theta]}_n(x) \bar{H}^{[\theta]}_m(x) \frac{e^{-\frac{1}{2} \frac{x^2}{\theta}}}{\sqrt{2\pi\theta}} \, dx = \delta_{nm}, \quad m, n \in \mathbb{N}, \quad \theta > 0, \quad (8)
\]

and follows the martingale equality

\[
\mathbb{E} \left[ \bar{H}^{[T]}_n(W_T) \big| \mathcal{F}_t \right] = \left( \frac{t}{T} \right)^{n/2} \bar{H}^{[t]}_n(W_t), \quad 0 \leq t \leq T. \quad (9)
\]

\[
\partial_x \bar{H}^{[\theta]}_n(x) = \left( \frac{n}{\theta} \right)^{1/2} \bar{H}^{[\theta]}_{n-1}(x), \quad \theta > 0, \quad n > 0. \quad (10)
\]
Decomposition of the BSDE

In \((\bar{H}^t)_{t \in [0, T]}\), for a fixed \(t \in [0, T]\)

\[
\begin{cases}
Y_t = \sum_{k \geq 0} \alpha_k(t) \bar{H}_k^t(W_t) \quad \text{a.s.,} \\
Z_t = \sum_{k \geq 0} \beta_k(t) \bar{H}_k^t(W_t) = \sum_{k \geq 0} \left(\frac{k + 1}{t}\right)^{1/2} \alpha_{k+1}(t) \bar{H}_k^t(W_t) \quad \text{a.s.,}
\end{cases}
\]

\(\alpha_k(t) := \mathbb{E} \left[ Y_t \bar{H}_k^t(W_t) \right], \quad \gamma_k(t) := \mathbb{E} \left[ g(t, W_t, Y_t, Z_t) \bar{H}_k^t(W_t) \right].\)
We know \[ \mathbb{E}(Y_T | \mathcal{F}_t) - Y_t + \mathbb{E}\left( \int_t^T g(s, W_s, Y_s, Z_s) \, ds | \mathcal{F}_t \right) = 0. \]

\[ \left( \frac{t}{T} \right)^{k/2} \alpha_k(T) - \alpha_k(t) + \int_t^T \left( \frac{t}{s} \right)^{k/2} \gamma_k(s, \alpha(s)) \, ds = 0, \quad k = 0, 1, 2, \ldots \]

we obtain the CODEs

\[ t \dot{\alpha}_k(t) - \frac{k}{2} \alpha_k(t) + t \gamma_k(t, \alpha(t)) = 0, \quad \text{with} \quad k = 0, 1, 2, \ldots, \quad t \in (0, T]. \quad (11) \]
Affine Example

Let us consider the example of the BSDE where the driver is given by

\[ g(t, x, y, z) = ay + bz, \quad a, b \in \mathbb{R}. \]

\( \gamma_k(t) \) represents \( g \) at \( t \),

\[ \gamma_k(t) = a\alpha_k(t) + b \left( \frac{k + 1}{t} \right)^{1/2} \alpha_{k+1}(t), \quad t \in (0, T]. \]

Therefore we solve the countable systems of ODEs

\[ \dot{\alpha}_k(t) = (a - \frac{k}{2t})\alpha_k(t) + b \left( \frac{k + 1}{t} \right)^{1/2} \alpha_{k+1}(t), \quad k = 0, 1, 2, \ldots \]

The truncated solution of the corresponding BSDE is

\[ Y_t^{(N)} = e^{a(T-t)} \sum_{k=0}^{N} \alpha_k(T) \left( \frac{t}{T} \right)^{k/2} \tilde{H}_j[t] (W_t + b(T - t)) \rightarrow Y_t \]
One-Sided Lipschitz CODEs Problems

In the system \((\tilde{H}^t)_{t \in (0, T]}\), we formulate the following countable backward problem

\[
(I) \left\{ \begin{array}{l}
\dot{\alpha}(t) = f(t, \alpha(t)), \quad 0 \leq t < T \\
\alpha(T) \text{ is the terminal condition}
\end{array} \right.
\]

where \(\alpha(T) = (\alpha_k(T))_{k \geq 1}\) and \(f(t, \alpha(t))\) denotes an infinite dimensional vector where \(f_k(t, \alpha(t)) = -\frac{k}{2t}\alpha_k(t) + \gamma_k(t, \alpha(t))\) for each \(k \in \mathbb{N}\).
If the solution of the problem (I) exists, then the solution is unique on the time interval $[0, T]$.

Proof

$$
\Delta \alpha^{1,2}_k(t) = \alpha^1_k(t) - \alpha^2_k(t), \quad \Delta \beta^{1,2}_k(t) = \left( \frac{k + 1}{t} \right)^{1/2} \Delta \alpha^{1,2}_{k+1}(t).
$$

$$
|\Delta \alpha^{1,2}_k(t)| = \left| \int_t^T \left( \frac{t}{s} \right)^{k/2} (\gamma^1_k(s) - \gamma^2_k(s)) ds \right|,
$$

$$
= \left| \int_t^T \left( \frac{t}{s} \right)^{k/2} \mathbb{E} \left[ (g(s, W_s, Y^1_s, Z^1_s) - g(s, W_s, Y^2_s, Z^2_s)) \tilde{H}^{[t]}_k(W_s) \right] ds. \right.
$$

(13)
Proof

By the Lipschitz property of $g$,

$$|\Delta \alpha^{1,2}_k(t)| \leq K \int_t^T \left( \frac{t}{s} \right)^{k/2} |\Delta \alpha^{1,2}_k(s)|ds + K \int_t^T |\Delta \beta^{1,2}_k(s)|ds.$$ 

By the Gronwall inequality and the Cauchy-Schwartz inequality,

$$|\Delta \alpha^{1,2}_k(t)|^2 \leq K^2 T \exp(2(T-t)) \times \int_t^T |\Delta \beta^{1,2}_k(s)|^2 ds.$$ 

By Itô’s formula

$$\mathbb{E} \left( \left| \Delta Y^{1,2}_t \right|^2 + \int_t^T |\Delta Z^{1,2}_s|^2 ds \right) \leq 2K \int_t^T \mathbb{E} |\Delta Y^{1,2}_s|^2 ds$$

$$+ 2K \int_t^T \mathbb{E} |\Delta Y^{1,2}_s| |\Delta Z^{1,2}_s| ds.$$
Proof

By Young Inequality ( $\forall \epsilon \geq 0, \ 2ab \leq \frac{1}{\epsilon} a^2 + \epsilon b^2$), there exists a constant $C > 0$ such that

$$(1 - \epsilon K) \int_t^T \mathbb{E} |\Delta Z_{s}^{1,2}|^2 \, ds \leq K(2 + 1/\epsilon) \int_t^T \mathbb{E} |\Delta Y_{s}^{1,2}|^2 \, ds, \quad \forall \epsilon \geq 0.$$  

By choosing $\epsilon = 2/K$

$$\sum_{k \geq 0} \int_t^T |\Delta \beta_{k}^{1,2} (s)|^2 \, ds \leq C \int_t^T \sum_{k \geq 0} |\Delta \alpha_{k}^{1,2} (s)|^2 \, ds.$$  

We obtain from above,

$$\sum_{k \geq 0} |\Delta \alpha_{k}^{1,2} (t)|^2 \leq CK^2 T \exp 2(T - t) \times \int_t^T \sum_{k \geq 0} |\Delta \alpha_{k}^{1,2} (s)|^2 \, ds.$$  

By the Gronwall inequality

$$\sum_{k > 0} |\Delta \alpha_{k}^{1,2} (t)|^2 = 0, \quad \Rightarrow \quad \alpha_{k}^{1}(t) = \alpha_{k}^{2}(t), \quad k = 0, 1, 2, \ldots$$
Projected Problems (Stiff system of ODEs)

\[
(l_n) \begin{cases} 
\dot{\alpha}^n(t) = P_n f(t, \alpha^n(t)), & 0 \leq t < T \\
\alpha^n(T) = P_n \alpha(T), & t = T 
\end{cases}
\]

Lemma

If the functional vector \( f \) is continuous on the set \([0, T] \times l^2(\mathbb{N})\), then for \( \alpha, \beta \in l^2(\mathbb{N}) \), \( f \) satisfies the following quadratic inequality

\[
(f(t, \alpha) - f(t, \beta), \alpha - \beta) \leq K(1 + \frac{K}{2})|\alpha - \beta|^2, \quad \text{for all} \quad t \in [0, T]
\]
Convergence of the Truncated Solution

Lemma

Under the assumption $(H)$, the function $\alpha_k(.)$ solves the following equivalent ODE. For all $(k, t) \in \mathbb{N} \times [0, T],$

$$\dot{\alpha}_k(t) = f_k(t, \alpha(t)) = -\mathbb{E}\left( \partial_t F_k(t, W_t) + \frac{1}{2} \partial^2_{xx} F_k(t, W_t) \right)$$

where $F_k(t, x) = u(t, x) H_k^{[t]}(x)$ and $u$ solves the PDE (3).
Let us consider the previous family of the orthogonal projection operators \((P_n)_{n \geq 1}\) in the span of the first \(n\) first basis. The truncated solution \(\alpha^n\) of the system of ordinary differential equations \((I_n)\) converges uniformly to the true solution on the time interval \([0, T]\), when \(n \to \infty\).

The proof of the result is based on the Theorem 7.1. in Klaus Deimling.
Description of the algorithm

- **Initialisation**: Approximate the terminal condition $\bar{Y}_T = \phi(W_T)$ and compute the coefficients $\bar{\alpha}_k(T) = \alpha_k(T)$ and $\bar{\beta}_k(T) = \beta_k(T) = \alpha_{k+1}(T)\left(\frac{k+1}{T}\right)^{1/2}$ for $k = 0, 1, 2, \ldots$

- For $i = (N - 1)$ to 0, on each sub-interval $[t_i, t_{i+1}] \subset [0, T]$ with $t_i, t_{i+1} \in \pi$,
  - compute $\bar{\gamma}^*_{t_{i+1}}$ by the following projection

$$
\begin{align*}
\text{Find} \quad \bar{\gamma}^*_{t_{i+1}} &= (\bar{\gamma}_1(t_{i+1}), \bar{\gamma}_2(t_{i+1}), \bar{\gamma}_2(t_{i+1}), \bar{\gamma}_3(t_{i+1}), \ldots) \quad \text{such that}, \\
J(\bar{\gamma}^*_{t_{i+1}}) &= \inf_\xi \mathbb{E} \left| \xi \bar{H}_i(W_{t_{i+1}}) - g(t_{i+1}, W_{t_{i+1}}, \bar{Y}_{t_{i+1}}, \bar{Z}_{t_{i+1}}) \right|^2,
\end{align*}
$$

where the family of function $\bar{H}_i := \left(\bar{H}_0^{[t_{i+1}]}, \bar{H}_1^{[t_{i+1}]}, \bar{H}_2^{[t_{i+1}]}, \ldots\right)$. 
Description of the algorithm

- compute $\bar{\alpha}_{t_i}$ and $\bar{\beta}_k(t_i)$

\[
\begin{align*}
\bar{\alpha}_k(t_i) &= \left(\frac{t_i}{t_{i+1}}\right)^{k/2} \bar{\alpha}_k(t_{i+1}) + \Delta_i \left(\frac{t_i}{t_{i+1}}\right)^{k/2} \bar{\gamma}_k(t_{i+1}) = 0, \\
\bar{\beta}_k(t_i) &= \bar{\alpha}_{k+1}(t_i)(\frac{k+1}{t_i})^{1/2} \quad k = 0, 1, 2, \ldots
\end{align*}
\]

- compute

\[\bar{Y}_{t_i} = \sum_{k \geq 0} \bar{\alpha}_k(t_i) \bar{H}^{[t_i]}_k(W_{t_i}), \quad \bar{Z}_{t_i} = \sum_{k \geq 0} \bar{\beta}_k(t_i) \bar{H}^{[t_i]}_k(W_{t_i})\]

- **End of the algorithm**

The couple of coefficients $(\bar{\alpha}, \bar{\gamma})$ is the Euler approximation of the couple $(\alpha, \gamma)$. 
Theorem

Under the assumptions \((H)\) and considering the previous uniform subdivision \(\pi\) of the interval \([0, T]\), there exists a positive constant \(C\) independent of \(\pi\) such that

\[
\max_{0 \leq i < N} \mathbb{E} |Y_{t_i} - \bar{Y}_{t_i}|^2 + \mathbb{E} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}_{t_i}|^2 ds \leq C|\pi|. \tag{14}
\]

The couple of coefficients \((\bar{Y}, \bar{Z})\) are the Euler approximation of the couple \((Y, Z)\).
We consider the system

\[
\begin{cases}
-dY_t = f(s, W_s, Y_s, Z_s)dt - Z_t dW_t, & 0 \leq t < T, \\
Y_T = \phi(W_T), & t = T = 1,
\end{cases}
\] (15)

where

\[
\begin{align*}
\phi(x) &= x \arctan(x) - \ln(\sqrt{1 + x^2}) \\
f(t, W_t, Y_t, Z_t) &= -\frac{1}{2(1 + \tan^2(Z_t))}.
\end{align*}
\]

Thus,

\[
(Y_t, Z_t) = \left(-\frac{1}{2} \ln(1 + W_t^2) + W_t \arctan(W_t) , \arctan(W_t)\right) \quad \text{a.s.}
\]

\[
(Y_0, Z_0) = (0, 0).
\]
Log-Error Curve

\[ k = 4, \ M = 10^4 \]

Figure: Log-Error curve to estimate \((Y_0, Z_0) = (0, 0)\)
Example 2

We consider the system

\[
\begin{cases}
-dY_t = g(s, W_s, Y_s, Z_s)dt - Z_t dW_t, & 0 \leq t < T \\
Y_1 = \phi(W_1), & T = 1,
\end{cases}
\]

where the functions \( f \) and \( \phi \) are defined as follows;

\[
\phi(x) = \cos(x + 1), \quad x \in \mathbb{R},
\]

\[
g(t, X_t, Y_t, Z_t) = Z_t(Y_t + 1) - \frac{1}{2}(Y_t - \sin(2(t + W_t))) + \cos(t + W_t).
\]

The solution is

\[
(Y_t, Z_t) = (\cos(W_t + t), -\sin(W_t + t)), \quad a.s.
\]

\[
(Y_0, Z_0) = (1, 0).
\]
Log-Error Curve

\[ k = 4, \ M = 10^4 \]

Figure: Log-Error curve to estimate \((Y_0, Z_0) = (0, 0)\)
Summary & Perspectives

- **Advantages**
  - Simple Algorithm, Higher Scheme Order
  - High dimensions

- **Limits**
  - Global control of the propagation error
  - Curse of dimensionality

- **Perspectives**
  - Full diffusion Case, semimartingale + Jump case
Two-Step Valuation of the Unit Linked Contract with Mortality Risk

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15 March 2016

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Outline

1. Introduction
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4. Applications