Variable Annuity Semi-Static Hedging Strategy Testing with DSLs and GPUs

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Section 1: High Level Overview
Insurance Retirement Products and Systematic Financial Risk

- **Equity-Based Insurance Guarantees**
  - Investment Guarantees embedded in Life Insurance contracts
  - Modeled as complex long-term derivatives contracts
  - Examples
    - Variable Annuities, Equity-Indexed Annuities

- **Risk Management and Hedging**
  - These products or derivatives create market risks for insurers, e.g.
    - Equity market risk
    - Interest rate risk
    - Volatility risk
  - Systematic risk accretes as the insurer sells more product
  - These risks ultimately need to be transferred or hedged
The Hedging Simulation Process at a High Level

One of many paths

0 1 2

Steps

Time

The jth hedge point

Outer Loop
Real World Scenarios

At an Inner Loop path and step Pair

“Inner Loop” Risk Neutral Scenario Generator

Liability Models

Liability Values

Hedging Strategy

Results

Legend

GPU Accelerated

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Section 2: Our Approach to Running Complex Hedging Simulations
VA Hedging Simulation Issues and Solutions

- **Typical Computational Challenges Involved in Running Hedging Simulations Include:**
  - Long run times
  - Realistic modeling
  - Numerical stability
  - Heterogeneous Systems and Data Headaches

- **One Proposed Solution**
  - **Language-Oriented Programming Paradigm**
    - “Rather than solving problems in general-purpose programming languages, the programmer creates one or more **domain-specific languages (DSLs)** for the problem first, and solves the problems in those languages”¹
    - We propose that DSLs are an excellent fit for GPU programming for data parallel applications in specialized domains

Domain Specific Languages

- Aon Benfield’s PathWise Modeling Studio DSL for GPUs and Analytics Studio for Scripting
Domain Specific Language Continued

- A Sample Screenshot of Aon Benfield’s PathWise Modeling Studio DSL for GPUs
Domain Specific Language Benefits

- **Productivity**
  - Business Logic can be implemented by Subject Matter Experts (SMEs), without requiring any programming expertise
  - Programming experts can in turn develop and improve software infrastructure without disrupting the SMEs
  - One SME can implement a complex Monte Carlo model in a week versus 6-12 months if directly using general-purpose language, GPUs, grid middleware, and a cloud API

- **Models implemented in the DSL can be automatically targeted to execute on GPU hardware, grid middleware and cloud infrastructure**
  - Massive performance gains are essentially “free” for the DSL users

- **Auditing and debugging**
  - Auditors and SMEs can easily validate and debug business logic, without being exposed to programming complexities
Putting it All Together—A DSL+Scripting+GPUs

- **Computational Steering**
  - Implementing this logic in a **maintainable** and **efficient** manner is a major Software Design problem in itself

- **Proposed Solution**
  - Use a General-Purpose High Level Scripting Language
Computational Steering

- **Our Approach**
  - Providing necessary APIs to integrate seamlessly with DSL models and data
    - Grid / Cloud Middleware API
    - Data storage API
    - Large Scale Optimization Libraries
    - Bloomberg Open API
  - Python
    - High-level, interactive scripting languages (such as Python) have well documented productivity benefits for users
    - A large number of scientific computing tools are available out-of-the-box (e.g. numerical arrays, plotting, etc)
    - Libraries and APIs allow a vast majority of computations to be off-loaded to underlying C function calls
GPU Cloud Computing

- **Benefits**
  - Amazon EC2 offers Cluster GPUs On-Demand Instances and 10GigE interconnects
  - Highly economical when provisioning large clusters for short periods of time
  - Example: Quarterly Stochastic-on-Stochastic reporting (1 run per quarterly, 100 GPUs)

- GPU cloud solution compared to a traditional CPU cluster solution collocated in a data center achieves an estimated **performance per dollar cost efficiency of 1500x**
GPU Cloud Computing

- **Cloud Computing Challenges**
  - **Performance**
    - Cloud GPUs do not behave in the same way as bare-metal GPUs
    - Para-virtualization technology used by cloud providers leads to significant overheads, especially in CPU-GPU synchronization of critical sections of code
    - Our initial attempts to run our models on Amazon’s GPU cloud led to a 200% performance loss
    - Optimizations to our DSL compiler and runtime libraries allowed us to reduce this overhead to 10-20%
  - **Integration**
    - DSL runtime libraries and middleware had to be modified to integrate with cloud API
  - **Stability**
    - Fault-tolerance has to be built into the application in order to effectively use the cloud (especially if utilizing *spot* instances)
Section 3: Semi Static Hedging Simulations
Liu’s Semi-Static Hedging Strategy (LSSHS)

- We start by following the approach taken from “Pricing and Hedging the Guaranteed Minimum Withdrawal Benefits in Variable Annuities” (Yan Liu, 2010) to set up a semi-static hedging strategy on a GMWB.

- A GMWB is a kind of path-dependent insurance product. In order to hedge the net liability risk, the paper proposed a method: using a basket of put options to approximate the conditional net liability.

- A key idea in the paper is the notion of a conditional net liability at hedge point $t_i$ and it is defined as the expectation of future liability conditioned on the underlying index level at next hedge point $t_{i+1}$ is known.

- Suppose that there are $N$ hedge points: $t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N$, at time $t_{i+1}$, a portfolio of put options, with some predetermined strike levels, are selected via an optimization process to hedge the net liability risk for the GMWB product.

- A key requirement for implementation includes having knowledge about the future index distribution to generate bridged sample paths.

- A series of sample paths are generated in a consistent manner with each future index value, and a conditional net liability is then repeatedly calculated to obtain an expected conditional net liability.

- For a Black-Scholes world, the future index distribution is log-normal one, and a Brownian Bridge process is used to generate the required sample paths in the paper.
Practical Issues to consider when using LSSHS

- For some complicated scenario generation models it is difficult to know future index distribution.
- It is not easy either to implement a method of generating bridged sample paths, for multiple underlying indices, or for jump processes.
- Our approach
  - We leverage the power of PathWise and GPUs in flexible, low cost and transparent HPC environment, to run these complex simulations.
  - Instead of calculating a conditional net liability, we directly approximate the net liability using linear combination of put options.
  - At each hedge point on any sample path, we project 2000 possible values of the net Liability at a one year horizon. We then use an optimization process to select a basket of put options to hedge the liability risk, and repeat this process to the maturity of the contract.
  - At each hedge point, the option strike levels are automatically adjusted depending on the prevailing underlying index level.
Semi-Static Hedging Optimization Process

- Denote by $L_k$ as the net liability at any hedge point $t_k$.
- The hedging portfolio is the weighted sum of options with different strike levels—we use 6 put options with the following strikes: 50%, 60%, 70%, 80%, 90% and 100%—and denote $A_k$ as the portfolio value at any time $t_k$.
- Denote by $dL_k$ the liability increment between $t_k$ and $t_{k+1}$, by $dA_k$ the portfolio increment between $t_k$ and $t_{k+1}$.
- 1. to minimize the local risk increment to find the option weights:

$$E_{t_k}[(dL_k - dA_k)^2]$$

- Liu’s objective function:

$$E_{t_k}[(CNL_k - A_k)^2]$$

where $CNL_k$ is the conditional net liability at time $t_k$. 
Case Study - GMWB

- Index return: 5%
- Risk neutral rate: 5%
- Volatility: 20%
- Term: 15yr
- Premium: $100
- Annual withdrawal frequency: every four months
- Annual withdrawal rate: 6.67%
- Fair dividend rate: 0.4877%
- Hedge period: 1yr
- Option term: 1yr
- Number of sample paths: 500
- Number of projections: 2000
- Number of simulations: 5000
Sample Paths, Projections and Expected Net Liability Calculations

- We evaluate the hedging strategy over 500 paths with 15 steps, for a total of 7500 stochastic-on-stochastic calculations points.

<table>
<thead>
<tr>
<th>quarter</th>
<th>Index</th>
<th>Fund</th>
<th>Account</th>
<th>Withdrawal</th>
<th>Cumulative Charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>103.24</td>
<td>103.12</td>
<td>101.45</td>
<td>1.67</td>
<td>0.12</td>
</tr>
<tr>
<td>2</td>
<td>97.65</td>
<td>97.41</td>
<td>94.17</td>
<td>1.67</td>
<td>0.24</td>
</tr>
<tr>
<td>3</td>
<td>86.02</td>
<td>85.71</td>
<td>81.19</td>
<td>1.67</td>
<td>0.36</td>
</tr>
<tr>
<td>4</td>
<td>80.93</td>
<td>80.53</td>
<td>74.61</td>
<td>1.67</td>
<td>0.45</td>
</tr>
<tr>
<td>60</td>
<td>169.74</td>
<td>157.77</td>
<td>35.54</td>
<td>1.67</td>
<td>4.16</td>
</tr>
</tbody>
</table>

For each of these 2000 Projections 5000 sample paths are run to calculate the Expected Net Liability at each point.
The 2000 projection estimates are generated from log-normal distribution.

It takes 4 GPUs about 60 minutes to compute this whole 15-year SOS hedging test.
Diagram of the SOS Process for ABSSHS

One of 500 paths

The jth hedge point

The j+1 th hedge point

0 1 2 . . . 15 Years

Optimization Routine to find the right combination of put options, repeated for each path and step

The 2000 projections are generated from GMWB’s underlying financial model

Expected Net Liability is calculated here

<table>
<thead>
<tr>
<th># paths</th>
<th># time steps</th>
<th># projections</th>
<th># Exp. Calc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>x 15</td>
<td>x 2000</td>
<td>x 5000</td>
</tr>
</tbody>
</table>

75 Billion Unique Paths or 15 Million Expected Net Liability Calculations each using 5000 paths
Replication of Liu’s Key Results

- Replicate Conditional Expected Net Liability plot in Liu’s paper on pp. 69
- In optimization procedure, the portfolio option weights depends on the log-normal sample points. Here we show the option cost and Sum Squares Replication Error (SSER).

<table>
<thead>
<tr>
<th>Strike</th>
<th>Liu’s Put price</th>
<th>AB’s Put price</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.0003</td>
<td>0.0003</td>
</tr>
<tr>
<td>60</td>
<td>0.0113</td>
<td>0.0108</td>
</tr>
<tr>
<td>70</td>
<td>0.1262</td>
<td>0.1225</td>
</tr>
<tr>
<td>80</td>
<td>0.6872</td>
<td>0.6748</td>
</tr>
<tr>
<td>90</td>
<td>2.3101</td>
<td>2.2844</td>
</tr>
<tr>
<td>100</td>
<td>5.5735</td>
<td>5.5367</td>
</tr>
</tbody>
</table>

| Liu’s \(\theta^{LS}\) | 0.1931 | 0.1343 |
| AB’s \(\theta^{LS}\)  | 0.2151 | 0.1357 |

<table>
<thead>
<tr>
<th>Liu</th>
<th>Cost</th>
<th>Average SSRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>1.5778</td>
<td>6.0376</td>
</tr>
</tbody>
</table>

The table above shows the option cost and average SSRE for both Liu’s results and AB’s results.
PV of Profit & Loss Statistics

The following table shows the discounted cumulative profit and loss at maturity statistics for the two different semi-static hedging strategies.

<table>
<thead>
<tr>
<th>P&amp;L</th>
<th>Min</th>
<th>Max</th>
<th>Range</th>
<th>Mean</th>
<th>STD</th>
<th>95% pct.</th>
<th>5% pct.</th>
<th>CTE95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naked</td>
<td>-38.73</td>
<td>14.03</td>
<td>52.76</td>
<td>-0.18</td>
<td>9.00</td>
<td>8.66</td>
<td>-20.13</td>
<td>-24.92</td>
</tr>
<tr>
<td>LSSHS</td>
<td>-14.62</td>
<td>15.39</td>
<td>30.01</td>
<td>0.00</td>
<td>4.64</td>
<td>7.17</td>
<td>-8.09</td>
<td>-9.88</td>
</tr>
<tr>
<td>ABSSHS</td>
<td>-10.19</td>
<td>13.18</td>
<td>23.37</td>
<td>-0.16</td>
<td>4.22</td>
<td>7.3</td>
<td>-6.17</td>
<td>-7.45</td>
</tr>
</tbody>
</table>

- Note the ABSSHS has lower CTE95 number but a slightly higher implementation cost
Sample Paths Of Naked Net Liability

- The PV at time zero of the naked P&L of the net liability, i.e. the difference between charges and benefits. After the sample path turns to be flat, insurance companies start to be “on-the-risk” so to speak.

- Note the long-tailed risk associated with this product.
Density Function At Maturity Time

- LSSHS’ mean value is closer to zero but has a larger standard deviation than ABSSHS.

The following density functions are fitted with sample points by using Gaussian kernel functions. Scott's method is used to choose bandwidth.
Case Study – The Mean Cumulative NPV of Profit & Loss At Different Times

- After 4 years, LSSHS’ mean values are closer to zero than ABSSHS’ mean values
Case Study – STD of NPV of Cumulative Profit & Loss At Different Times

- After 4 years, ABSSHS’ standard deviations are slightly smaller than LSSHS’ standard deviations.
Run Time Numbers

- The following table shows the run time of the two semi-static stochastic-on-stochastic hedge strategy tests. The Simulation part represents all the computations and projections calculations. The Optimization part represents all the time spent running optimizations to find the right semi-static hedging strategies at each path and step pair.

<table>
<thead>
<tr>
<th>Elapsed Time (min)</th>
<th>LSSHS</th>
<th>ABSSHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation</td>
<td>61.57</td>
<td>92.47</td>
</tr>
<tr>
<td>Optimization</td>
<td>34.73</td>
<td>35.12</td>
</tr>
<tr>
<td>Total</td>
<td>96.30</td>
<td>127.59</td>
</tr>
<tr>
<td>Number of GPUs</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Number of GPUs: 4
The Delta-Hedging simulation does not require projecting 2000 index values at each time point.

However, the Delta-Hedging simulation requires more dense time grids for accuracy than the semi-static hedging simulations.

A Base, Up and Down calculation is performed for each path and step pair.

It takes 4 GPUs about 5 minutes to compute whole 15-year monthly SOS hedging simulation.
## A Hybrid Hedging Strategy (HHS)

### Time grid (month) | Strategy
---|---
0 | Mixed
1 | 50% Delta Hedge
2 | 50% Delta Hedge
| |  
11 | 50% D
12 | Mixed
13 | 50% Delta Hedge
| |  
167 | 50% Delta Hedge
168 | Mixed
169 | 50% Delta Hedge
| |  
179 | 50% Delta Hedge
180 | Clear All Asset Positions

- At each semi-static hedge point, project 2000 future index points.
- Simulate delta hedging between the semi static hedge points using a monthly time step.
- Assume 50% of the risk was delta hedged over the course of the one year period.
- Use a semi-static hedge strategy then to cover off all the remaining risk.

There are two types of hedge time grids:

- The pink rows are Semi-Static hedge points.
- The grey rows are Delta hedge points.

At maturity, all positions are cleared.
A Hybrid Hedging Strategy - Results

- We use a hybrid hedging strategy which is a mix of delta and semi-static hedging strategies.
- Please see that table below for cumulative NPV of profit across all paths

<table>
<thead>
<tr>
<th>P&amp;L</th>
<th>Min</th>
<th>Max</th>
<th>Range</th>
<th>Mean</th>
<th>STD</th>
<th>95% pct</th>
<th>5% pct</th>
<th>CTE95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naked</td>
<td>-35.88</td>
<td>26.26</td>
<td>62.14</td>
<td>-0.05</td>
<td>9.11</td>
<td>8.26</td>
<td>-20.91</td>
<td>-25.26</td>
</tr>
<tr>
<td>Delta Hedge</td>
<td>-13.73</td>
<td>6.65</td>
<td>20.39</td>
<td>-0.17</td>
<td>2.72</td>
<td>2.63</td>
<td>-6.21</td>
<td>-8.33</td>
</tr>
<tr>
<td>ABSSHS</td>
<td>-16.54</td>
<td>24.60</td>
<td>41.14</td>
<td>-0.16</td>
<td>4.83</td>
<td>6.62</td>
<td>-8.34</td>
<td>-11.13</td>
</tr>
<tr>
<td>HHS</td>
<td>-11.92</td>
<td>17.92</td>
<td>29.85</td>
<td>-0.27</td>
<td>3.99</td>
<td>7.21</td>
<td>-6.44</td>
<td>-7.62</td>
</tr>
</tbody>
</table>

- Note how the downside tail risk, as measure by CTE95, is significantly reduced using the Hybrid Hedging Strategy

- Are all your eggs in one basket? The results seem to suggest there may be some practical diversification benefits associated with combining different types of hedging strategies on the same risk.
Hedge Efficiency Plot Over The First Year

- This plot contains 500 points and compares the change in value of the Net Liability, including intermediate cashflows, versus the change in value of the Hedge Portfolio from time zero to the end of the first year.
- Note the different R-Squared statistics and the straight line, hockey stick, and mixed profile for the HHS.
- Hedging Efficiency is defined as the r-squared from OLS regression of the change in value of the Net Liability versus the change in value of the Hedge Portfolio.
## Monthly Hedge Efficiency for a Single Path

### R Square Statistics of Monthly Hedging across all the 500 Paths

<table>
<thead>
<tr>
<th></th>
<th>R-SQR.</th>
<th>min</th>
<th>max</th>
<th>range</th>
<th>mean</th>
<th>std.</th>
<th>95%pct</th>
<th>5%pct.</th>
<th>cte95</th>
</tr>
</thead>
<tbody>
<tr>
<td>DHS</td>
<td>0.48</td>
<td>0.99</td>
<td>0.52</td>
<td>0.85</td>
<td>0.11</td>
<td>0.98</td>
<td>0.66</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>ABSSHS</td>
<td>0.08</td>
<td>0.91</td>
<td>0.83</td>
<td>0.62</td>
<td>0.17</td>
<td>0.85</td>
<td>0.25</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>HHS</td>
<td>0.27</td>
<td>0.93</td>
<td>0.66</td>
<td>0.79</td>
<td>0.09</td>
<td>0.90</td>
<td>0.62</td>
<td>0.57</td>
<td></td>
</tr>
</tbody>
</table>

- Semi-Static Hedging may produce very low monthly hedging efficiency numbers
- Such an approach may introduce financial statement volatility
Yearly Hedge Efficiency for a Single Path

R Square Statistics of Yearly Hedging across all the 500 Paths

<table>
<thead>
<tr>
<th></th>
<th>min</th>
<th>max</th>
<th>range</th>
<th>mean</th>
<th>std.</th>
<th>95%pct</th>
<th>5%pct.</th>
<th>cte95</th>
</tr>
</thead>
<tbody>
<tr>
<td>DHS</td>
<td>0.06</td>
<td>0.99</td>
<td>0.94</td>
<td>0.96</td>
<td>0.09</td>
<td>0.99</td>
<td>0.87</td>
<td>0.63</td>
</tr>
<tr>
<td>ABSSHS</td>
<td>0.10</td>
<td>0.99</td>
<td>0.89</td>
<td>0.71</td>
<td>0.21</td>
<td>0.96</td>
<td>0.29</td>
<td>0.22</td>
</tr>
<tr>
<td>HHS</td>
<td>0.16</td>
<td>0.99</td>
<td>0.84</td>
<td>0.87</td>
<td>0.12</td>
<td>0.98</td>
<td>0.66</td>
<td>0.51</td>
</tr>
</tbody>
</table>

- Hedging efficiency measure looks better when matched to hedge strategy horizon
- If you are rewarded based on your hedging efficiency statistic, you will favor dynamic hedging over static hedging
Correlation Structure of Different Hedging Strategies

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- Cumulative Correlation Structure
  Plot jumps every year as new semi-static hedging portfolios are created, with different option holdings

- There is a relatively low level of positive correlation between DHS and ABSSHS

- The correlation between HHS and ABSSHS grows over time

<table>
<thead>
<tr>
<th>Terminal</th>
<th>DHS</th>
<th>ABSSHS</th>
<th>HHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>DHS</td>
<td>1.00</td>
<td>0.53</td>
<td>0.68</td>
</tr>
<tr>
<td>ABSSHS</td>
<td>0.53</td>
<td>1.00</td>
<td>0.96</td>
</tr>
<tr>
<td>HHS</td>
<td>0.68</td>
<td>0.96</td>
<td>1.00</td>
</tr>
</tbody>
</table>

P&L Correlation of Three Hedging Strategies

Time (yr)
Section 4: Conclusions
Key Points

- GPUs are well suited to performing difficult VA hedging simulation calculations
- PathWise abstracts away complex low level coding issues, leverages a common and powerful scripting language, and integrates with large scale optimization software, in a transparent, version controlled and flexible fashion
- PathWise is designed to work with Amazon’s GPU cloud, which may save users a lot of time and money for these types of calculations
- In this presentation we
  - Used GPUs to extend the period of stochastic analysis until the maturity of the contract
  - Demonstrated a generalized approach to creating semi-static hedging portfolios which better lends itself to industry settings and can be extended to work with such things as Stochastic Rates and Volatilities, Jumps, Market Impact Costs, Basket of Sub Accounts, Different Products, etc
  - Introduced hybrid semi-static hedging strategy into the analysis
  - Other possible areas of future practical research include the following:
    - Back Testing Using Real Market Numbers
    - Using More Complex Product Designs and Scenario Generators
    - Including Market Impact and Commissions Costs in the Analyses
REFERENCES

- Static Hedging of Standard Options, Peter Carr and Liuren Wu, Working paper, 2009